WS 2014/15

Harald Räcke

Fakultät für Informatik
TU München

http://www14.in.tum.de/lehre/2014WS/ea/

Winter Term 2014/15

Part I

Organizational Matters

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- Modul: IN2003
- Name: “Efficient Algorithms and Data Structures”
  “Effiziente Algorithmen und Datenstrukturen”
- ECTS: 8 Credit points
- Lectures:
  - 4 SWS
    Mon 10:30–12:00 (Room Interim2)
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    “Einführung in die Informatik 1/2”
  - IN0007
    “Fundamentals of Algorithms and Data Structures”
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    “Basic Theoretic Informatics”
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The Lecturer

- Harald Räcke
- Email: raecke@in.tum.de
- Room: 03.09.044
- Office hours: (by appointment)

Tutorials

- Tutors:
  - Chintan Shah
  - chintan.shah@tum.de
  - Room: 03.09.059
  - Office hours: Wed 11:30–12:30

  - Dario Frascaria
  - frascari@in.tum.de
  - Room: 03.09.035
  - Office hours: (by appointment)

Tutorials

- Monday 16-18 (MI 00.08.038)
  Chintan
- Tuesday 14-16 (MI 00.08.038)
  Dario
- Thursday 10-12 (MI 00.08.038)
  Dario
- Friday 12-14 (MI 00.13.009A)
  Chintan

Assignment sheets

In order to pass the module you need to pass a 3 hour exam.

Assessment

Assignment Sheets:

- An assignment sheet is usually made available on Monday on the module webpage.
- Solutions have to be handed in in the following week before the lecture on Monday.
- You can hand in your solutions by putting them in the right folder in front of room 03.09.019A.
- Solutions have to be given in English.
- Solutions will be discussed in the tutorial of the week when the sheet has been handed in, i.e, sheet may not be corrected by this time.
- You can submit solutions in groups of up to 3 people.

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Assessment

Assignment can be used to improve you grade

- If you obtain 50% of the points on the first half and 50% on the second half of assignments your grade will improve according to the following function

- It will improve by 0.3 or 0.7 respectively.

Examples:

Assessment

Assignment can be used to improve you grade

- If you obtain 50% of the points on the first half **and** 50% on the second half of assignments your grade will improve according to the following function

$$f(x) = \begin{cases} \frac{1}{10}\text{round}\left(10\left(\frac{\text{round}(3x)-1}{3}\right)\right) & 1 < x \leq 4 \\ x & \text{otw.} \end{cases}$$

- It will improve by 0.3 or 0.4, respectively. Examples:

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- It will improve by 0.3 or 0.4, respectively.
  Examples:
  - $3.3 \rightarrow 3.0$
  - $2.0 \rightarrow 1.7$
  - $3.7 \rightarrow 3.3$
  - $1.0 \rightarrow 1.0$
  - > 4.0 no improvement

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Alfred V. Aho, John E. Hopcroft, Jeffrey D. Ullman:
The design and analysis of computer algorithms,
Addison-Wesley Publishing Company: Reading (MA), 1974


Thomas H. Cormen, Charles E. Leiserson, Ron L. Rivest, Clifford Stein:
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2 Literatur


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2. Auflage, Vieweg, 2003


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Addison-Wesley, 2005


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The art of computer programming. Vol. 1: Fundamental Algorithms,
3. Auflage, Addison-Wesley Publishing Company: Reading (MA), 1997

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The art of computer programming. Vol. 3: Sorting and Searching,
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Uwe Schöning:
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Steven S. Skiena:
The Algorithm Design Manual,
Springer, 1998

Part II

Foundations

Vocabularies

$a \cdot b$ "a times b"

"a multiplied by b"

"a into b"

$\frac{a}{b}$ "a divided by b"

"a by b"

"a over b"

(a: numerator (Zähler), b: denominator (Nenner))

a^b "a raised to the b-th power"

"a to the b-th"

"a raised to the power of b"

"a to the power of b"

"a raised to b"

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Vocabularies

$n!$ "n factorial"

$\binom{n}{k}$ "n choose k"

x_i "x subscript i"

"x sub i"

"x i"

$\log_b a$ "log to the base b of a"

"log a to the base b"

$$f : X \to Y, x \mapsto x^2$$

f is a function that maps from domain (Definitionsbereich) X to codomain (Zielmenge) Y. The set $\{y \in Y \mid \exists x \in X : f(x) = y\}$ is the image or the range of the function (Bildbereich/Wertebereich).

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3 Goals

- Gain knowledge about efficient algorithms for important problems, i.e., learn how to solve certain types of problems efficiently.
- Learn how to analyze and judge the efficiency of algorithms.
- Learn how to design efficient algorithms.

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4 Modelling Issues

1 What do you measure?

- Memory requirement
- Running time
- Number of comparisons
- Number of multiplications
- Number of hard-disc accesses
- Program size
- Power consumption
- ...

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4 Modelling Issues

How do you measure?

- Implementing and testing on representative inputs
  - How do you choose your inputs?
  - May be very time-consuming.
  - Very reliable results if done correctly.
  - Results only hold for a specific machine and for a specific set of inputs.
- Theoretical analysis in a specific model of computation.
  - Gives a mathematical bound on the running time of the algorithm.
  - Typically focuses on the worst case.
  - Can give lower bounds like "any comparison-based sorting algorithm needs at least $\Omega(n \log n)$ comparisons in the worst case".

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  - Gives [illegible] like "this algorithm always runs in time $O(n^3)$".
  - Typically focuses on the [illegible].
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- Theoretical analysis in a specific model of computation.
  - Gives a mathematical proof that this algorithm always runs in time at most ...
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  - Can give lower bounds like "any comparison-based sorting algorithm needs at least ... comparisons in the worst case".

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Input length

The theoretical bounds are usually given by a function $f : \mathbb{N} \to \mathbb{N}$ that maps the input length to the running time (or storage space, comparisons, multiplications, program size etc.).

The input length may e.g. be

- the size of the input (number of bits)
- the number of arguments

Example:

Suppose n numbers from the interval $[1, \ldots, n^2]$ have to be sorted. In this case we usually say that the input size is n instead of e.g. $n \log n$ which would be the number of bits required to encode the input.

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Example:

Suppose we numbers from the interval $[0, \ldots, k]$ have to be sorted. In this case we usually say that the input size is n. Instead of e.g. $n \log k$, which would be the number of bits required to encode the input.

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Model of Computation

How to measure performance

- Calculate running time and storage space etc. on a simplified, idealized model of computation, e.g. Random Access Machine (RAM), Turing Machine (TM), ...
- Calculate number of certain basic operations: comparisons, multiplications, harddisc accesses, ...

Version 1 is often easier, but focusing on one type of operation makes it more difficult to obtain meaningful results.

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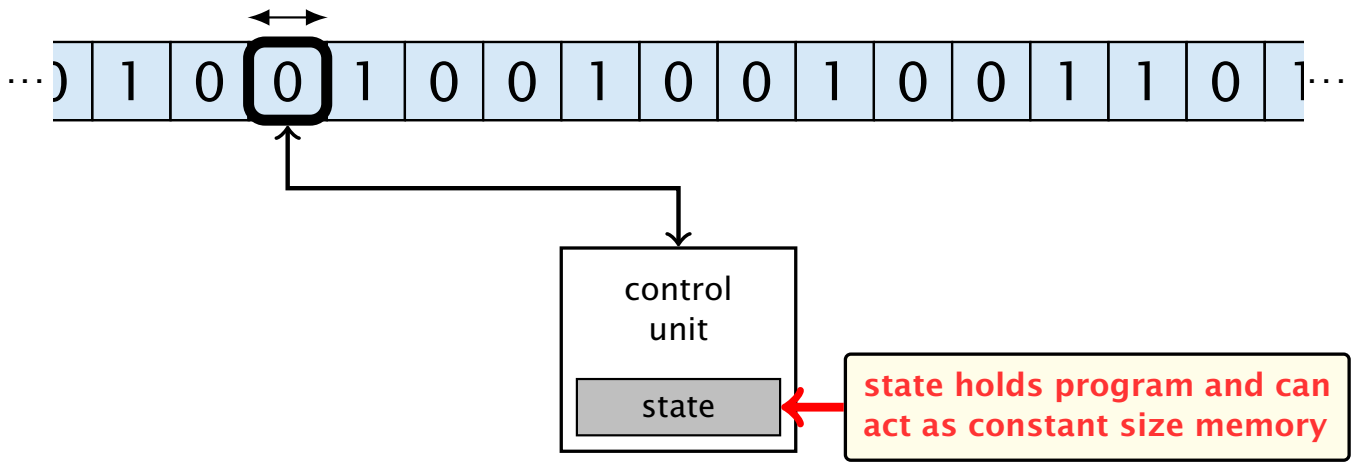
Turing Machine

- Very simple model of computation.
- Only the "current" memory location can be altered.
- Very good model for discussing computabiliy, or polynomial vs. exponential time.
- Some simple problems like recognizing whether input is of the form xx, where x is a string, have quadratic lower bound.

$\Longrightarrow$ **Not a good model for developing efficient algorithms.**



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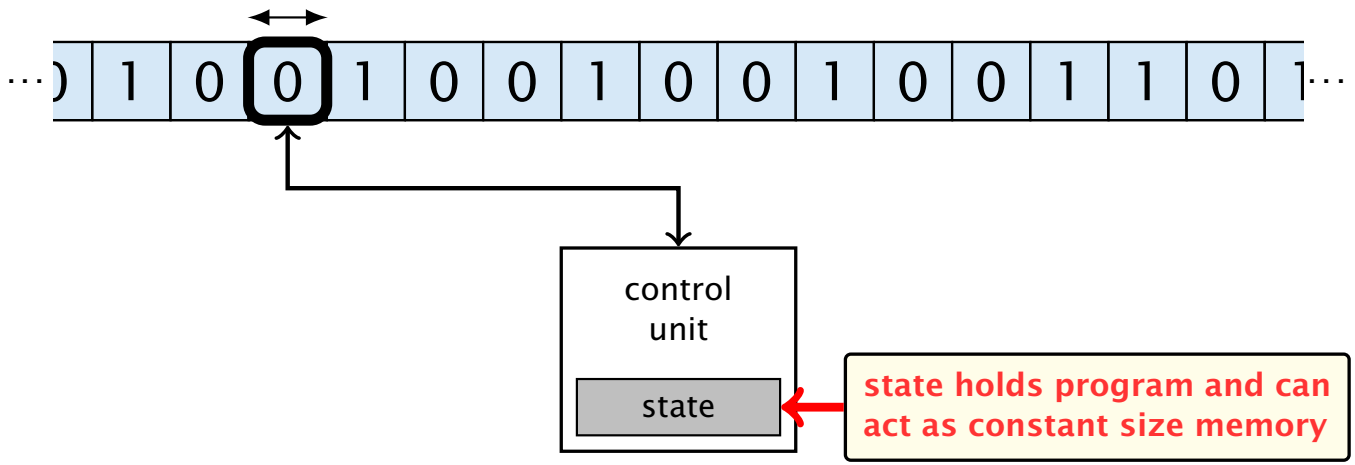

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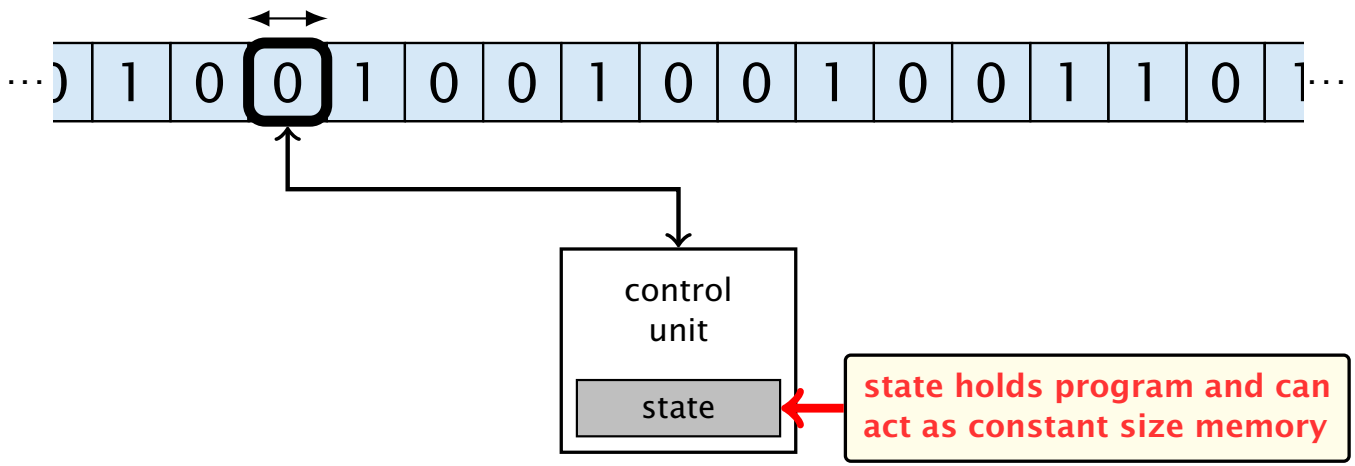

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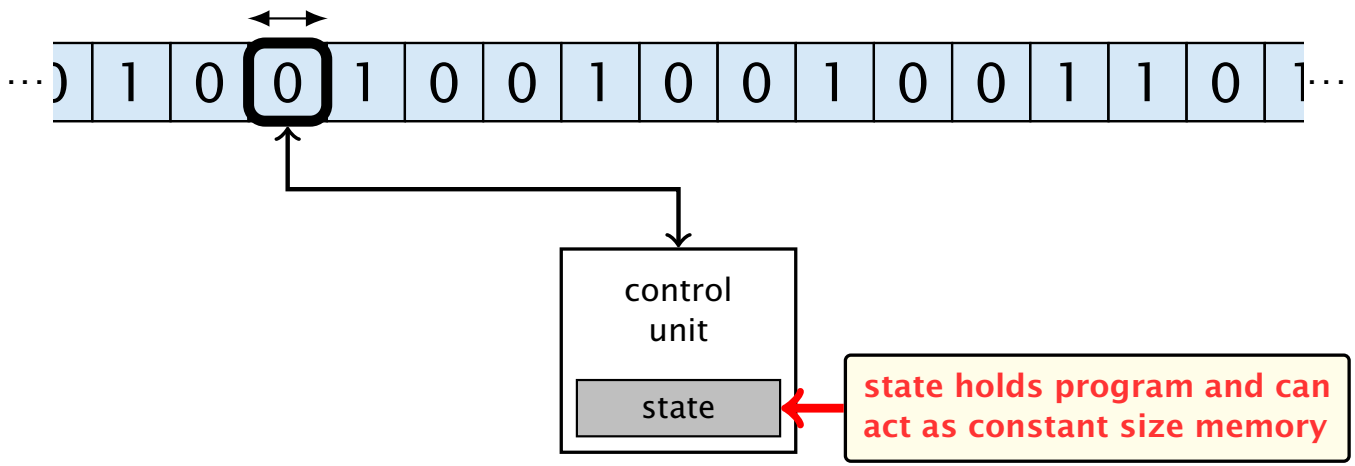

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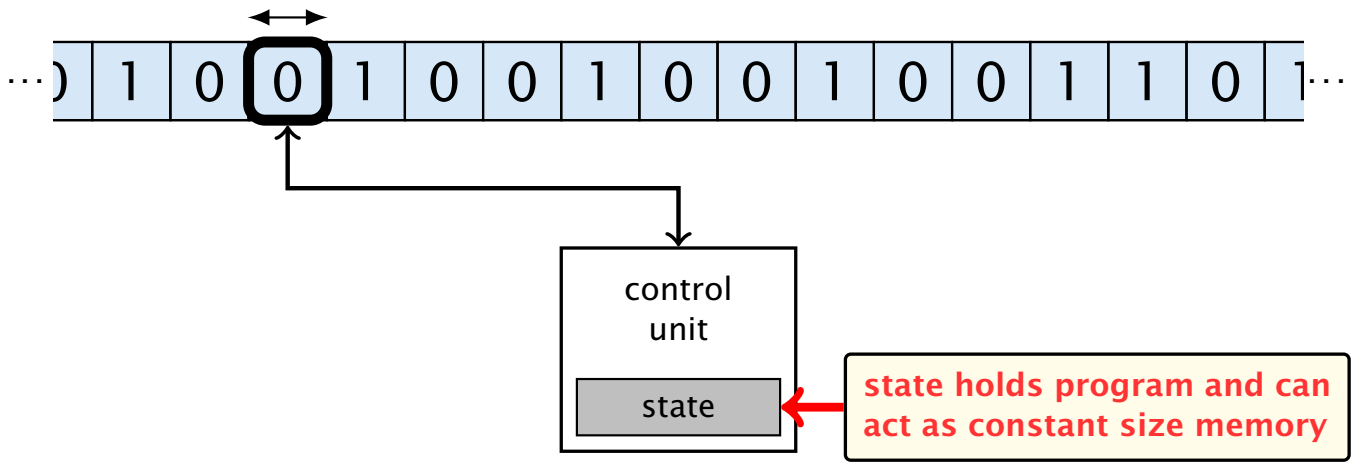
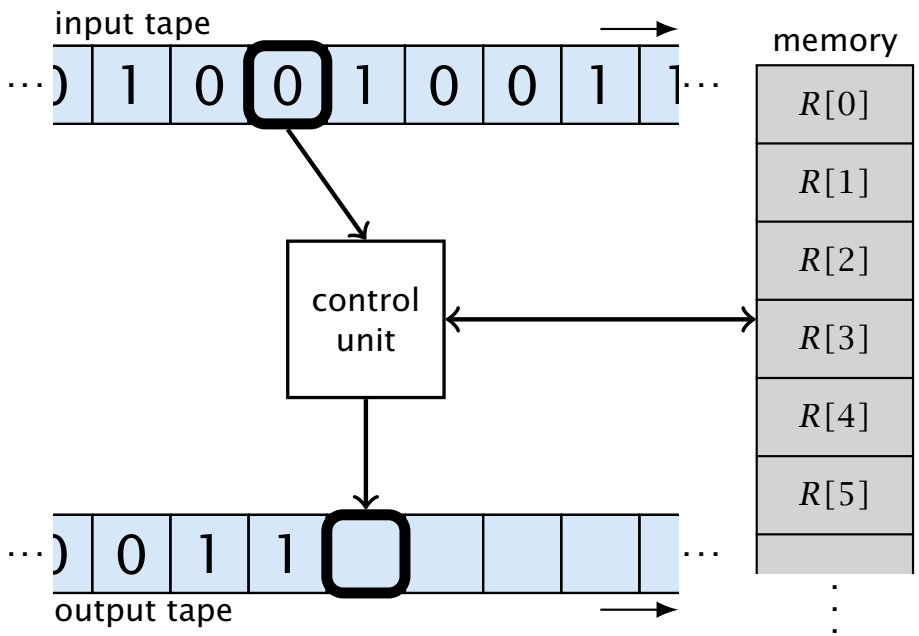


Random Access Machine (RAM)

- Input tape and output tape (sequences of zeros and ones; unbounded length).
- Memory unit: infinite but countable number of registers $R[0], R[1], R[2], \ldots$.
- Registers hold integers.
- Indirect addressing.



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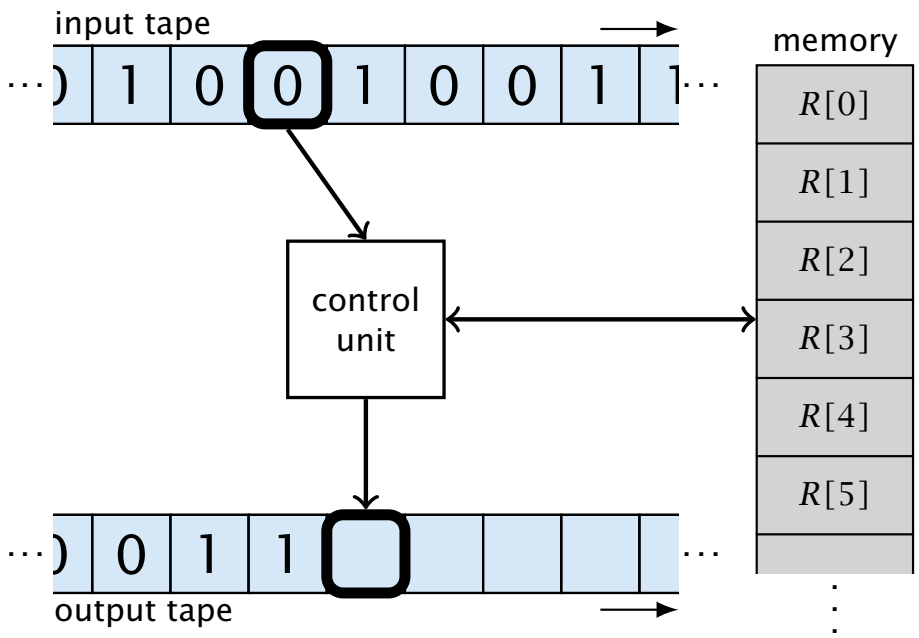

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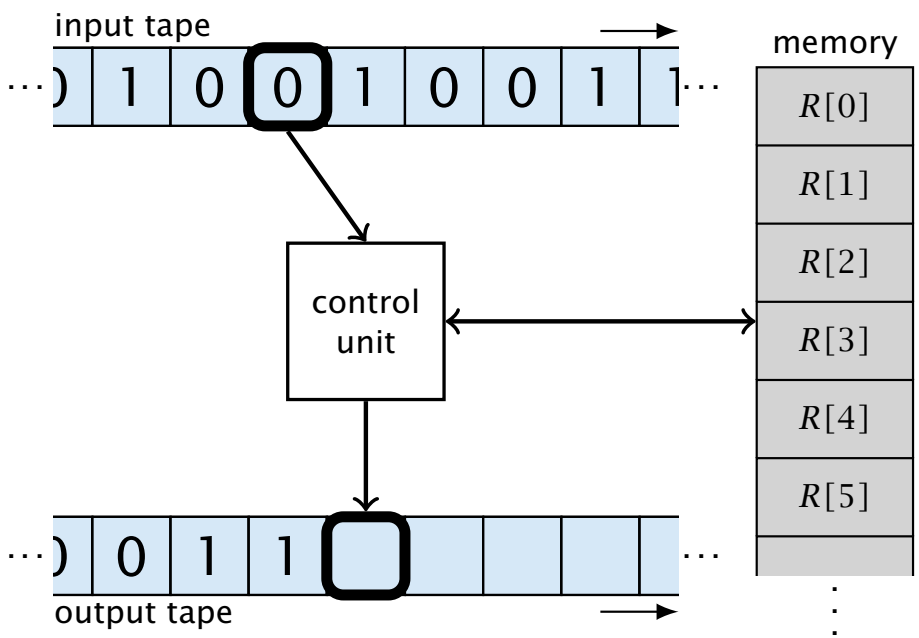

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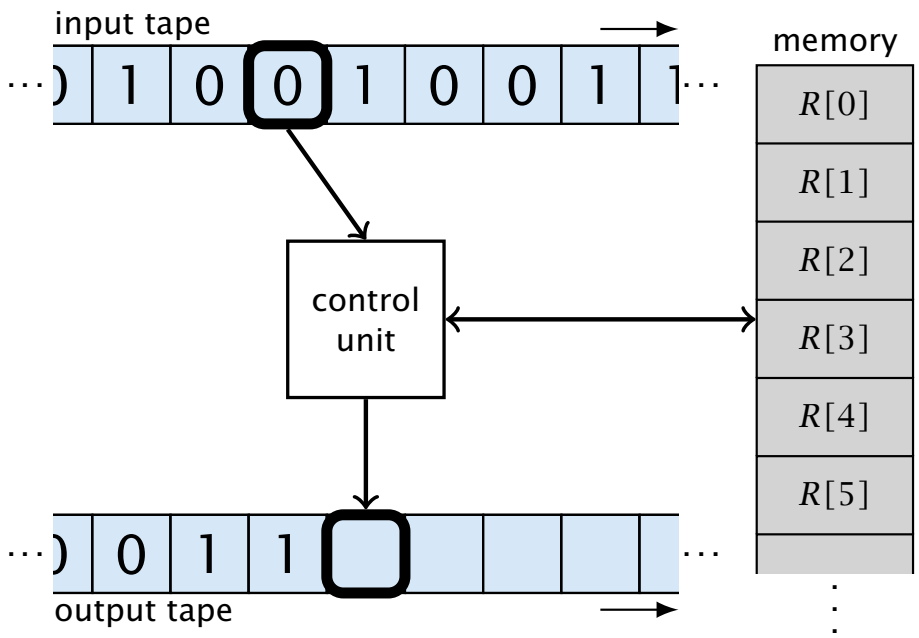

Random Access Machine (RAM)

Operations

- input operations (input tape $\rightarrow R[i]$)
  - READ i
- output operations ($R[i] \rightarrow$ output tape)
  - WRITE i
- register-register transfers
  - $R[i] := R[j]$
  - $R[i] := j$
- **indirect** addressing
  - $R[i] := R[R[j]]$

    loads the content of the j-th register into the i-th register.
  - $R[R[i]] := R[j]$

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Random Access Machine (RAM)

Operations

- branching (including loops) based on comparisons
  - jump x
    jumps to position x in the program;
    sets instruction counter to x;
    reads the next operation to perform from register $R[x]$
  - jumpz x $R[i]$
    jump to x if $R[i] = 0$
    if not the instruction counter is increased by 1;
  - jumpi i
    jump to $R[i]$ (indirect jump);
- arithmetic instructions: $+$, $-$, $\times$, $/$

Random Access Machine (RAM)

Operations

- branching (including loops) based on comparisons
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Model of Computation

- uniform cost model

  Every operation takes time 1.

- logarithmic cost model

  The cost depends on the content of memory cells:

  - The time for a step is equal to the largest operand involved.
  - The storage space of a register is equal to the length (in bits) of the largest value ever stored in it.

Bounded word RAM model: cost is uniform but the largest value stored in a register may not exceed 2^w, where usually $w = \log_2 n$.

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4 Modelling Issues

Example 2

Algorithm 1 RepeatedSquaring(n)

1: $r \leftarrow 2$;
2: **for** $i = 1 \rightarrow n$ **do**
3: $\quad r \leftarrow r^2$
4: **return** r

- running time:
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There are different types of complexity bounds:

- best-case complexity:

$$C_{\text{bc}}(n) := \min\{C(x) \mid |x| = n\}$$

  Usually easy to analyze, but not very meaningful.

- worst-case complexity:

$$C_{\text{wc}}(n) := \max\{C(x) \mid |x| = n\}$$

  Usually moderately easy to analyze; sometimes too pessimistic.

- average case complexity:

$$C_{\text{avg}}(n) := \frac{1}{|I_n|} \sum_{|x|=n} C(x)$$

  more general: probability measure μ

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There are different types of complexity bounds:

- amortized complexity:
  The average cost of data structure operations over a worst case sequence of operations.
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We are usually not interested in exact running times, but only in an asymptotic classification of the running time, that ignores constant factors and constant additive offsets.

- We are usually interested in the running times for large values of n. Then constant additive terms do not play an important role.
- An exact analysis (e.g. exactly counting the number of operations in a RAM) may be hard, but would not lead to more precise results as the computational model is already quite a distance from reality.
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Formal Definition

Let f denote functions from $\mathbb{N}$ to $\mathbb{R}^+$.

- $\mathcal{O}(f) = \{g \mid \exists c > 0\ \exists n_0 \in \mathbb{N}_0\ \forall n \geq n_0 : [g(n) \leq c \cdot f(n)]\}$
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There is an equivalent definition using limes notation (assuming that the respective limes exists). f and g are functions from $\mathbb{N}_0$ to $\mathbb{R}_0^+$.

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1. People write $f = \mathcal{O}(g)$, when they mean $f \in \mathcal{O}(g)$. This is **not** an equality (how could a function be equal to a set of functions).
2. People write $f(n) = \mathcal{O}(g(n))$, when they mean $f \in \mathcal{O}(g)$, with $f : \mathbb{N} \to \mathbb{R}^+, n \mapsto f(n)$, and $g : \mathbb{N} \to \mathbb{R}^+, n \mapsto g(n)$.
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Asymptotic Notation in Equations

How do we interpret an expression like:

$$2n^2 + 3n + 1 = 2n^2 + \Theta(n)$$

Here, $\Theta(n)$ stands for an anonymous function in the set $\Theta(n)$ that makes the expression true.

Note that $\Theta(n)$ is on the right hand side, otw. this interpretation is wrong.

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Careful!

"It is understood" that every occurence of an Θ-symbol (or $\Theta, \Omega, o, \omega$) on the left represents one anonymous function.

Hence, the left side is **not** equal to

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We can view an expression containing asymptotic notation as generating a set:

$$n^2 \cdot \mathcal{O}(n) + \mathcal{O}(\log n)$$

represents

$$\Big\{ f : \mathbb{N} \to \mathbb{R}^+ \mid f(n) = n^2 \cdot g(n) + h(n) \quad \text{with } g(n) \in \mathcal{O}(n) \text{ and } h(n) \in \mathcal{O}(\log n) \Big\}$$

Asymptotic Notation in Equations

Then an asymptotic equation can be interpreted as containement btw. two sets:

$$n^2 \cdot \mathcal{O}(n) + \mathcal{O}(\log n) = \Theta(n^2)$$

represents

$$n^2 \cdot \mathcal{O}(n) + \mathcal{O}(\log n) \subseteq \Theta(n^2)$$

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Lemma 3

Let f, g be functions with the property $\exists n_0 > 0\, \forall n \geq n_0 : f(n) > 0$ (the same for g). Then

- $c \cdot f(n) \in \Theta(f(n))$ *for any constant* c
- $\mathcal{O}(f(n)) + \mathcal{O}(g(n)) = \mathcal{O}(f(n) + g(n))$
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The expressions also hold for Ω. Note that this means that $f(n) + g(n) \in \Theta(\max\{f(n), g(n)\})$.

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Comments

- Do not use asymptotic notation within induction proofs.
- For any constants a, b we have $\log_a n = \Theta(\log_b n)$. Therefore, we will usually ignore the base of a logarithm within asymptotic notation.
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In general asymptotic classification of running times is a good measure for comparing algorithms:

- If the running time analysis is tight and actually occurs in practise (i.e., the asymptotic bound is not a purely theoretical worst-case bound), then the algorithm that has better asymptotic running time will always outperform a weaker algorithm for large enough values of n.
- However, suppose that I have two algorithms:
  - Algorithm A: Running time $\ldots$
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  Clearly, $\ldots$ is better. However, as long as $\ldots$ Algorithm B will be more efficient.

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6 Recurrences

Algorithm 2 mergesort(list L)

1: $n \leftarrow \text{size}(L)$
2: **if** $n \leq 1$ **return** L
3: $L_1 \leftarrow L[1 \cdots \lfloor \frac{n}{2} \rfloor]$
4: $L_2 \leftarrow L[\lfloor \frac{n}{2} \rfloor + 1 \cdots n]$
5: mergesort(L_1)
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This algorithm requires

$$T(n) = T\left(\left\lceil \frac{n}{2} \right\rceil\right) + T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + \mathcal{O}(n) \leq 2T\left(\left\lceil \frac{n}{2} \right\rceil\right) + \mathcal{O}(n)$$

comparisons when $n > 1$ and 0 comparisons when $n \leq 1$.

Recurrences

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Methods for Solving Recurrences

1. **Guessing+Induction**
   Guess the right solution and prove that it is correct via induction. It needs experience to make the right guess.
2. **Master Theorem**
   For a lot of recurrences that appear in the analysis of algorithms this theorem can be used to obtain tight asymptotic bounds. It does not provide exact solutions.
3. **Characteristic Polynomial**
   Linear homogenous recurrences can be solved via this method.

Methods for Solving Recurrences

4. **Generating Functions**
   A more general technique that allows to solve certain types of linear inhomogenous relations and also sometimes non-linear recurrence relations.
5. **Transformation of the Recurrence**
   Sometimes one can transform the given recurrence relations so that it e.g. becomes linear and can therefore be solved with one of the other techniques.

6.1 Guessing+Induction

First we need to get rid of the $\mathcal{O}$-notation in our recurrence:

$$T(n) \leq \begin{cases} 2T(\lceil \frac{n}{2} \rceil) + cn & n \geq 2 \\ 0 & \text{otherwise} \end{cases}$$

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One way of solving such a recurrence is to guess a solution, and check that it is correct by plugging it in.

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Formally one would make an induction proof, where the above is the induction step. The base case is usually trivial.

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  Hence, statement is true if we choose $d \geq c$.

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If we do not do this we instead consider the following recurrence:

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Note that we can do this as for constant-sized inputs the running time is always some constant (b in the above case).

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$\frac{n}{2} + 1 \leq \frac{9}{16}n$

$$\leq dn \log\left(\frac{9}{16}n\right) + 2d\log n + cn$$

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$\log n \leq \frac{n}{4}$

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\log n \leq \tfrac{n}{4} \qquad &\leq dn\log n + (\log 9 - 3.5)dn + cn
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&\leq dn\log n - 0.33dn + cn \\
&\leq dn\log n
\end{aligned}
$$

for a suitable choice of d.

6.2 Master Theorem

Lemma 4

Let $a \geq 1, b \geq 1$ and $\epsilon > 0$ denote constants. Consider the recurrence

$$T(n) = aT\left(\frac{n}{b}\right) + f(n) \ .$$

Case 1.

If $f(n) = \mathcal{O}(n^{\log_b(a)-\epsilon})$ then $T(n) = \Theta(n^{\log_b a})$.

Case 2.

If $f(n) = \Theta(n^{\log_b(a)} \log^k n)$ then $T(n) = \Theta(n^{\log_b a} \log^{k+1} n)$, $k \geq 0$.

Case 3.

If $f(n) = \Omega(n^{\log_b(a)+\epsilon})$ and for sufficiently large n $af(\frac{n}{b}) \leq cf(n)$ for some constant $c < 1$ then $T(n) = \Theta(f(n))$.

6.2 Master Theorem

We prove the Master Theorem for the case that n is of the form b^ℓ, and we assume that the non-recursive case occurs for problem size 1 and incurs cost 1.

The Recursion Tree

The running time of a recursive algorithm can be visualized by a recursion tree:

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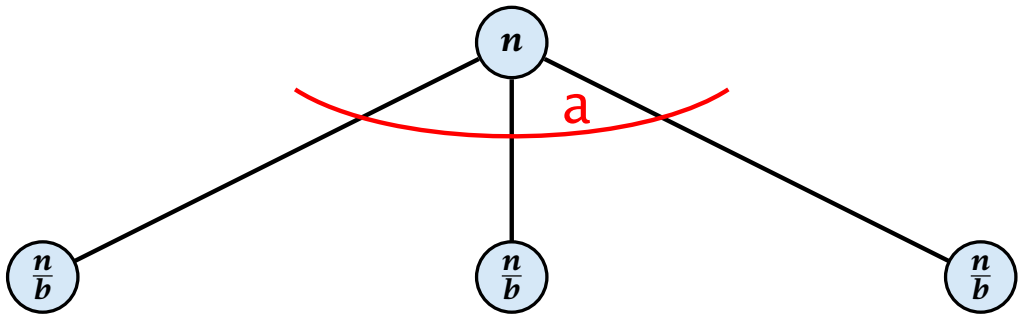

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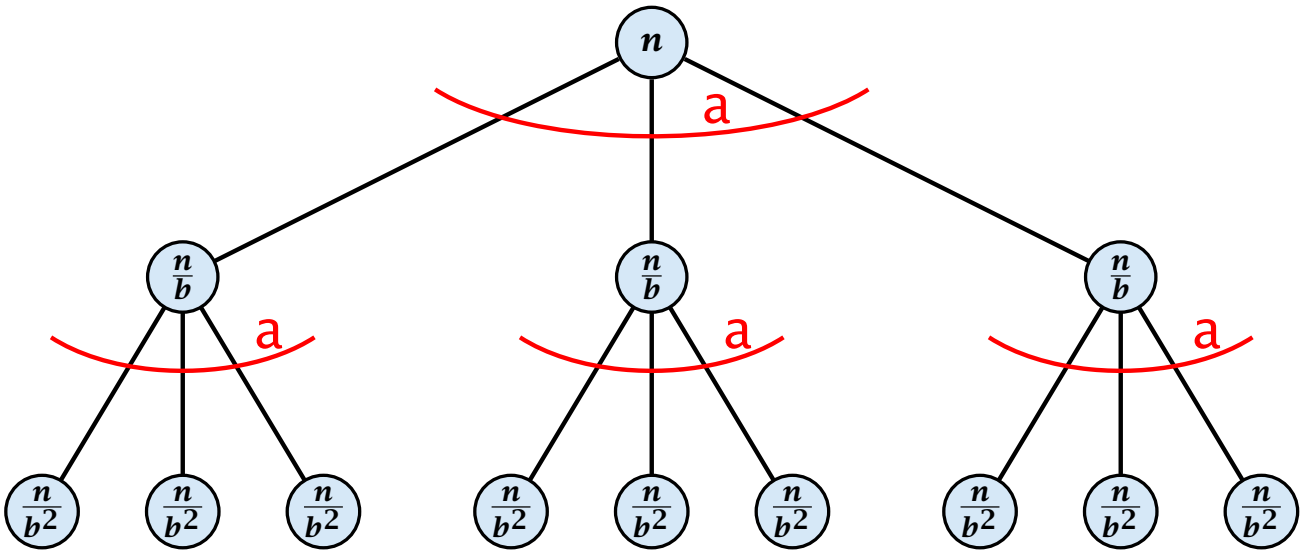

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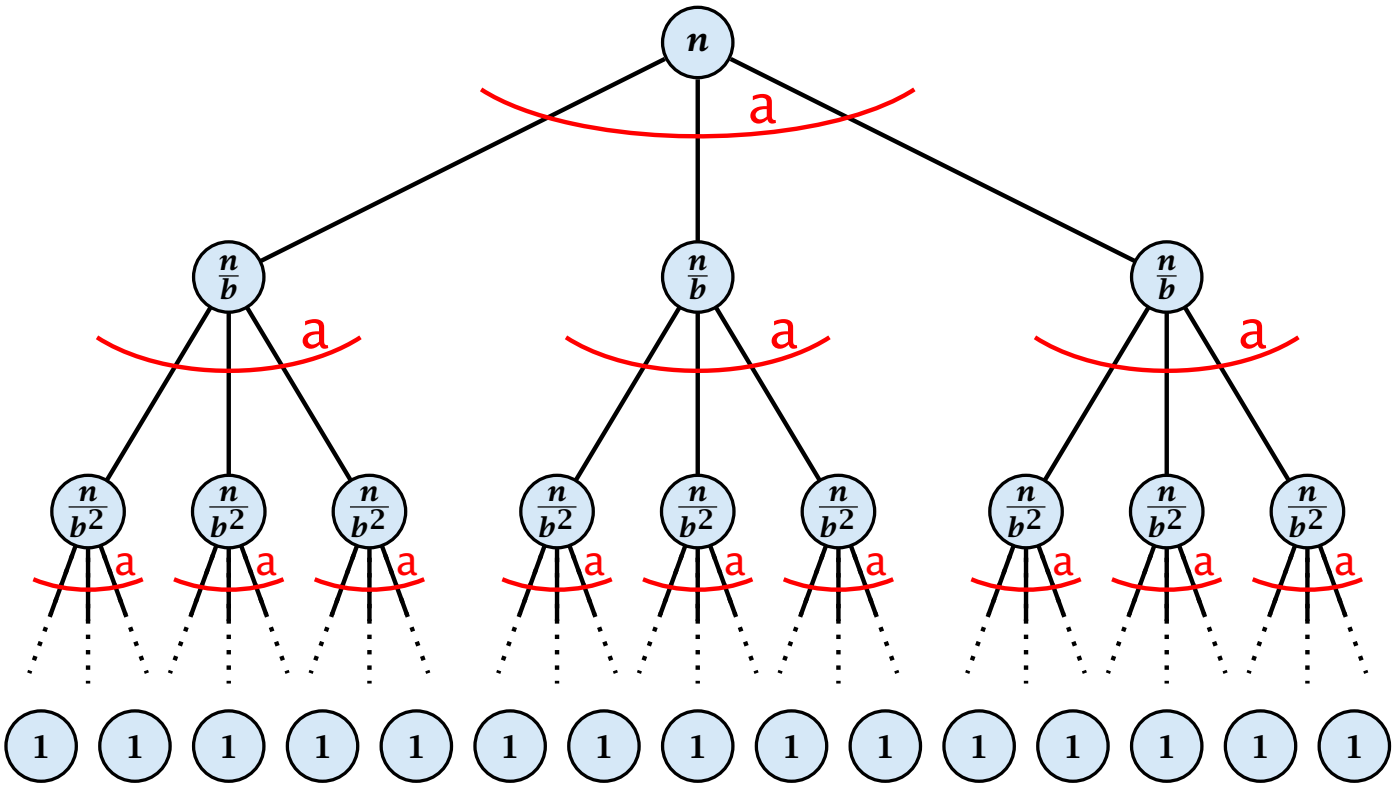

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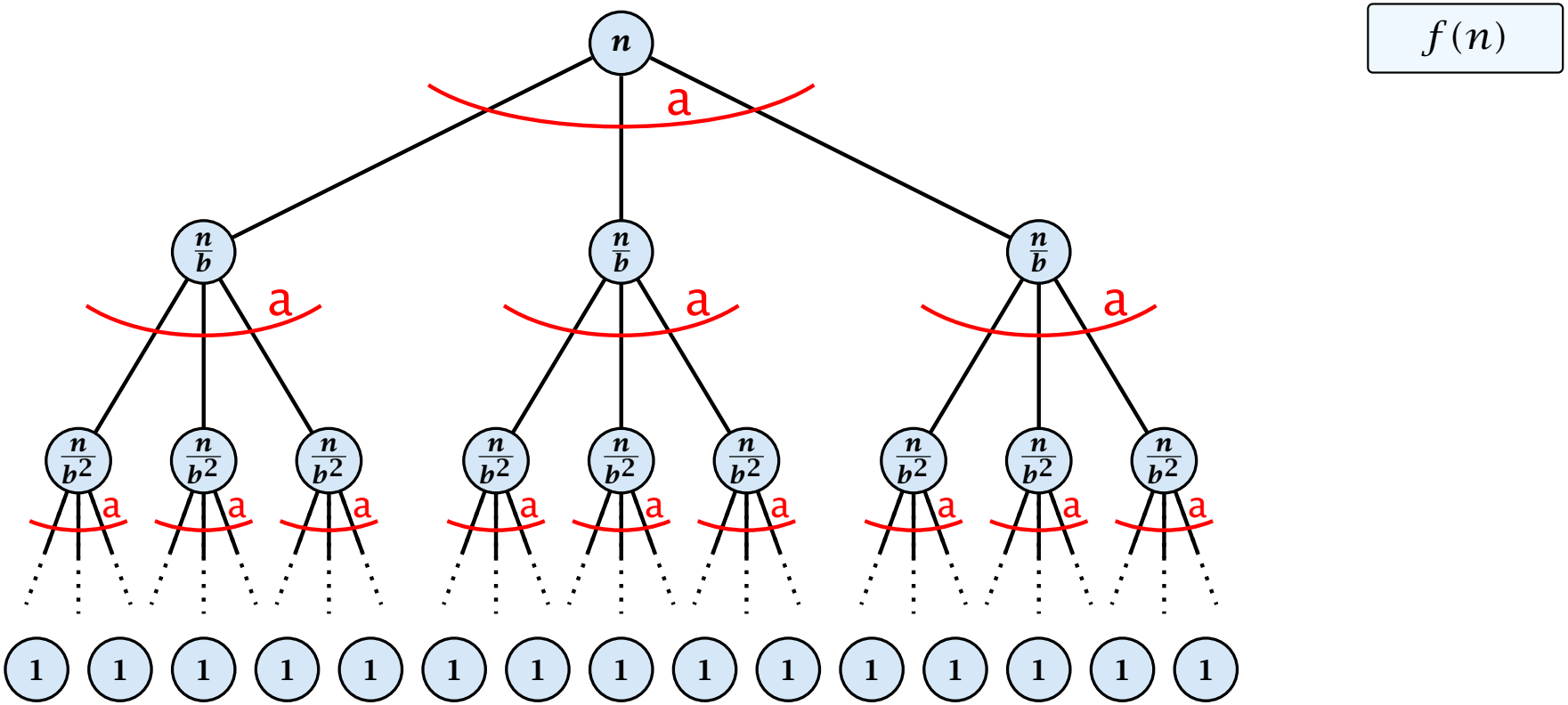

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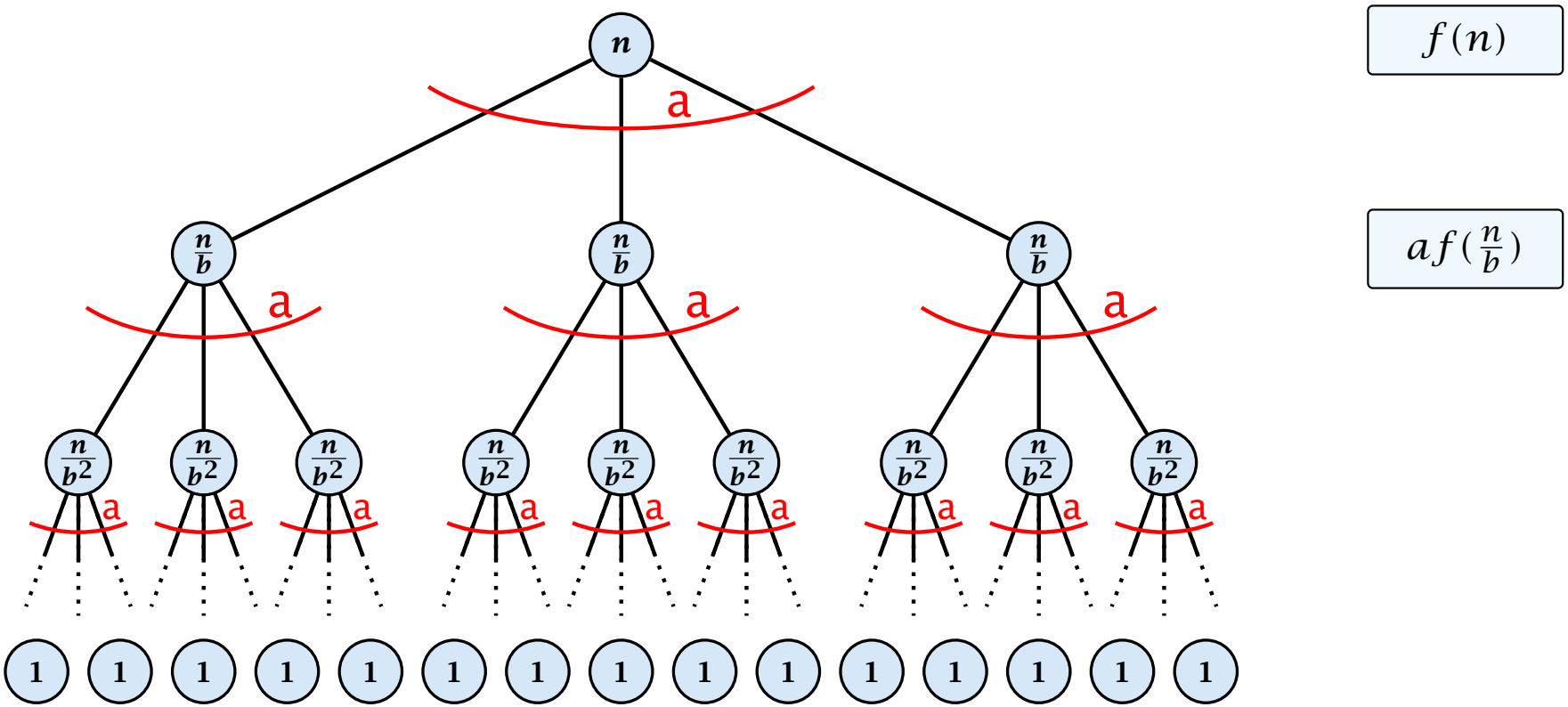

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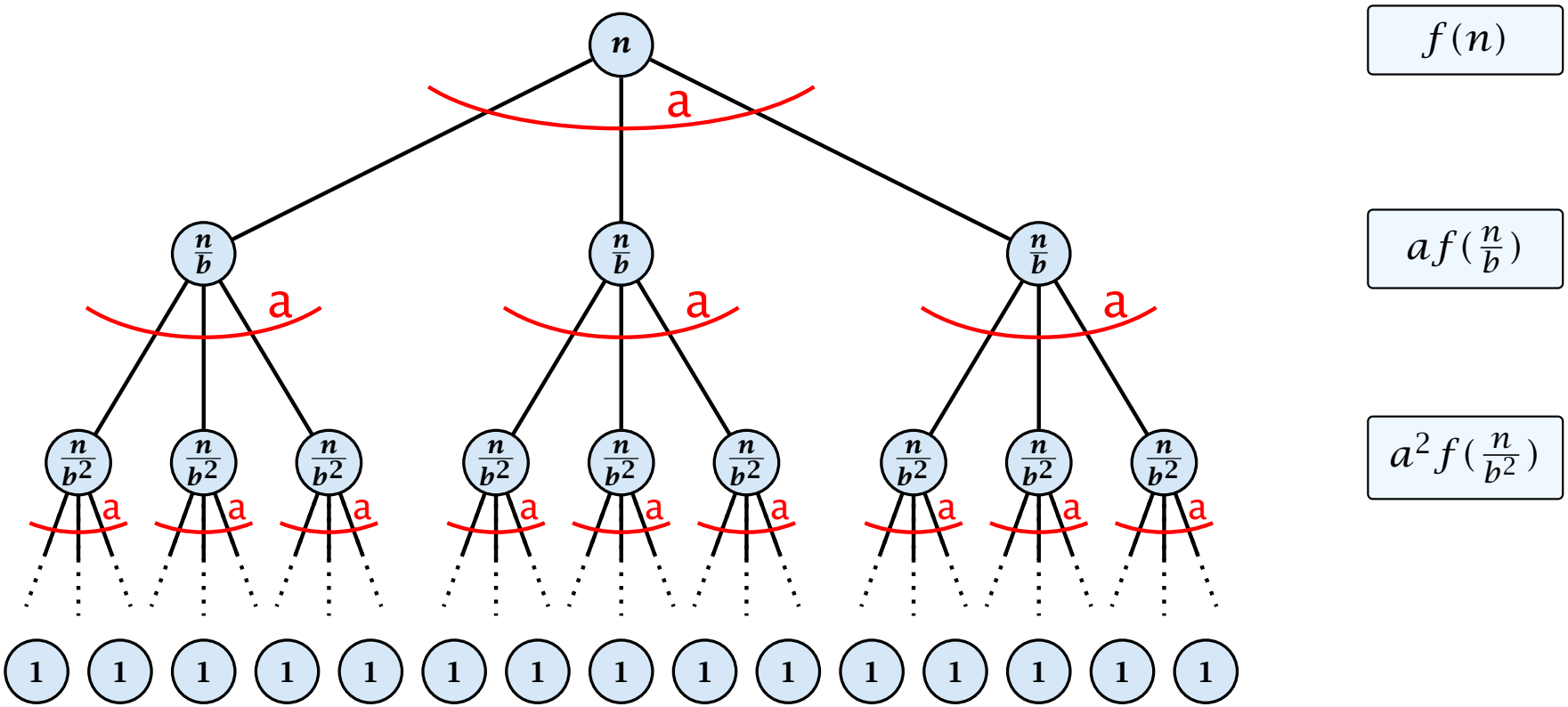

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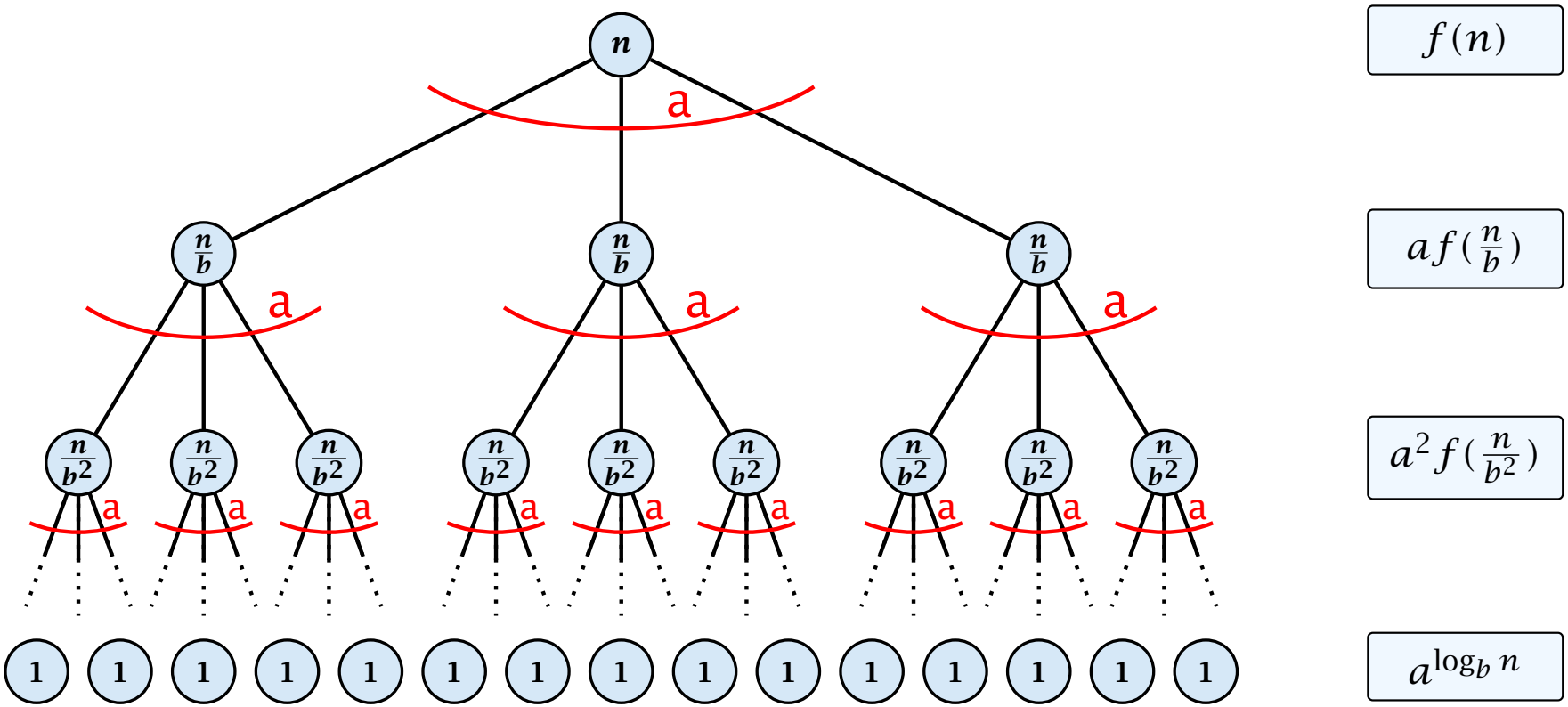
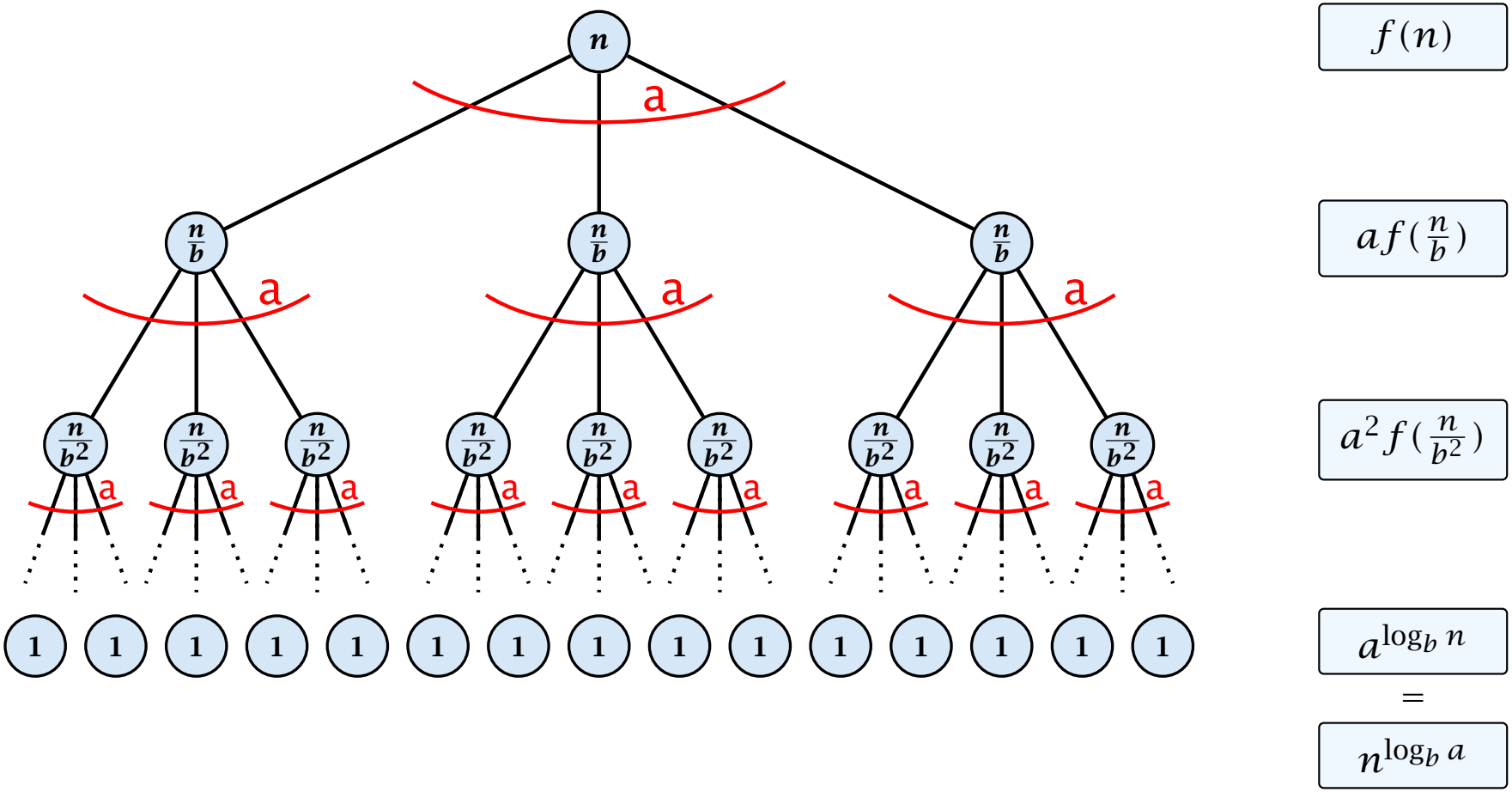


The Recursion Tree

The running time of a recursive algorithm can be visualized by a recursion tree:



6.2 Master Theorem

This gives

$$T(n) = n^{\log_b a} + \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) .$$

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$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n-1} a^i f\left(\frac{n}{b^i}\right)$$

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$$b^{-i(\log_b a - \epsilon)} = b^{\epsilon i} (b^{\log_b a})^{-i} = b^{\epsilon i} a^{-i}$$

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Hence,

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Hence,

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$$= cn^{\log_b a} \boxed{\sum_{i=1}^{\ell} i^k} \approx \tfrac{1}{k}\ell^{k+1}$$

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$$\Rightarrow T(n) = \mathcal{O}(n^{\log_b a} \log^{k+1} n).$$

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Hence,

$$T(n) \leq \mathcal{O}(f(n))$$

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$$q < 1 : \sum_{i=0}^{n} q^i = \frac{1-q^{n+1}}{1-q} \leq \frac{1}{1-q}$$

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Hence,

$$T(n) \leq \mathcal{O}(f(n)) \qquad \Rightarrow T(n) = \Theta(f(n)).$$

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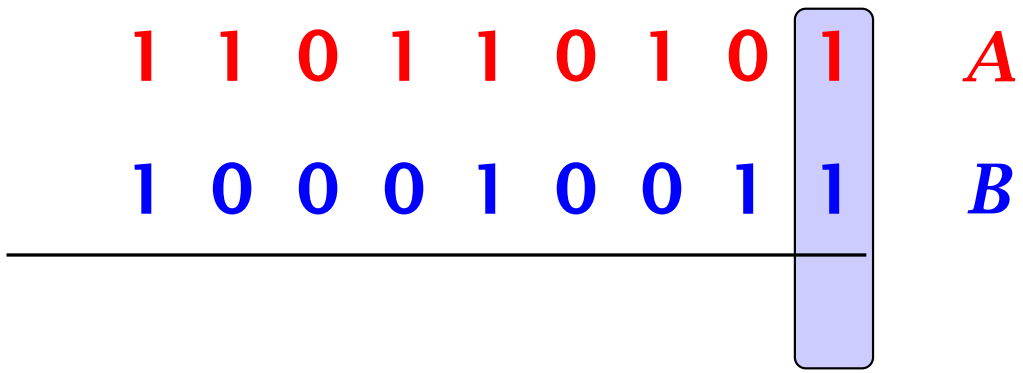

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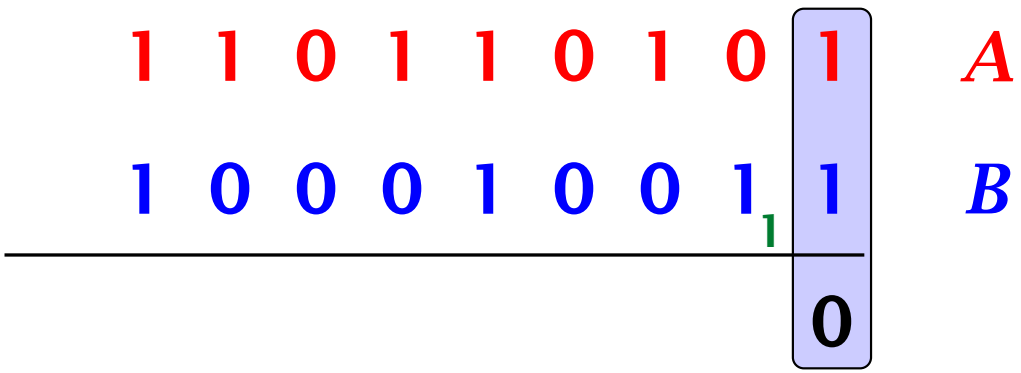

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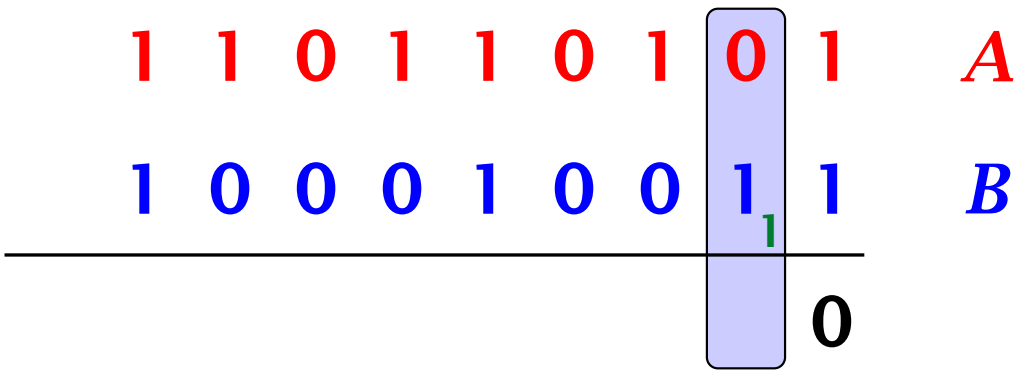

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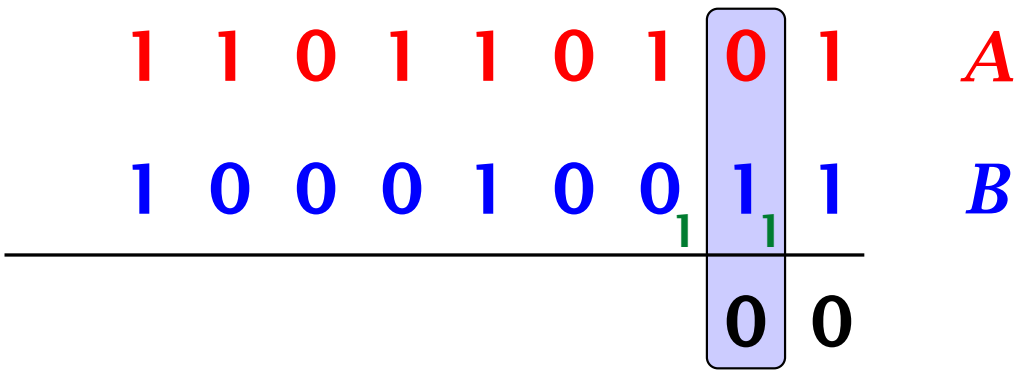

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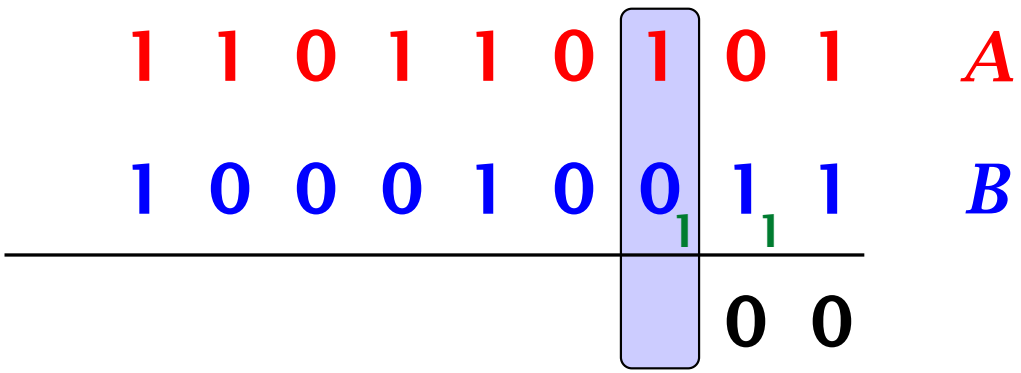

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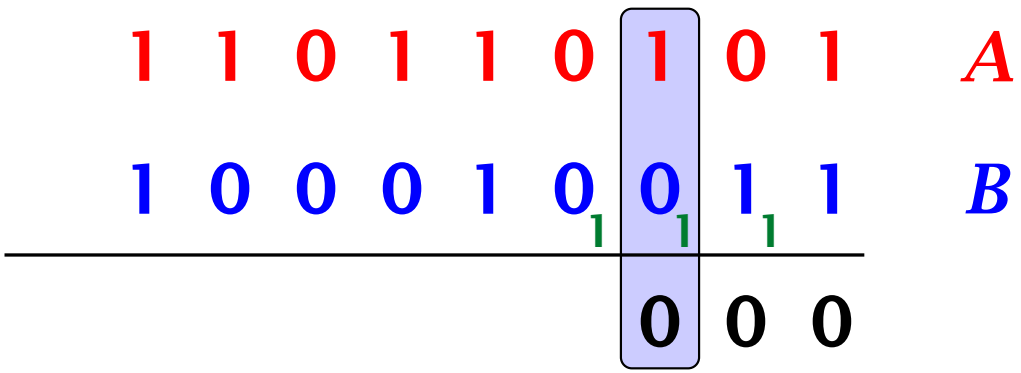

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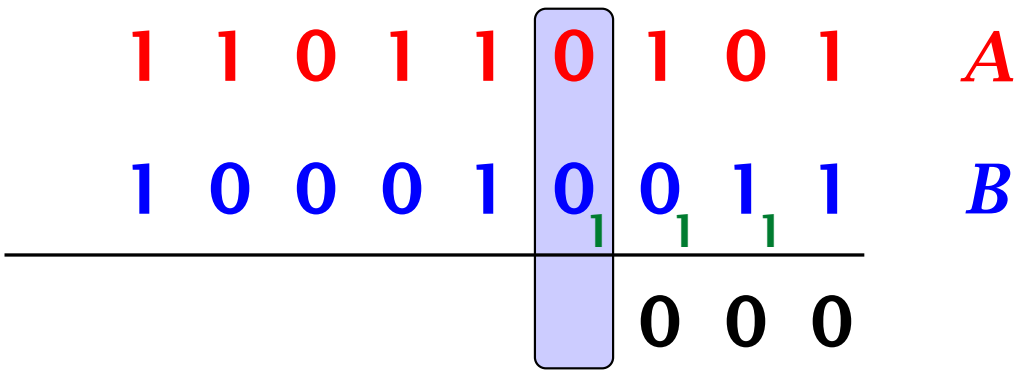

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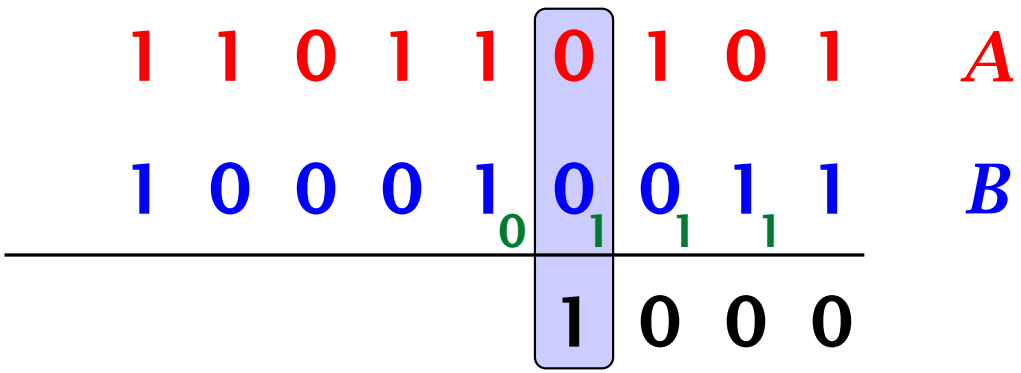

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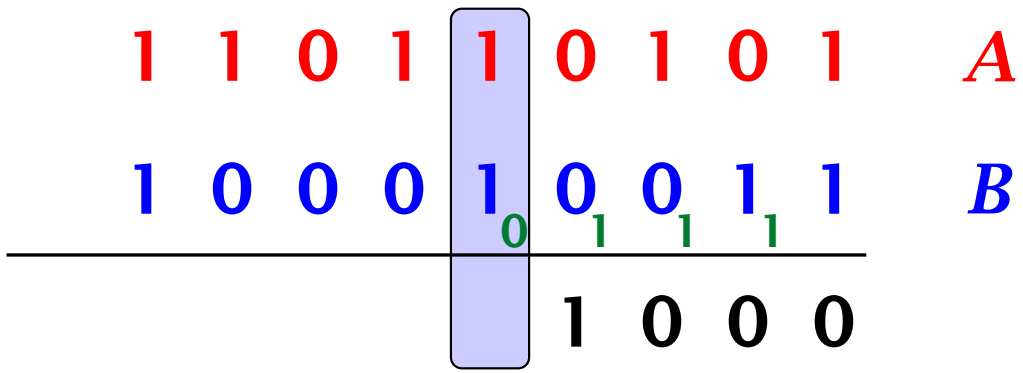

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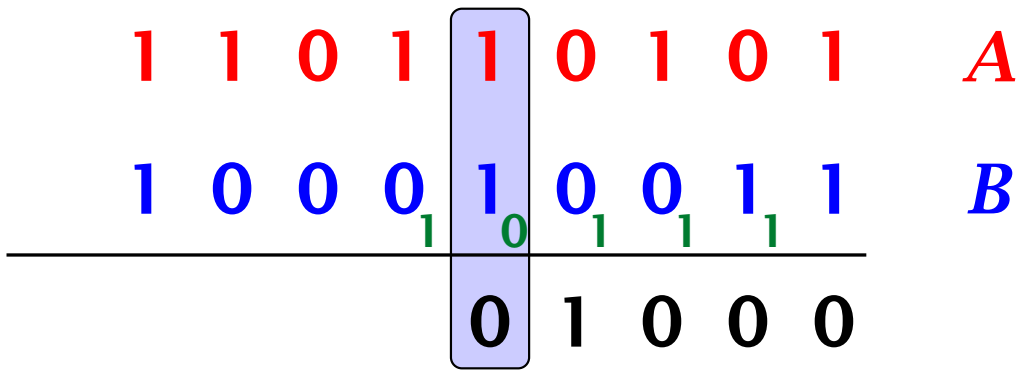

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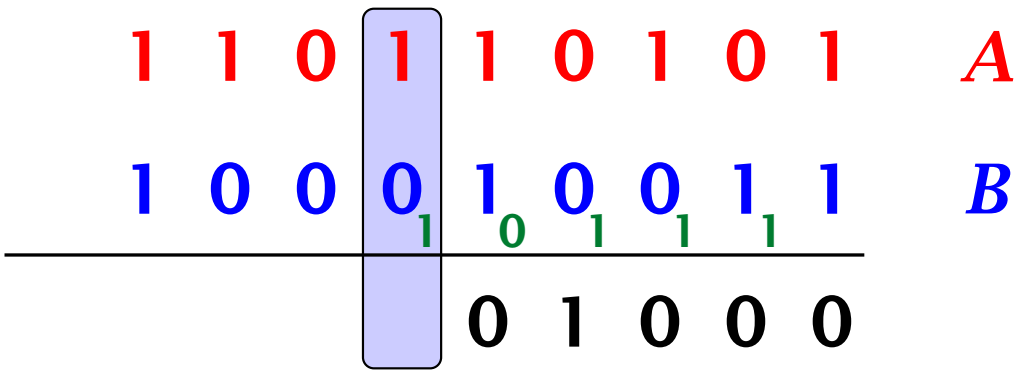

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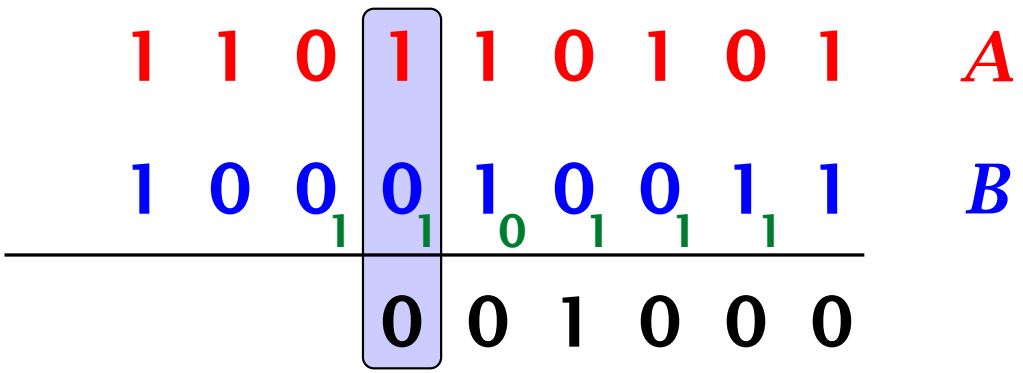

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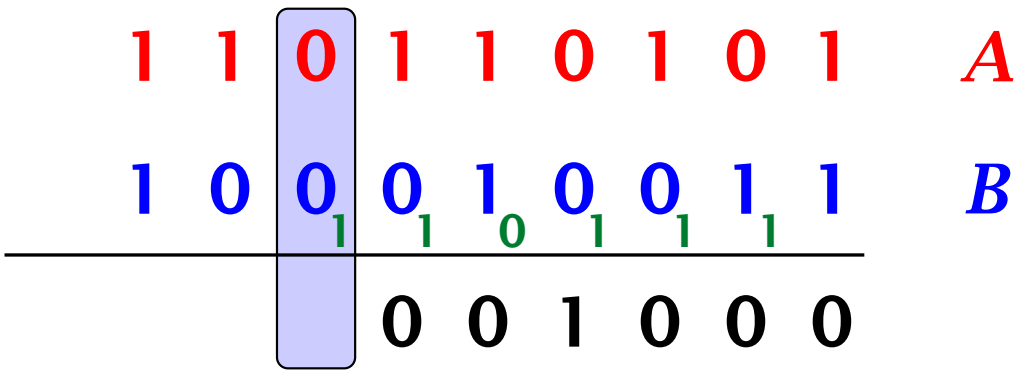

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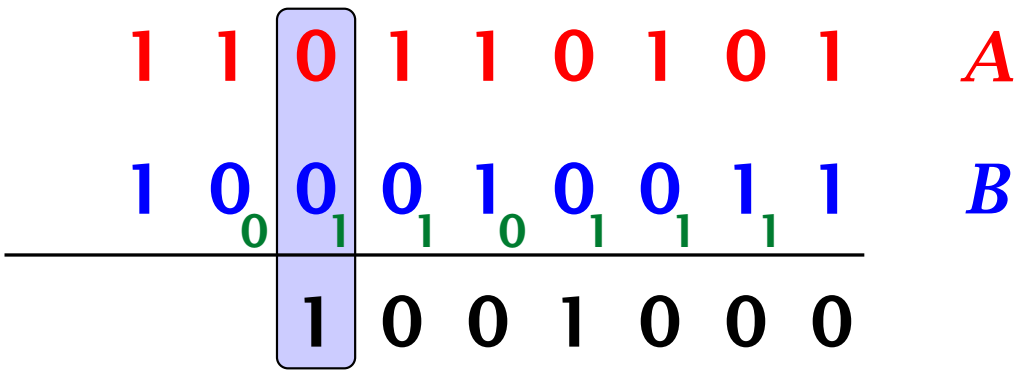

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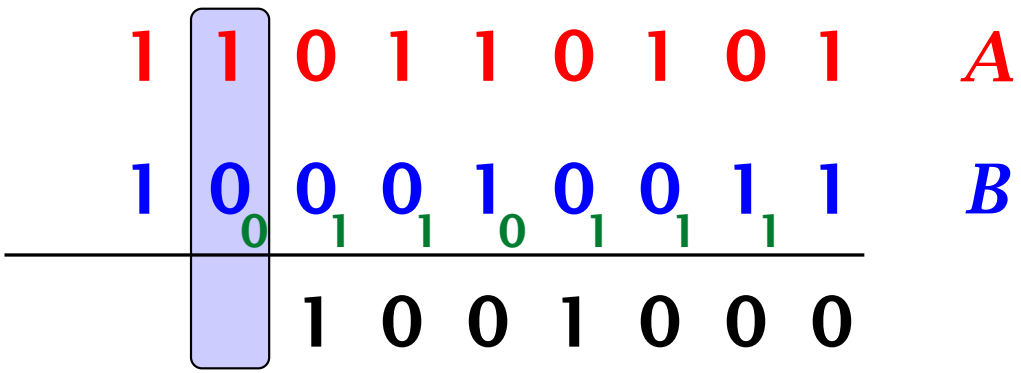

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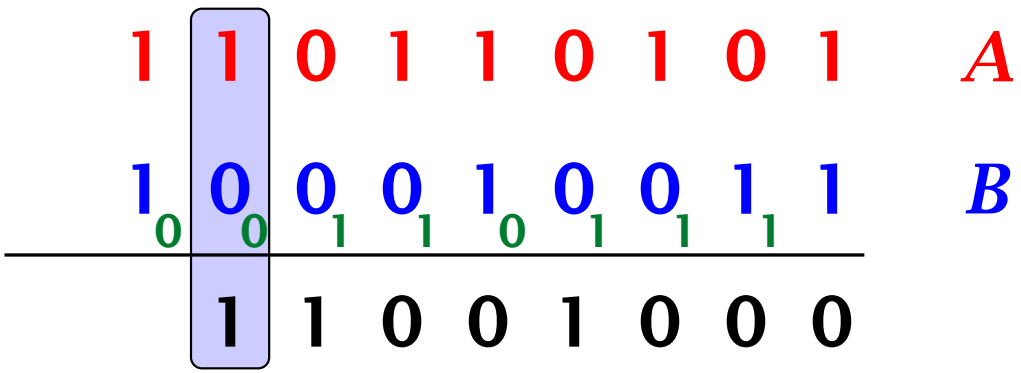

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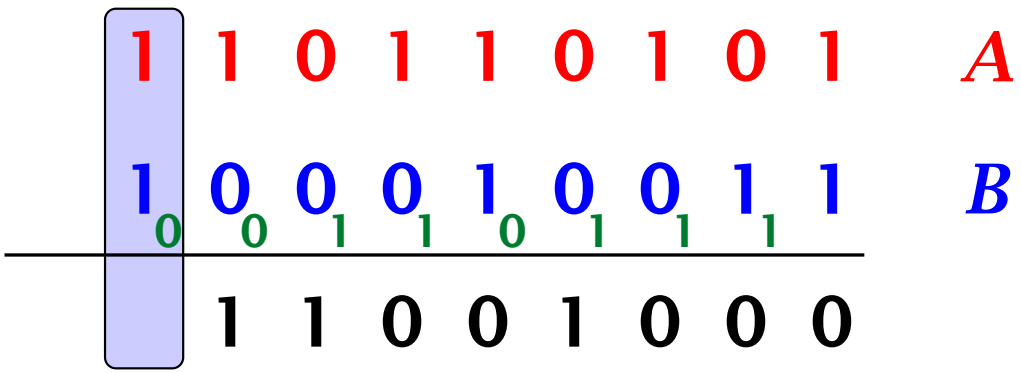

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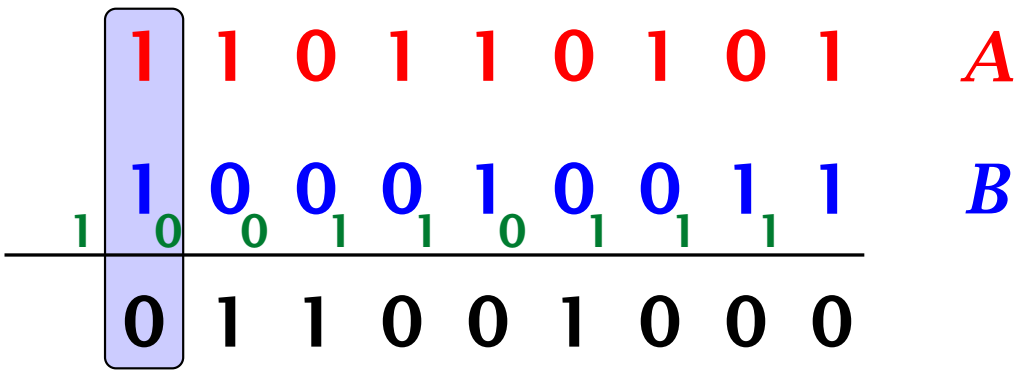

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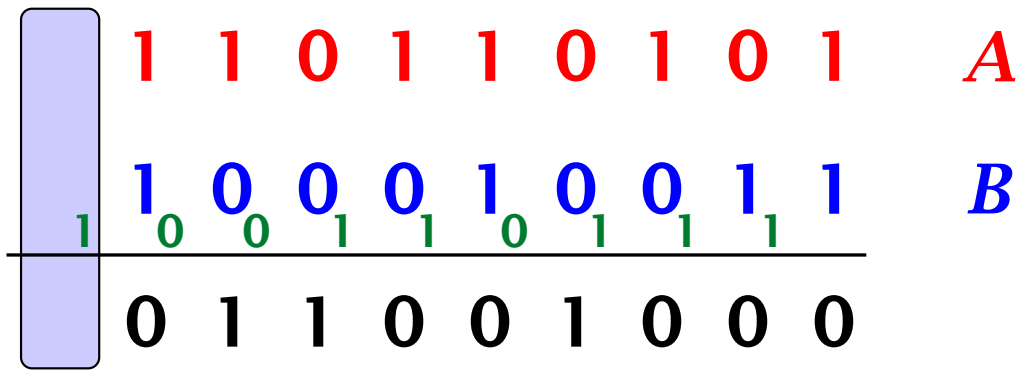

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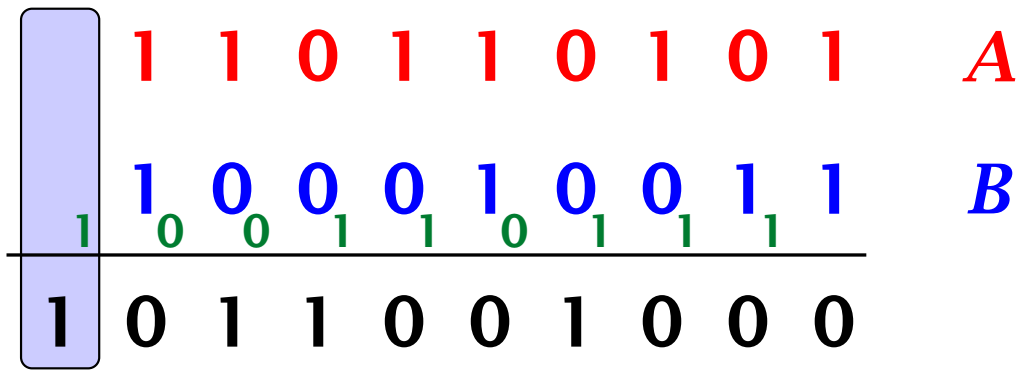

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This gives that two n-bit integers can be added in time $\mathcal{O}(n)$.

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- Computing intermediate results: $\mathcal{O}(nm)$.
- Adding m numbers of length $\leq 2n$: $\mathcal{O}((m+n)m) = \mathcal{O}(nm)$.

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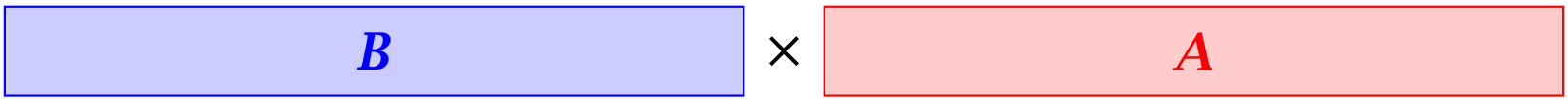

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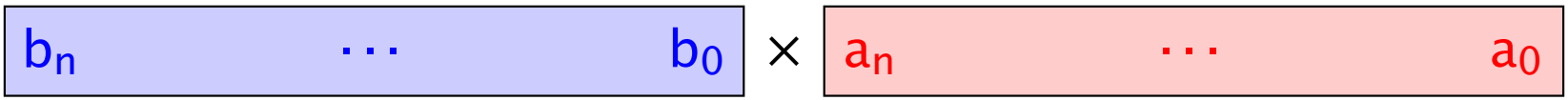

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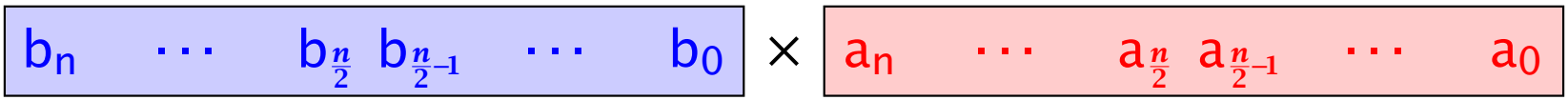

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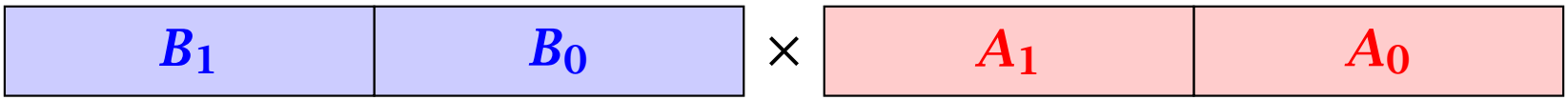

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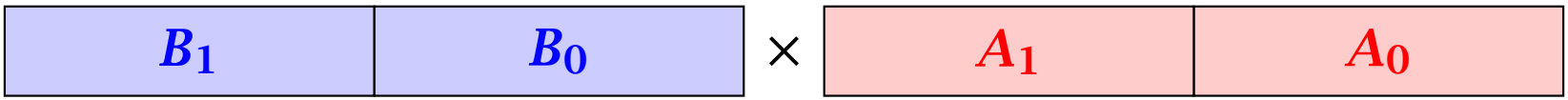

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$$A = A_1 \cdot 2^{\frac{n}{2}} + A_0 \text{ and } B = B_1 \cdot 2^{\frac{n}{2}} + B_0$$

Example: Multiplying Two Integers

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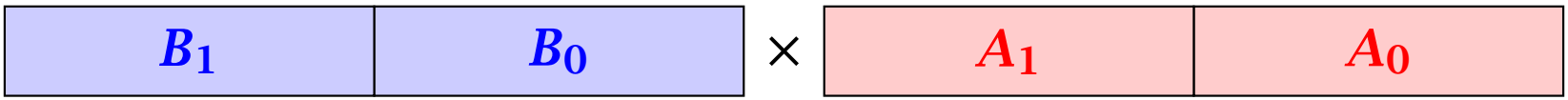

Then it holds that

$$A = A_1 \cdot 2^{\frac{n}{2}} + A_0 \text{ and } B = B_1 \cdot 2^{\frac{n}{2}} + B_0$$

Hence,

$$A \cdot B = A_1 B_1 \cdot 2^n + (A_1 B_0 + A_0 B_1) \cdot 2^{\frac{n}{2}} + A_0 \cdot B_0$$

Example: Multiplying Two Integers

Algorithm 3 mult(A, B)

1: **if** $|A| = |B| = 1$ **then**
2: **return** $a_0 \cdot b_0$
3: split A into A_0 and A_1
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5: $Z_2 \leftarrow \text{mult}(A_1, B_1)$
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1: **if** $|A| = |B| = 1$ **then** $\mathcal{O}(1)$
2: **return** $a_0 \cdot b_0$
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We get the following recurrence:

$$T(n) = 4T\left(\frac{n}{2}\right) + \mathcal{O}(n) \ .$$

Example: Multiplying Two Integers

Master Theorem: Recurrence: $T[n] = aT(\frac{n}{b}) + f(n)$.

- Case 1: $f(n) = \mathcal{O}(n^{\log_b a - \epsilon})$ $\quad T(n) = \Theta(n^{\log_b a})$
- Case 2: $f(n) = \Theta(n^{\log_b a} \log^k n)$ $\quad T(n) = \Theta(n^{\log_b a} \log^{k+1} n)$
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In our case $a = 4$, $b = 2$, and $f(n) = \Theta(n)$. Hence, we are in Case 1, since $n = \mathcal{O}(n^{2-\epsilon}) = \mathcal{O}(n^{\log_b a - \epsilon})$.

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$\Longrightarrow$ Not better then the "school method".

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Hence,

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1: if |A| = |B| = 1 then
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```
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Example: Multiplying Two Integers

We get the following recurrence:

$$T(n) = 3T\left(\frac{n}{2}\right) + \mathcal{O}(n) \ .$$

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Again we are in Case 1. We get a running time of $\Theta(n^{\log_2 3}) \approx \Theta(n^{1.59})$.

A huge improvement over the "school method".

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Example: Multiplying Two Integers

We get the following recurrence:

$$T(n) = 3T\left(\frac{n}{2}\right) + \mathcal{O}(n) \ .$$

Master Theorem: Recurrence: $T[n] = aT(\frac{n}{b}) + f(n)$.

- Case 1: $f(n) = \mathcal{O}(n^{\log_b a - \epsilon})$ $\quad T(n) = \Theta(n^{\log_b a})$
- Case 2: $f(n) = \Theta(n^{\log_b a} \log^k n)$ $\quad T(n) = \Theta(n^{\log_b a} \log^{k+1} n)$
- Case 3: $f(n) = \Omega(n^{\log_b a + \epsilon})$ $\quad T(n) = \Theta(f(n))$

Again we are in Case 1. We get a running time of $\Theta(n^{\log_2 3}) \approx \Theta(n^{1.59})$.

A huge improvement over the "school method".

6.3 The Characteristic Polynomial

Consider the recurrence relation:

$$c_0 T(n) + c_1 T(n-1) + c_2 T(n-2) + \cdots + c_k T(n-k) = f(n)$$

This is the general form of a linear recurrence relation of order k with constant coefficients ($c_0, c_k \neq 0$).

- $T(n)$ only depends on the k preceding values. This means the recurrence relation is of order k.
- The recurrence is linear as there are no products of $T(n)$'s.
- If $f(n) = 0$ then the recurrence relation becomes a linear homogeneous recurrence relation of order k.

Note that we ignore boundary conditions for the moment.

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Observations:

- The solution $T[1], T[2], T[3], \ldots$ is completely determined by a set of boundary conditions that specify values for $T[1], \ldots, T[k]$.
- In fact, any k consecutive values completely determine the solution.
- k non-concecutive values might not be an appropriate set of boundary conditions (depends on the problem).

Approach:

- First determine all solutions that satisfy recurrence relation.
- Then pick the right one by analyzing boundary conditions.
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The Homogenous Case

The solution space

$$S = \left\{ \mathcal{T} = T[1], T[2], T[3], \ldots \;\middle|\; \mathcal{T} \text{ fulfills recurrence relation} \right\}$$

is a vector space. This means that if $\mathcal{T}_1, \mathcal{T}_2 \in S$, then also $\alpha\mathcal{T}_1 + \beta\mathcal{T}_2 \in S$, for arbitrary constants α, β.

How do we find a non-trivial solution?

We guess that the solution is of the form λ^n, $\lambda \neq 0$, and see what happens. In order for this guess to fulfill the recurrence we need

$$c_0\lambda^n + c_1\lambda^{n-1} + c_2 \cdot \lambda^{n-2} + \cdots + c_k \cdot \lambda^{n-k} = 0$$

for all $n \geq k$.

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Dividing by λ^{n-k} gives that all these constraints are identical to

$$c_0\lambda^k + c_1\lambda^{k-1} + c_2 \cdot \lambda^{k-2} + \cdots + c_k = 0$$

This means that if λ_i is a root (Nullstelle) of $P[\lambda]$ then $T[n] = \lambda_i^n$ is a solution to the recurrence relation.

Let $\lambda_1, \ldots, \lambda_k$ be the k (complex) roots of $P[\lambda]$. Then, because of the vector space property

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Lemma 5

Assume that the characteristic polynomial has k distinct roots $\lambda_1, \ldots, \lambda_k$. Then all solutions to the recurrence relation are of the form

$$\alpha_1 \lambda_1^n + \alpha_2 \lambda_2^n + \cdots + \alpha_k \lambda_k^n \ .$$

Proof.

There is one solution for every possible choice of boundary conditions for $T[1], \ldots, T[k]$.

We show that the above set of solutions contains one solution for every choice of boundary conditions.

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Proof (cont.).

Suppose I am given boundary conditions $T[i]$ and I want to see whether I can choose the $\alpha_i' s$ such that these conditions are met:

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We show that the column vectors are linearly independent. Then the above equation has a solution.

Computing the Determinant

$$\begin{vmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_{k-1} & \lambda_k \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_{k-1}^2 & \lambda_k^2 \\ \vdots & \vdots & & \vdots & \vdots \\ \lambda_1^k & \lambda_2^k & \cdots & \lambda_{k-1}^k & \lambda_k^k \end{vmatrix} =$$

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$$= \prod_{i=1}^{k} \lambda_i \cdot \begin{vmatrix} 1 & \lambda_1 & \cdots & \lambda_1^{k-2} & \lambda_1^{k-1} \\ 1 & \lambda_2 & \cdots & \lambda_2^{k-2} & \lambda_2^{k-1} \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & \lambda_k & \cdots & \lambda_k^{k-2} & \lambda_k^{k-1} \end{vmatrix}$$

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$$\begin{vmatrix} 1 & \lambda_1 - \lambda_1 \cdot 1 & \cdots & \lambda_1^{k-2} - \lambda_1 \cdot \lambda_1^{k-3} & \lambda_1^{k-1} - \lambda_1 \cdot \lambda_1^{k-2} \\ 1 & \lambda_2 - \lambda_1 \cdot 1 & \cdots & \lambda_2^{k-2} - \lambda_1 \cdot \lambda_2^{k-3} & \lambda_2^{k-1} - \lambda_1 \cdot \lambda_2^{k-2} \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & \lambda_k - \lambda_1 \cdot 1 & \cdots & \lambda_k^{k-2} - \lambda_1 \cdot \lambda_k^{k-3} & \lambda_k^{k-1} - \lambda_1 \cdot \lambda_k^{k-2} \end{vmatrix}$$

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$$\begin{vmatrix} 1 & 0 & \cdots & 0 & 0 \\ 1 & (\lambda_2 - \lambda_1) \cdot 1 & \cdots & (\lambda_2 - \lambda_1) \cdot \lambda_2^{k-3} & (\lambda_2 - \lambda_1) \cdot \lambda_2^{k-2} \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & (\lambda_k - \lambda_1) \cdot 1 & \cdots & (\lambda_k - \lambda_1) \cdot \lambda_k^{k-3} & (\lambda_k - \lambda_1) \cdot \lambda_k^{k-2} \end{vmatrix}$$

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$$\begin{vmatrix} 1 & 0 & \cdots & 0 & 0 \\ 1 & (\lambda_2-\lambda_1)\cdot 1 & \cdots & (\lambda_2-\lambda_1)\cdot\lambda_2^{k-3} & (\lambda_2-\lambda_1)\cdot\lambda_2^{k-2} \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & (\lambda_k-\lambda_1)\cdot 1 & \cdots & (\lambda_k-\lambda_1)\cdot\lambda_k^{k-3} & (\lambda_k-\lambda_1)\cdot\lambda_k^{k-2} \end{vmatrix} =$$

$$\prod_{i=2}^{k}(\lambda_i-\lambda_1)\cdot\begin{vmatrix} 1 & \lambda_2 & \cdots & \lambda_2^{k-3} & \lambda_2^{k-2} \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & \lambda_k & \cdots & \lambda_k^{k-3} & \lambda_k^{k-2} \end{vmatrix}$$

Computing the Determinant

Repeating the above steps gives:

$$\begin{vmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_{k-1} & \lambda_k \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_{k-1}^2 & \lambda_k^2 \\ \vdots & \vdots & & \vdots & \vdots \\ \lambda_1^k & \lambda_2^k & \cdots & \lambda_{k-1}^k & \lambda_k^k \end{vmatrix} = \prod_{i=1}^{k} \lambda_i \cdot \prod_{i>\ell} (\lambda_i - \lambda_\ell)$$

Hence, if all λ_i's are different, then the determinant is non-zero.

The Homogeneous Case

What happens if the roots are not all distinct?

Suppose we have a root λ_i with multiplicity (Vielfachheit) at least 2. Then not only is λ_i^n a solution to the recurrence but also $n\lambda_i^n$.

To see this consider the polynomial

$$P[\lambda] \cdot \lambda^{n-k} = c_0\lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \cdots + c_k\lambda^{n-k}$$

Since λ_i is a root we can write this as $Q[\lambda] \cdot (\lambda - \lambda_i)^2$.
Calculating the derivative gives a polynomial that still has root λ_i.

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This means

$$c_0 n\lambda_i^{n-1} + c_1(n-1)\lambda_i^{n-2} + \cdots + c_k(n-k)\lambda_i^{n-k-1} = 0$$

Hence,

$$c_0 \underbrace{n\lambda_i^{n}}_{T[n]} + c_1 \underbrace{(n-1)\lambda_i^{n-1}}_{T[n-1]} + \cdots + c_k \underbrace{(n-k)\lambda_i^{n-k}}_{T[n-k]} = 0$$

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Suppose λ_i has multiplicity j. We know that

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(after taking the derivative; multiplying with λ; plugging in λ_i)

Doing this again gives

$$c_0 n^2\lambda_i^n + c_1(n-1)^2\lambda_i^{n-1} + \cdots + c_k(n-k)^2\lambda_i^{n-k} = 0$$

We can continue $j-1$ times.

Hence, $n^\ell\lambda_i^n$ is a solution for $\ell \in 0,\ldots,j-1$.

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Hence, $n^\ell\lambda_i^n$ is a solution for $\ell \in 0,\ldots,j-1$.

The Homogeneous Case

Lemma 6

Let $P[\lambda]$ denote the characteristic polynomial to the recurrence

$$c_0 T[n] + c_1 T[n-1] + \cdots + c_k T[n-k] = 0$$

Let λ_i, $i = 1, \ldots, m$ be the (complex) roots of $P[\lambda]$ with multiplicities ℓ_i. Then the general solution to the recurrence is given by

$$T[n] = \sum_{i=1}^{m} \sum_{j=0}^{\ell_i - 1} \alpha_{ij} \cdot (n^j \lambda_i^n) \ .$$

The full proof is omitted. We have only shown that any choice of α_{ij}'s is a solution to the recurrence.

Example: Fibonacci Sequence

$$T[0] = 0$$

$$T[1] = 1$$

$$T[n] = T[n-1] + T[n-2] \text{ for } n \geq 2$$

The characteristic polynomial is

$$\lambda^2 - \lambda - 1$$

Finding the roots, gives

$$\lambda_{1/2} = \frac{1}{2} \pm \sqrt{\frac{1}{4} + 1} = \frac{1}{2}\left(1 \pm \sqrt{5}\right)$$

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$$\alpha\left(\frac{1+\sqrt{5}}{2}\right) + \beta\left(\frac{1-\sqrt{5}}{2}\right) = 1 \Longrightarrow \alpha - \beta = \frac{2}{\sqrt{5}}$$

Example: Fibonacci Sequence

Hence, the solution is

$$\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

The Inhomogeneous Case

Consider the recurrence relation:

$$c_0 T(n) + c_1 T(n-1) + c_2 T(n-2) + \cdots + c_k T(n-k) = f(n)$$

with $f(n) \neq 0$.

While we have a fairly general technique for solving homogeneous, linear recurrence relations the inhomogeneous case is different.

The Inhomogeneous Case

The general solution of the recurrence relation is

$$T(n) = T_h(n) + T_p(n) \ ,$$

where T_h is any solution to the homogeneous equation, and T_p is one particular solution to the inhomogeneous equation.

There is no general method to find a particular solution.

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The Inhomogeneous Case

Example:

$$T[n] = T[n-1] + 1 \qquad T[0] = 1$$

Then,

$$T[n-1] = T[n-2] + 1 \qquad (n \geq 2)$$

Subtracting the first from the second equation gives,

$$T[n] - T[n-1] = T[n-1] - T[n-2] \qquad (n \geq 2)$$

or

$$T[n] = 2T[n-1] - T[n-2] \qquad (n \geq 2)$$

I get a completely determined recurrence if I add $T[0] = 1$ and $T[1] = 2$.

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$T[1] = 2$ gives $1 + \beta = 2 \Longrightarrow \beta = 1$.

The Inhomogeneous Case

If $f(n)$ is a polynomial of degree r this method can be applied $r+1$ times to obtain a homogeneous equation:

$$T[n] = T[n-1] + n^2$$

Shift:

$$T[n-1] = T[n-2] + (n-1)^2$$

Difference:

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Shift:

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and so on...

6.4 Generating Functions

Definition 7 (Generating Function)

Let $(a_n)_{n\geq 0}$ be a sequence. The corresponding

- generating function (Erzeugendenfunktion) is

$$F(z) := \sum_{n\geq 0} a_n z^n;$$

- exponential generating function (exponentielle Erzeugendenfunktion) is

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6.4 Generating Functions

There are two different views:

A generating function is a formal power series (formale Potenzreihe).

Then the generating function is an algebraic object.

Let $f = \sum_{n\geq 0} a_n z^n$ and $g = \sum_{n\geq 0} b_n z^n$.

- **Equality:** f and g are equal if $a_n = b_n$ for all n.
- **Addition:** $f + g := \sum_{n\geq 0}(a_n + b_n)z^n$.
- **Multiplication:** $f \cdot g := \sum_{n\geq 0} c_n z^n$ with $c_n = \sum_{p=0}^{n} a_p b_{n-p}$.

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There are no convergence issues here.

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There are two different views:

A generating function is a formal power series (formale Potenzreihe).

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What does $\sum_{n\geq 0} z^n = \frac{1}{1-z}$ mean in the algebraic view?

It means that the power series $1-z$ and the power series $\sum_{n\geq 0} z^n$ are invers, i.e.,

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Hence, the generating function of the sequence $a_n = n+1$ is $1/(1-z)^2$.

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The generating function of the sequence $a_n = \binom{n+k}{k}$ is $\frac{1}{(1-z)^{k+1}}$.

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The generating function of the sequence $a_n = n$ is $\frac{z}{(1-z)^2}$.

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We know

$$\sum_{n\geq 0} y^n = \frac{1}{1-y}$$

Hence,

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$$= 1 + 2z \cdot A(z)$$

Example: $a_n = 2a_{n-1}, a_0 = 1$

3. Transform right hand side so that infinite sums can be replaced by $A(z)$ or by simple function.

$$A(z) = 1 + \sum_{n\geq 1} (2a_{n-1})z^n$$

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$$\sum_{n\geq 0} a_n z^n = A(z) = \frac{1}{1-2z} = \sum_{n\geq 0} 2^n z^n$$

Example: $a_n = 3a_{n-1} + n, a_0 = 1$

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$$\begin{aligned} A(z) &= \sum_{n\geq 0} a_n z^n \\ &= a_0 + \sum_{n\geq 1} a_n z^n \\ &= 1 + \sum_{n\geq 1} (3a_{n-1} + n) z^n \end{aligned}$$

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&= 1 + 3z \sum_{n\geq 0} a_n z^n + \sum_{n\geq 0} n z^n \\
&= 1 + 3zA(z) + \frac{z}{(1-z)^2}
\end{aligned}
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$$A(z) = \frac{(1-z)^2 + z}{(1-3z)(1-z)^2} = \frac{z^2 - z + 1}{(1-3z)(1-z)^2}$$

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$$\frac{z^2 - z + 1}{(1-3z)(1-z)^2} \stackrel{!}{=} \frac{A}{1-3z} + \frac{B}{1-z} + \frac{C}{(1-z)^2}$$

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This gives

$$\begin{aligned} z^2 - z + 1 &= A(1-z)^2 + B(1-3z)(1-z) + C(1-3z) \\ &= A(1-2z+z^2) + B(1-4z+3z^2) + C(1-3z) \\ &= (A+3B)z^2 + (-2A-4B-3C)z + (A+B+C) \end{aligned}$$

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This leads to the following conditions:

$$A + B + C = 1$$
$$2A + 4B + 3C = 1$$
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This leads to the following conditions:

$$A + B + C = 1$$
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which gives

$$A = \frac{7}{4} \quad B = -\frac{1}{4} \quad C = -\frac{1}{2}$$

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6. This means $a_n = \frac{7}{4} 3^n - \frac{1}{2} n - \frac{3}{4}$.

6.5 Transformation of the Recurrence

Example 9

$$f_0 = 1$$

$$f_1 = 2$$

$$f_n = f_{n-1} \cdot f_{n-2} \text{ for } n \geq 2 \,.$$

6.5 Transformation of the Recurrence

Example 9

$$f_0 = 1$$
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Define

$$g_n := \log f_n .$$

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$$g_n = F_n \ (n\text{-th Fibonacci number})$$
$$f_n = 2^{F_n}$$

6.5 Transformation of the Recurrence

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Define

$$g_k := f_{2^k}\ .$$

Then:

$$g_0 = 1$$
$$g_k = 3g_{k-1} + 2^k,\ k \geq 1$$

We get

$$g_k = 3\left[g_{k-1}\right] + 2^k$$

We get

$$\begin{aligned} g_k &= 3\left[g_{k-1}\right] + 2^k \\ &= 3\left[3g_{k-2} + 2^{k-1}\right] + 2^k \end{aligned}$$

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$$\begin{aligned} g_k &= 3\left[g_{k-1}\right] + 2^k \\ &= 3\left[3g_{k-2} + 2^{k-1}\right] + 2^k \\ &= 3^2\left[g_{k-2}\right] + 32^{k-1} + 2^k \end{aligned}$$

We get

$$
\begin{aligned}
g_k &= 3\left[g_{k-1}\right] + 2^k \\
&= 3\left[3g_{k-2} + 2^{k-1}\right] + 2^k \\
&= 3^2\left[g_{k-2}\right] + 32^{k-1} + 2^k \\
&= 3^2\left[3g_{k-3} + 2^{k-2}\right] + 32^{k-1} + 2^k
\end{aligned}
$$

We get

$$
\begin{aligned}
g_k &= 3\left[g_{k-1}\right] + 2^k \\
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&= 3^3 g_{k-3} + 3^2 2^{k-2} + 32^{k-1} + 2^k
\end{aligned}
$$

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g_k &= 3\left[g_{k-1}\right] + 2^k \\
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&= 3^3 g_{k-3} + 3^2 2^{k-2} + 32^{k-1} + 2^k \\
&= 2^k \cdot \sum_{i=0}^{k} \left(\frac{3}{2}\right)^i \\
&= 2^k \cdot \frac{(\frac{3}{2})^{k+1} - 1}{1/2}
\end{aligned}
$$

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\begin{aligned}
g_k &= 3\left[g_{k-1}\right] + 2^k \\
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&= 3^3 g_{k-3} + 3^2 2^{k-2} + 32^{k-1} + 2^k \\
&= 2^k \cdot \sum_{i=0}^{k} \left(\frac{3}{2}\right)^i \\
&= 2^k \cdot \frac{(\frac{3}{2})^{k+1} - 1}{1/2} = 3^{k+1} - 2^{k+1}
\end{aligned}
$$

Let $n = 2^k$:

$$g_k = 3^{k+1} - 2^{k+1}, \text{ hence}$$
$$f_n = 3 \cdot 3^k - 2 \cdot 2^k$$

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&= 3(2^{\log 3})^k - 2 \cdot 2^k \\
&= 3(2^k)^{\log 3} - 2 \cdot 2^k \\
&= 3n^{\log 3} - 2n \ .
\end{aligned}
$$

Part III

Data Structures

Abstract Data Type

An abstract data type (ADT) is defined by an interface of operations or methods that can be performed and that have a defined behavior.
The data types in this lecture all operate on objects that are represented by a [key, value] pair.

- The key comes from a totally ordered set, and we assume that there is an efficient comparison function.
- The value can be anything; it usually carries satellite information important for the application that uses the ADT.

Dynamic Set Operations

- $S.\,\mathbf{search}(k)$: Returns pointer to object x from S with $\mathrm{key}[x] = k$ or null.
- $S.\,\mathbf{insert}(x)$: Inserts object x into set S. $\mathrm{key}[x]$ must not currently exist in the data-structure.
- $S.\,\mathbf{delete}(x)$: Given pointer to object x from S, delete x from the set.
- $S.\,\mathbf{minimum}()$: Return pointer to object with smallest key-value in S.
- $S.\,\mathbf{maximum}()$: Return pointer to object with largest key-value in S.
- $S.\,\mathbf{successor}(x)$: Return pointer to the next larger element in S or null if x is maximum.
- $S.\,\mathbf{predecessor}(x)$: Return pointer to the next smaller element in S or null if x is minimum.

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Dynamic Set Operations

- $S.\textbf{search}(k)$: Returns pointer to object x from S with $\text{key}[x] = k$ or null.
- $S.\textbf{insert}(x)$: Inserts object x into set S. $\text{key}[x]$ must not currently exist in the data-structure.
- $S.\textbf{delete}(x)$: Given pointer to object x from S, delete x from the set.
- $S.\textbf{minimum}()$: Return pointer to object with smallest key-value in S.
- $S.\textbf{maximum}()$: Return pointer to object with largest key-value in S.
- $S.\textbf{successor}(x)$: Return pointer to the next larger element in S or null if x is maximum.
- $S.\textbf{predecessor}(x)$: Return pointer to the next smaller element in S or null if x is minimum.

Dynamic Set Operations

- $S.\textbf{union}(S')$: Sets $S := S \cup S'$. The set S' is destroyed.
- $S.\textbf{merge}(S')$: Sets $S := S \cup S'$. Requires $S \cap S' = \emptyset$.
- $S.\textbf{split}(k, S')$:
  $S := \{x \in S \mid \text{key}[x] \leq k\}$, $S' := \{x \in S \mid \text{key}[x] > k\}$.
- $S.\textbf{concatenate}(S')$: $S := S \cup S'$.
  Requires $\text{key}[S.\text{maximum}()] \leq \text{key}[S'.\text{minimum}()]$.

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Examples of ADTs

Stack:

- $S.\,\mathbf{push}(x)$: Insert an element.
- $S.\,\mathbf{pop}()$: Return the element from S that was inserted most recently; delete it from S.
- $S.\,\mathbf{empty}()$: Tell if S contains any object.

Queue:

- $S.\,\mathbf{enqueue}(x)$: Insert an element.
- $S.\,\mathbf{dequeue}()$: Return the element that is longest in the structure; delete it from S.
- $S.\,\mathbf{empty}()$: Tell if S contains any object.

Priority-Queue:

- $S.\,\mathbf{insert}(x)$: Insert an element.
- $S.\,\mathbf{delete\text{-}min}()$: Return the element with lowest key-value; delete it from S.

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7 Dictionary

Dictionary:

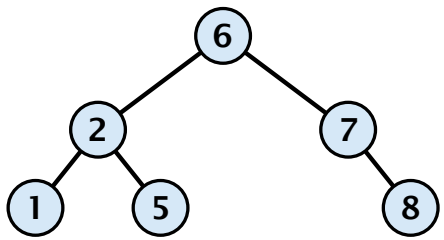
- $S.\,\mathbf{insert}(x)$: Insert an element x.
- $S.\,\mathbf{delete}(x)$: Delete the element pointed to by x.
- $S.\,\mathbf{search}(k)$: Return a pointer to an element e with $\mathrm{key}[e] = k$ in S if it exists; otherwise return null.

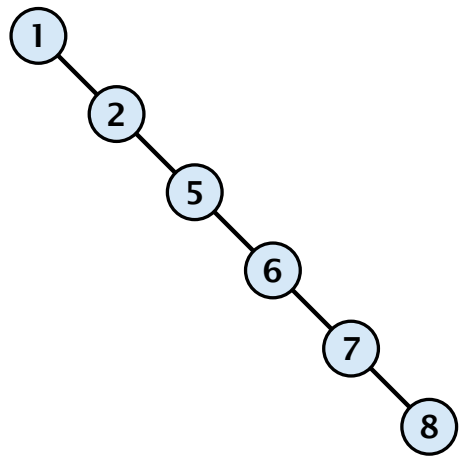
7.1 Binary Search Trees

An (internal) binary search tree stores the elements in a binary tree. Each tree-node corresponds to an element. All elements in the left sub-tree of a node v have a smaller key-value than $\text{key}[v]$ and elements in the right sub-tree have a larger-key value. We assume that all key-values are different.

(External Search Trees store objects only at leaf-vertices)

Examples:



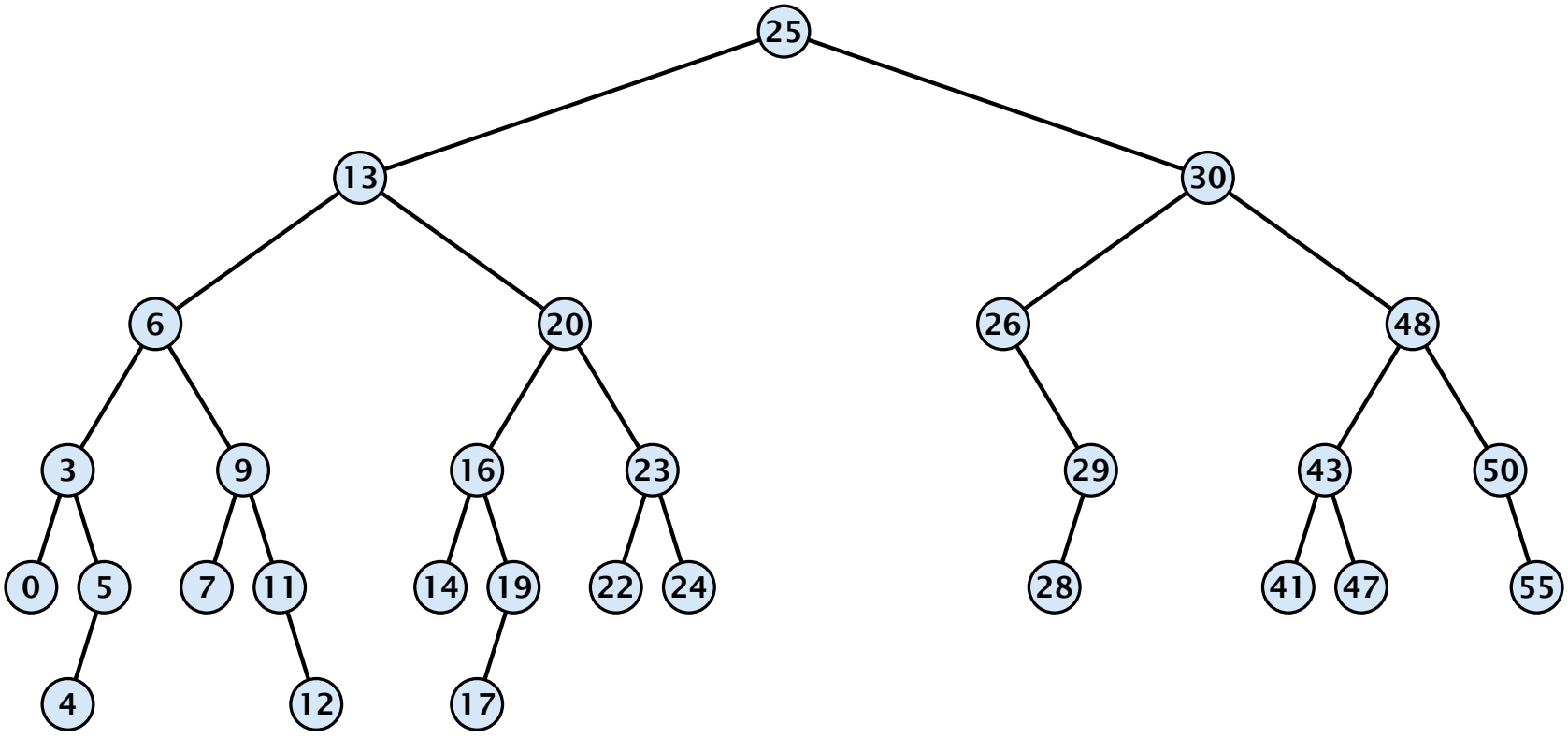


7.1 Binary Search Trees

We consider the following operations on binary search trees. Note that this is a super-set of the dictionary-operations.

- $T.\mathrm{insert}(x)$
- $T.\mathrm{delete}(x)$
- $T.\mathrm{search}(k)$
- $T.\mathrm{successor}(x)$
- $T.\mathrm{predecessor}(x)$
- $T.\mathrm{minimum}()$
- $T.\mathrm{maximum}()$

Binary Search Trees: Searching

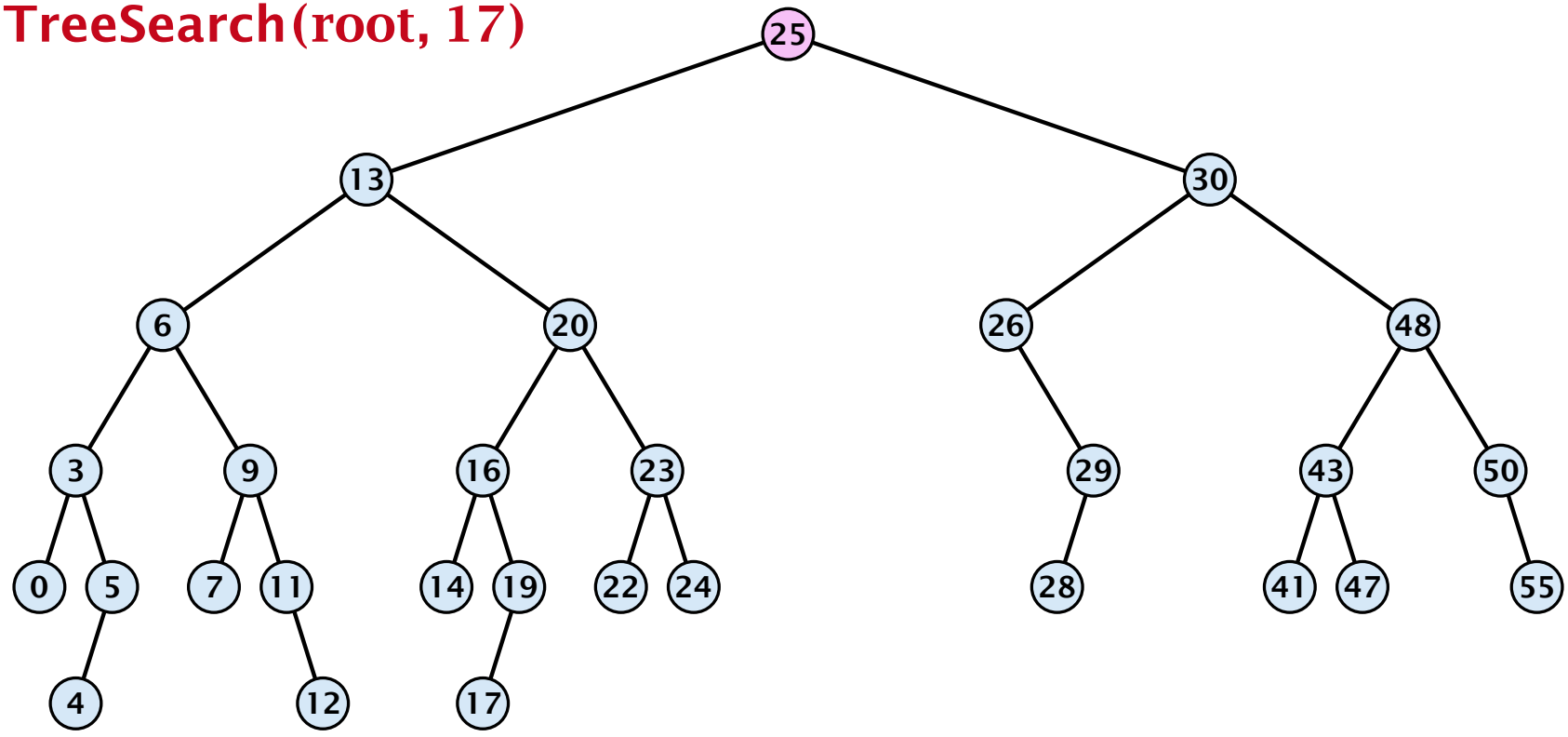


Algorithm 5 TreeSearch(x, k)

1: **if** $x = \text{null}$ **or** $k = \text{key}[x]$ **return** x
2: **if** $k < \text{key}[x]$ **return** TreeSearch$(\text{left}[x], k)$
3: **else return** TreeSearch$(\text{right}[x], k)$

Binary Search Trees: Searching

TreeSearch(root, 17)



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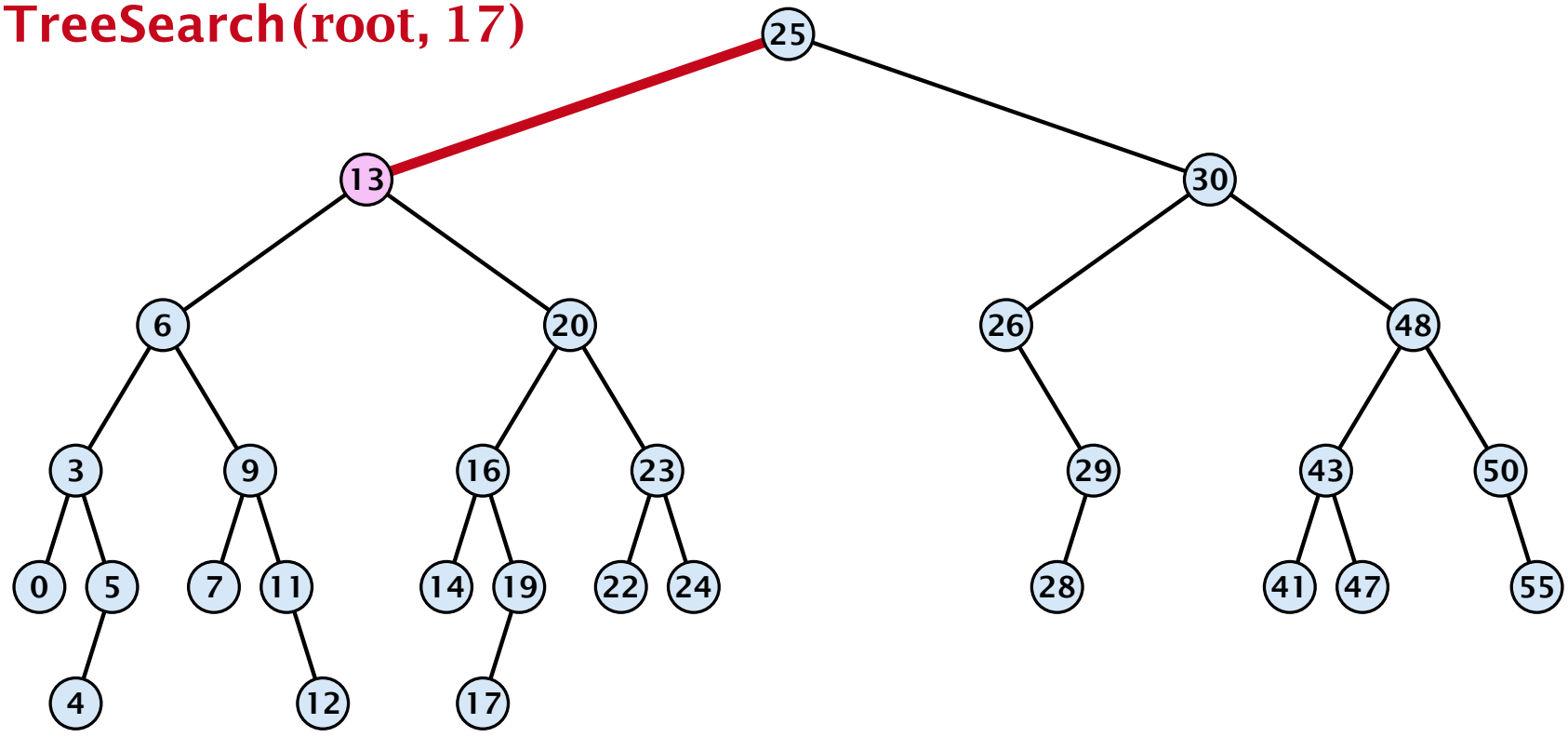

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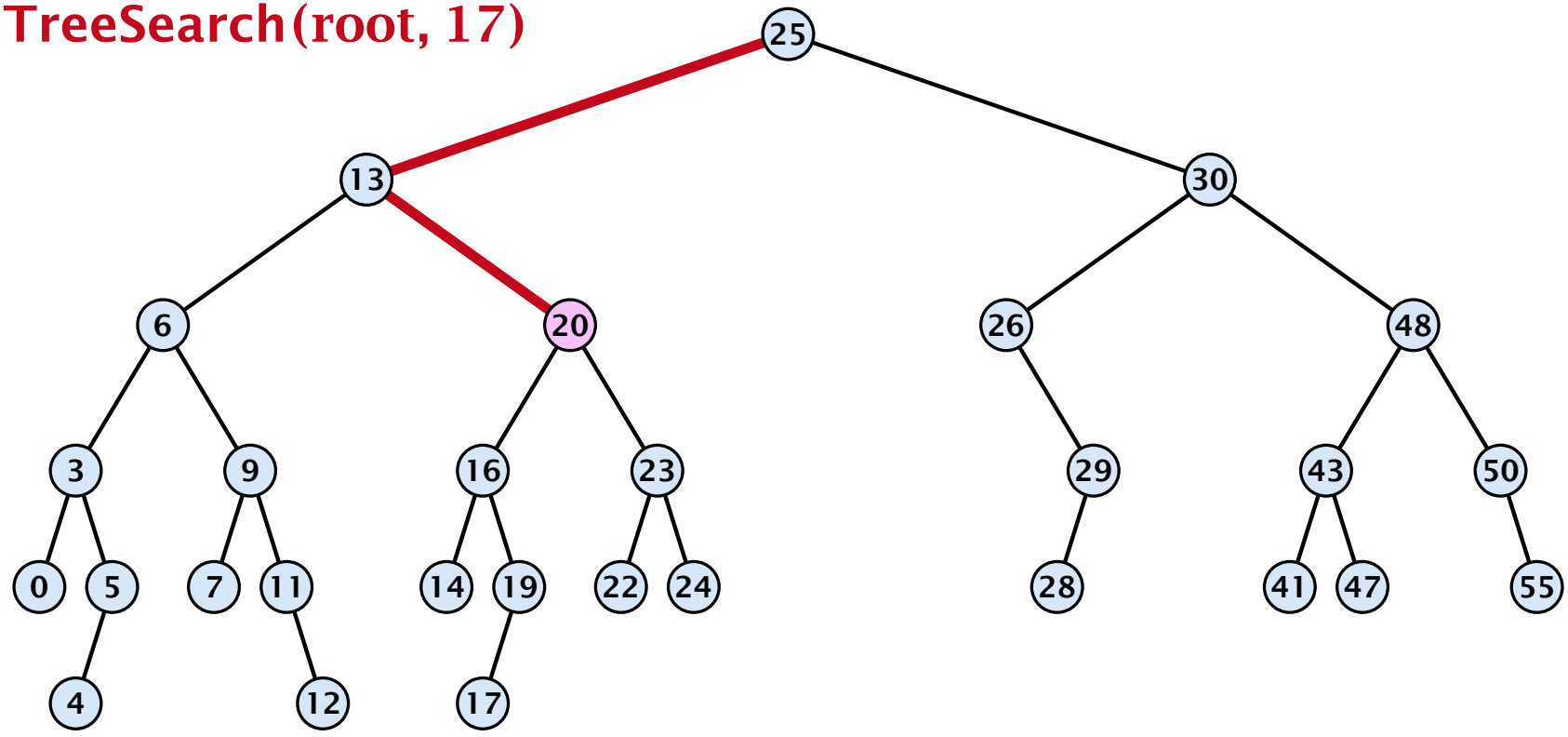

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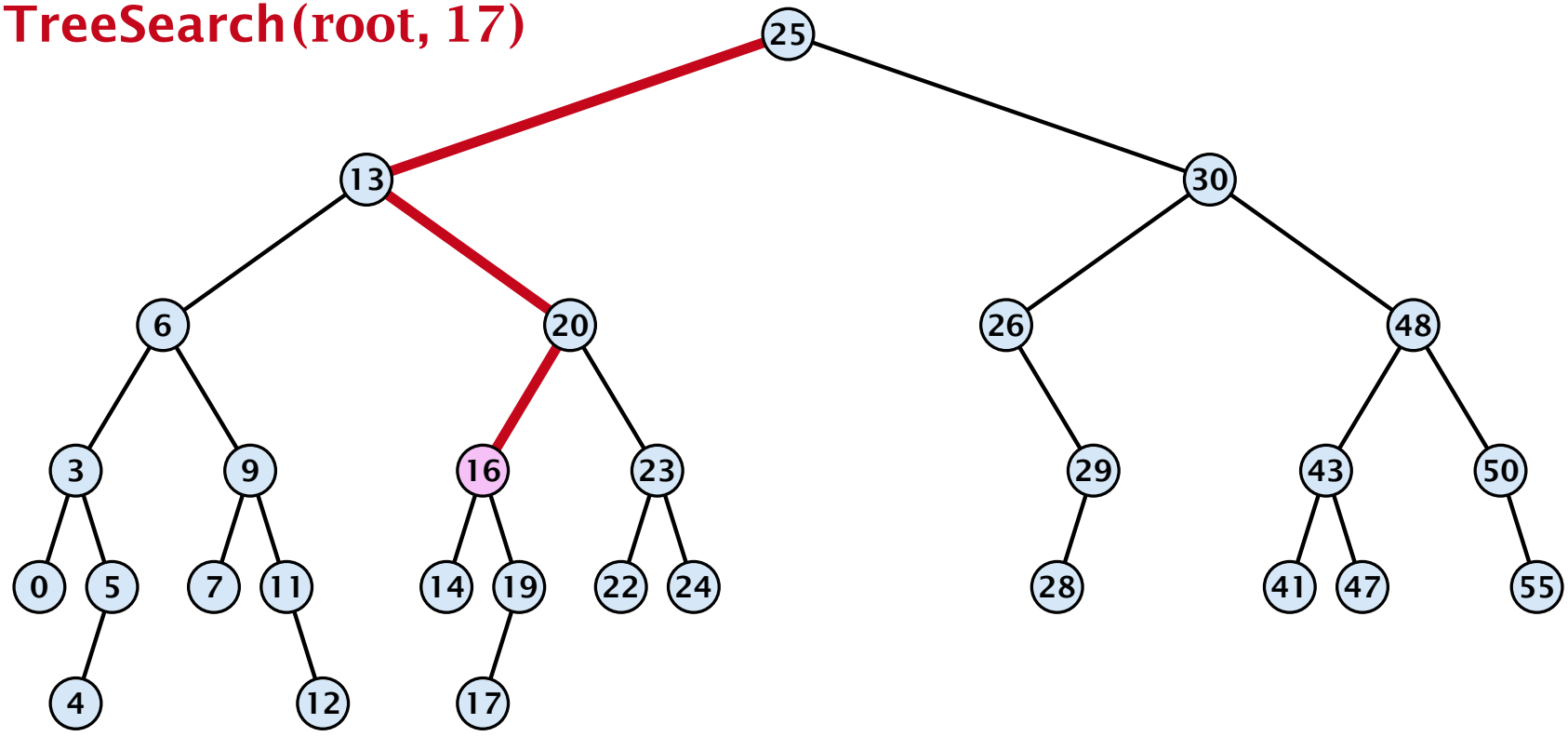

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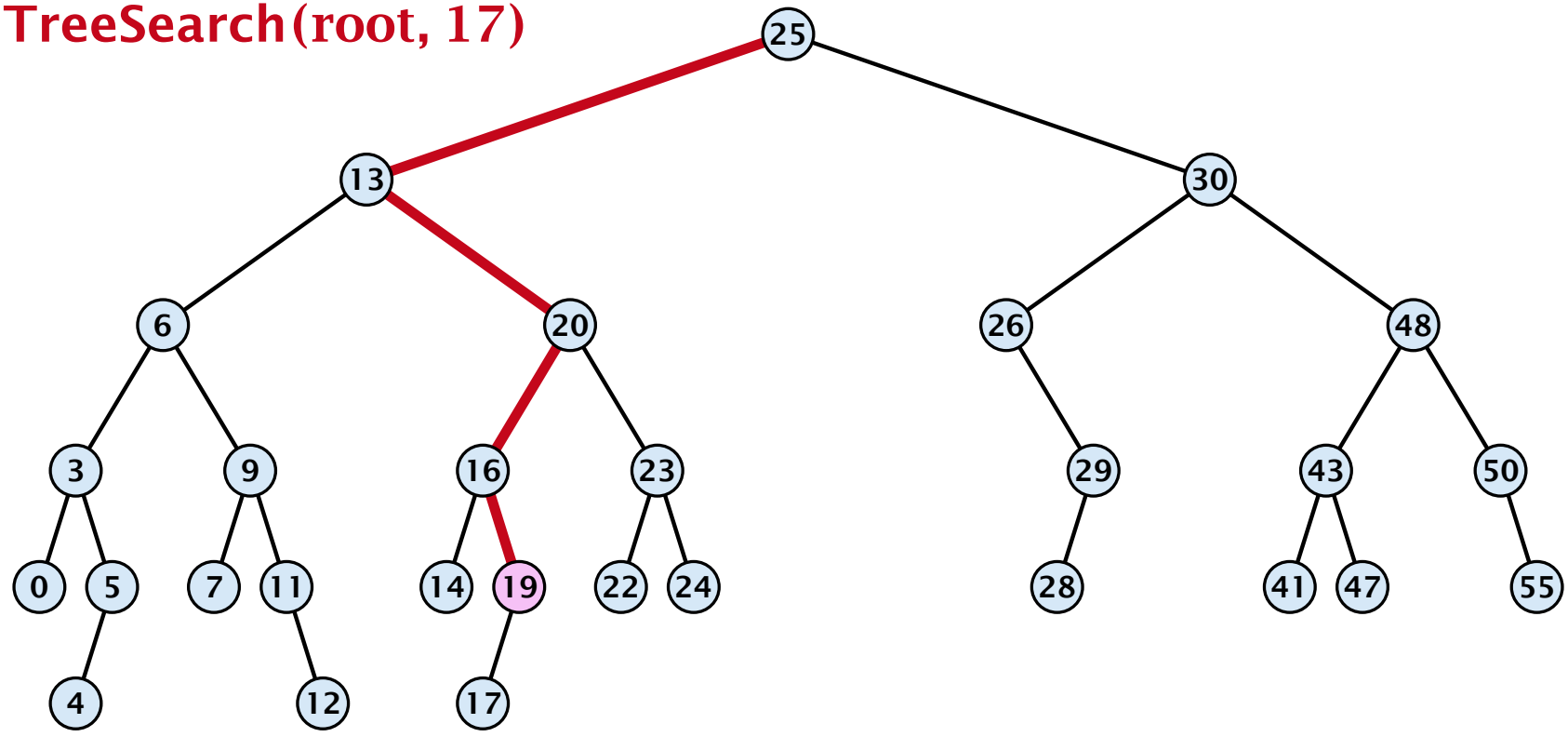

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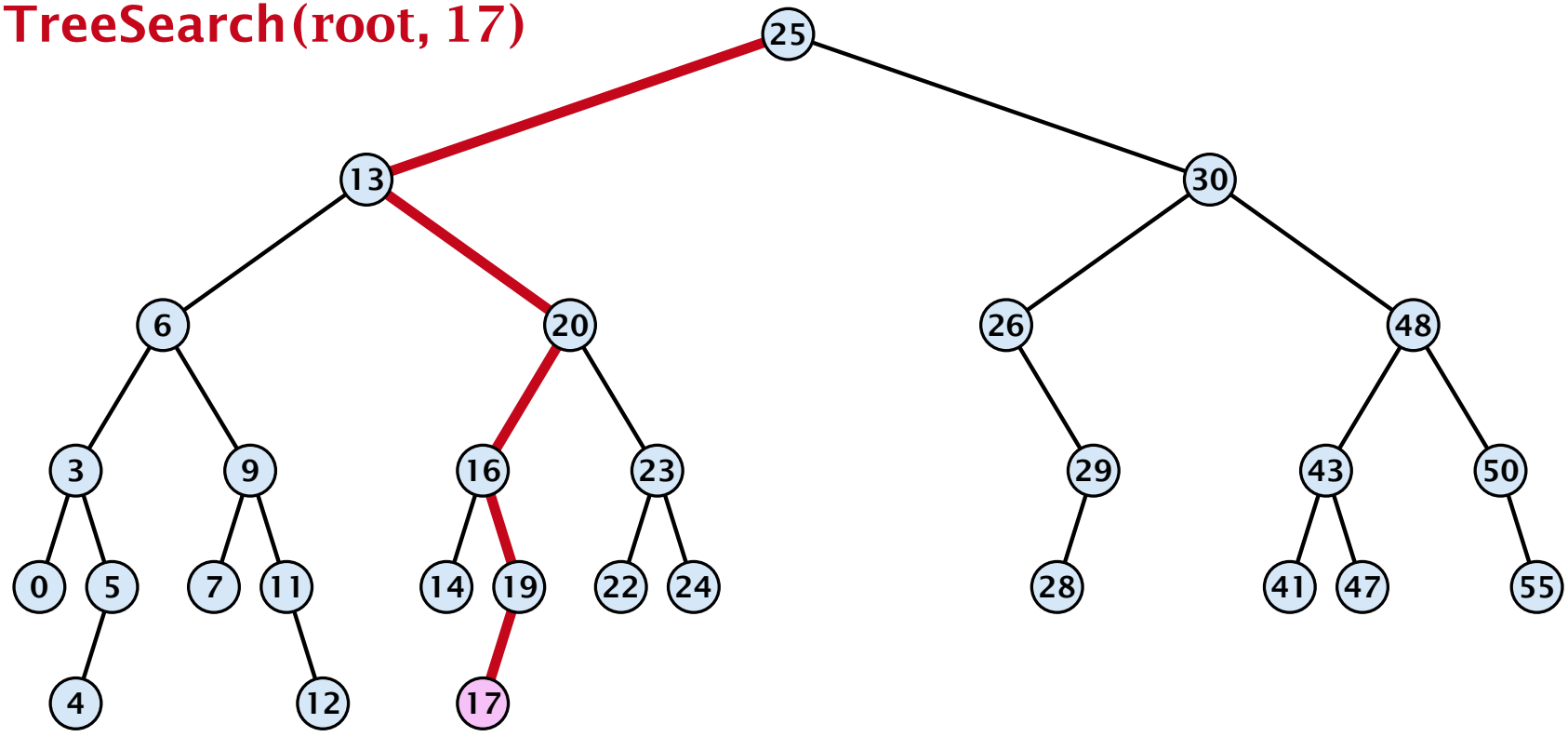

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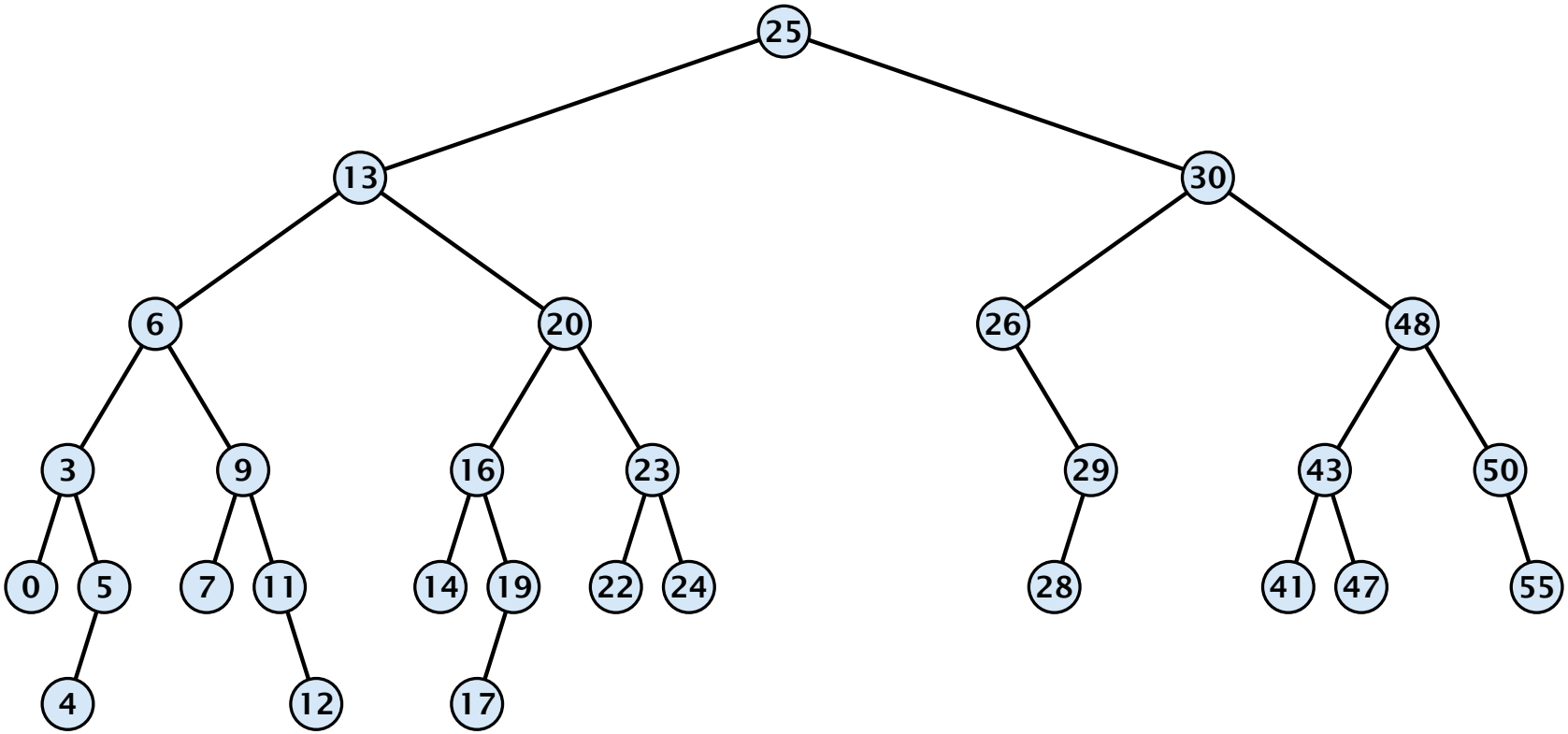

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Binary Search Trees: Searching

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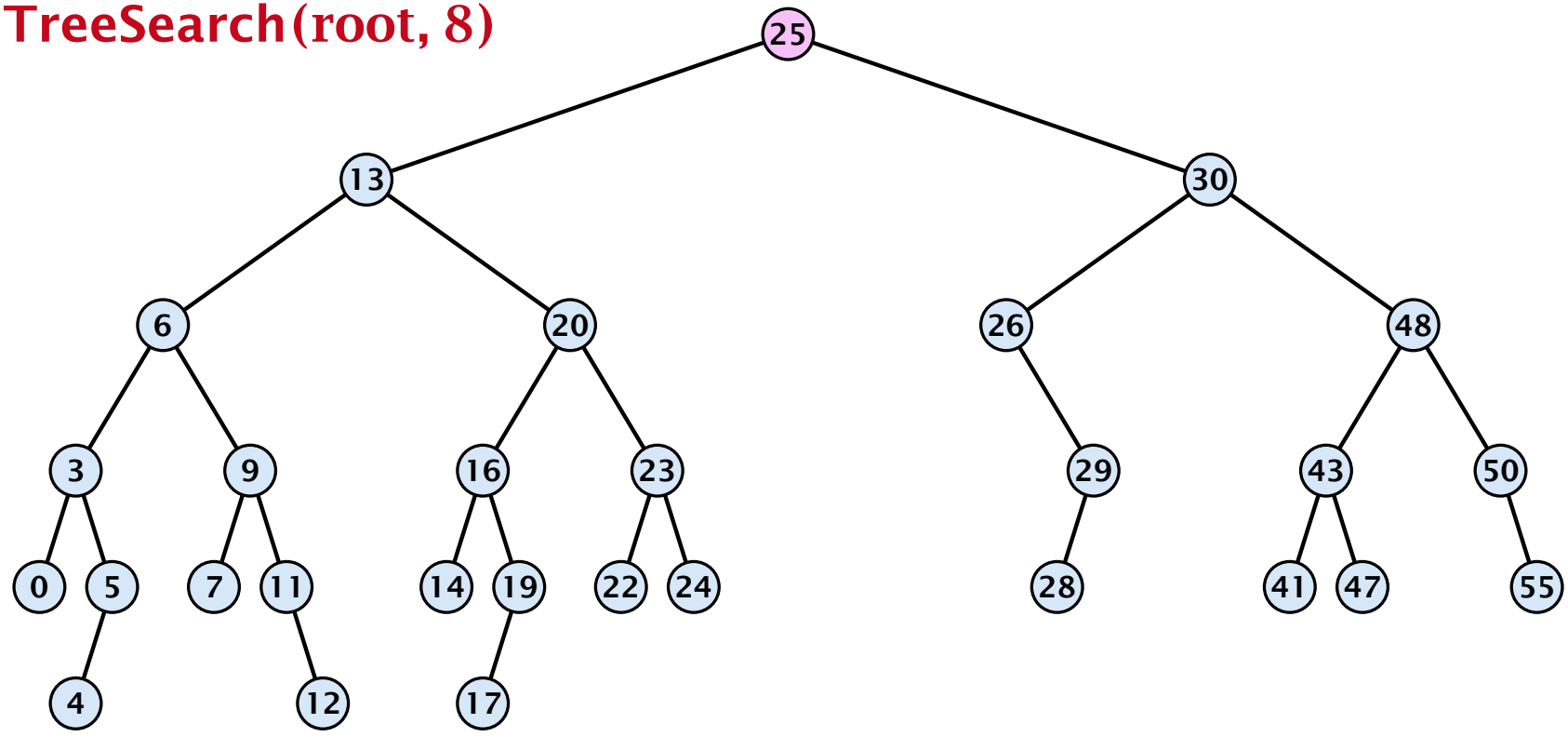

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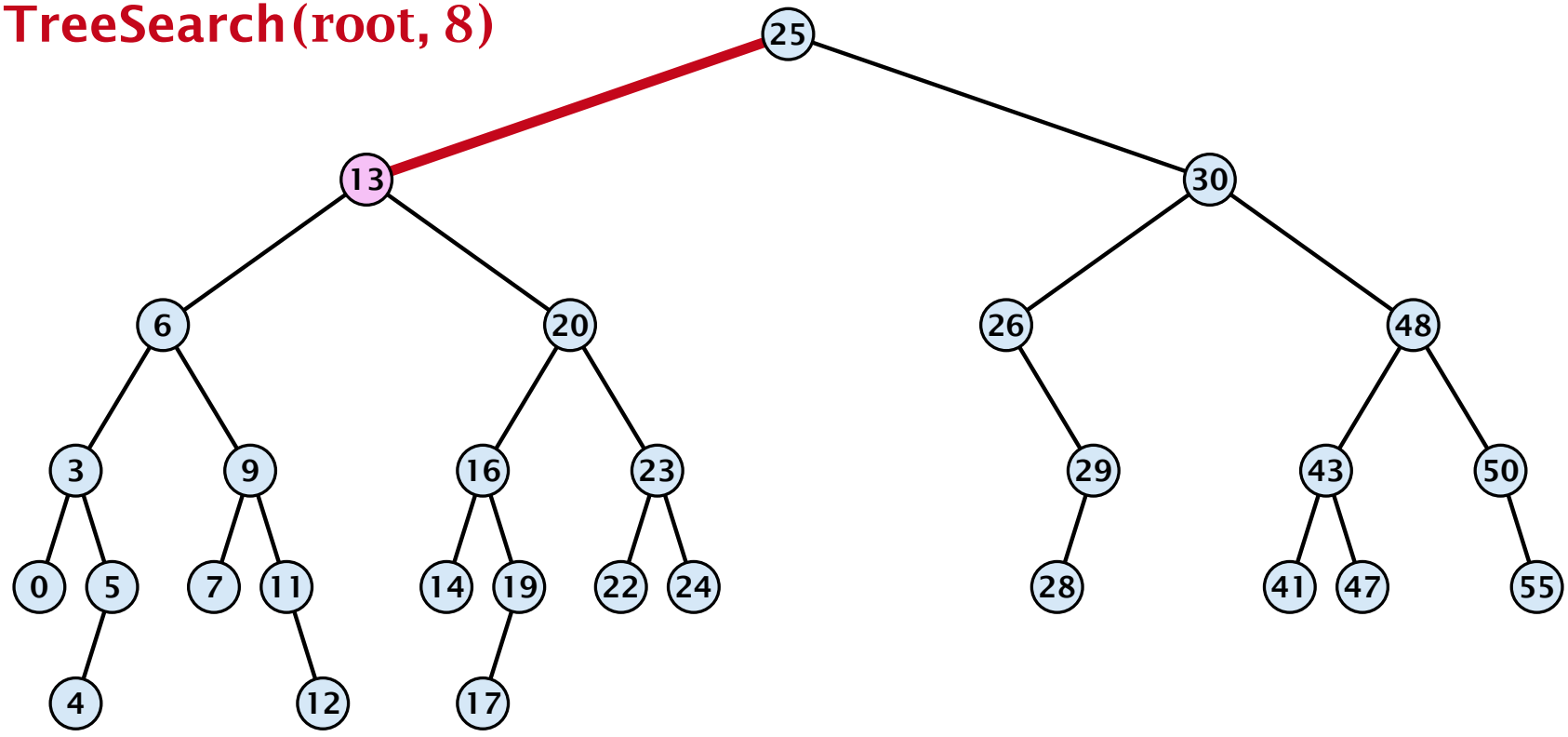

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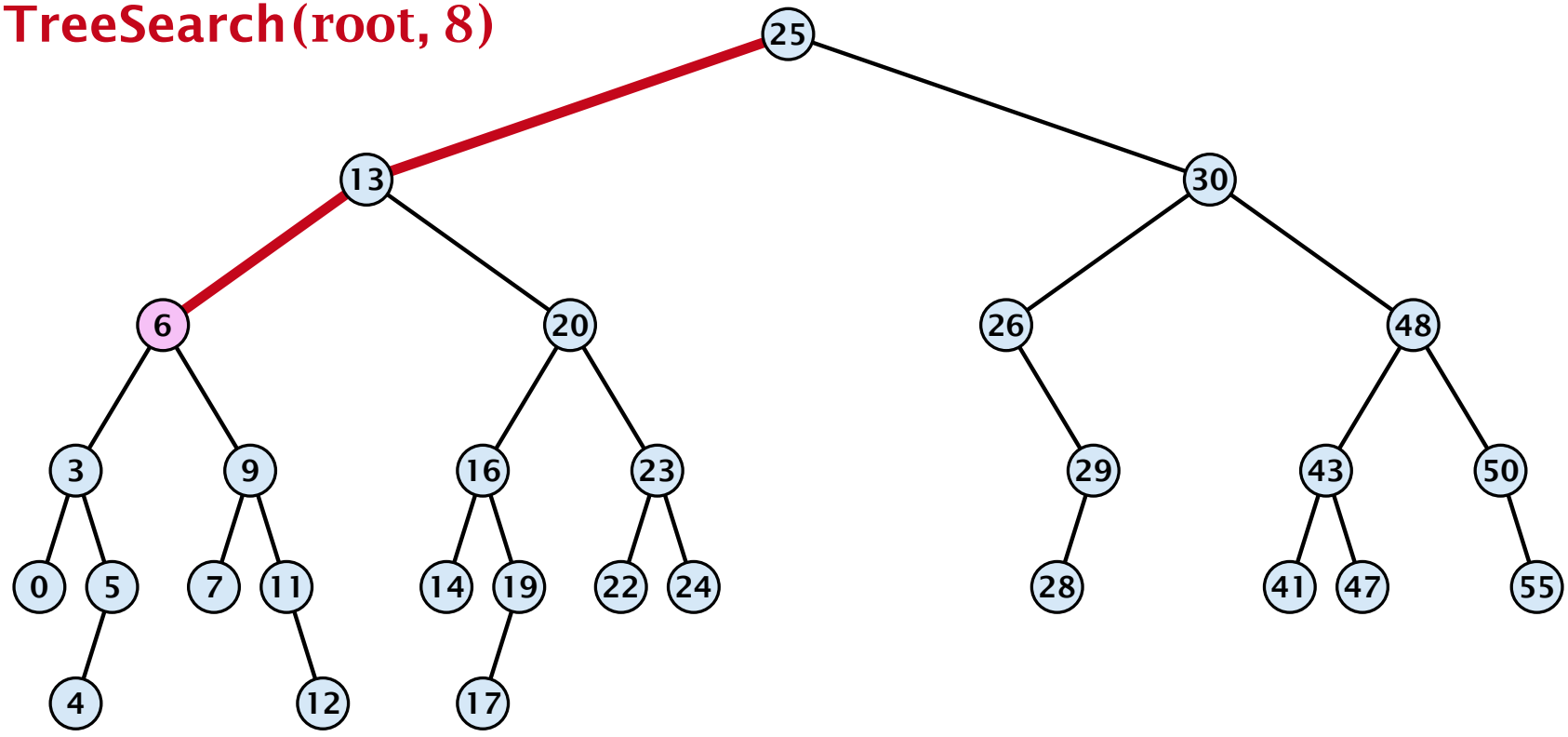

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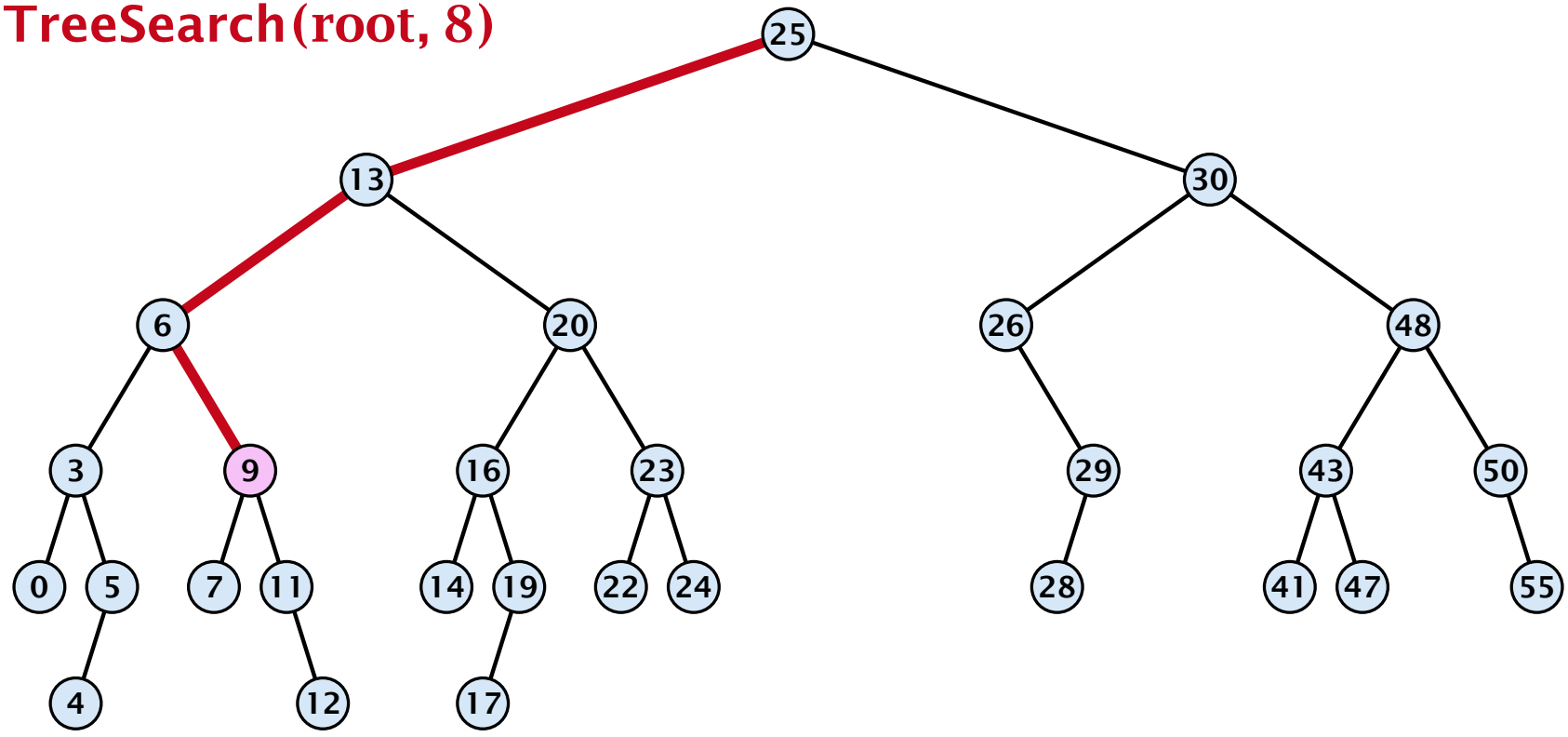

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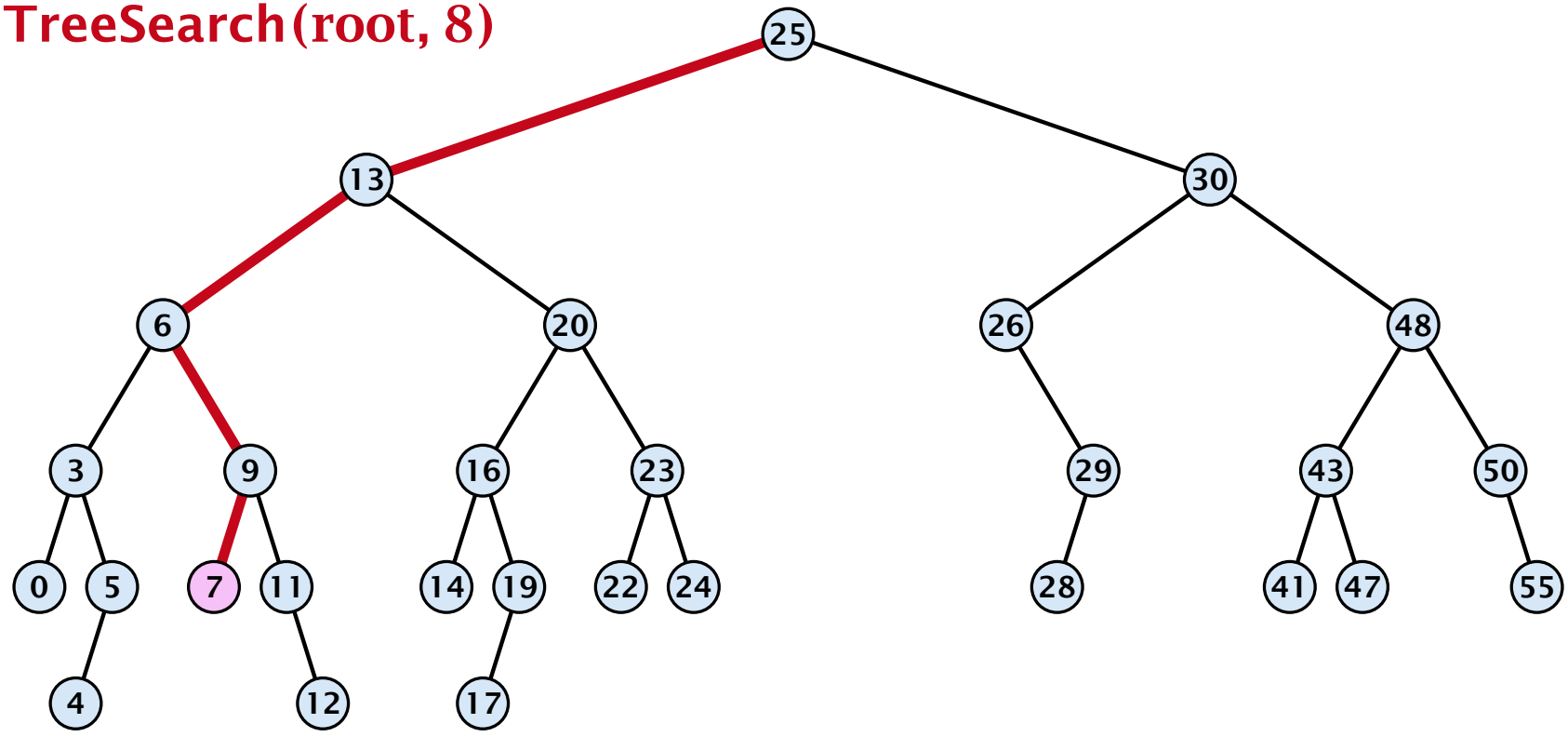

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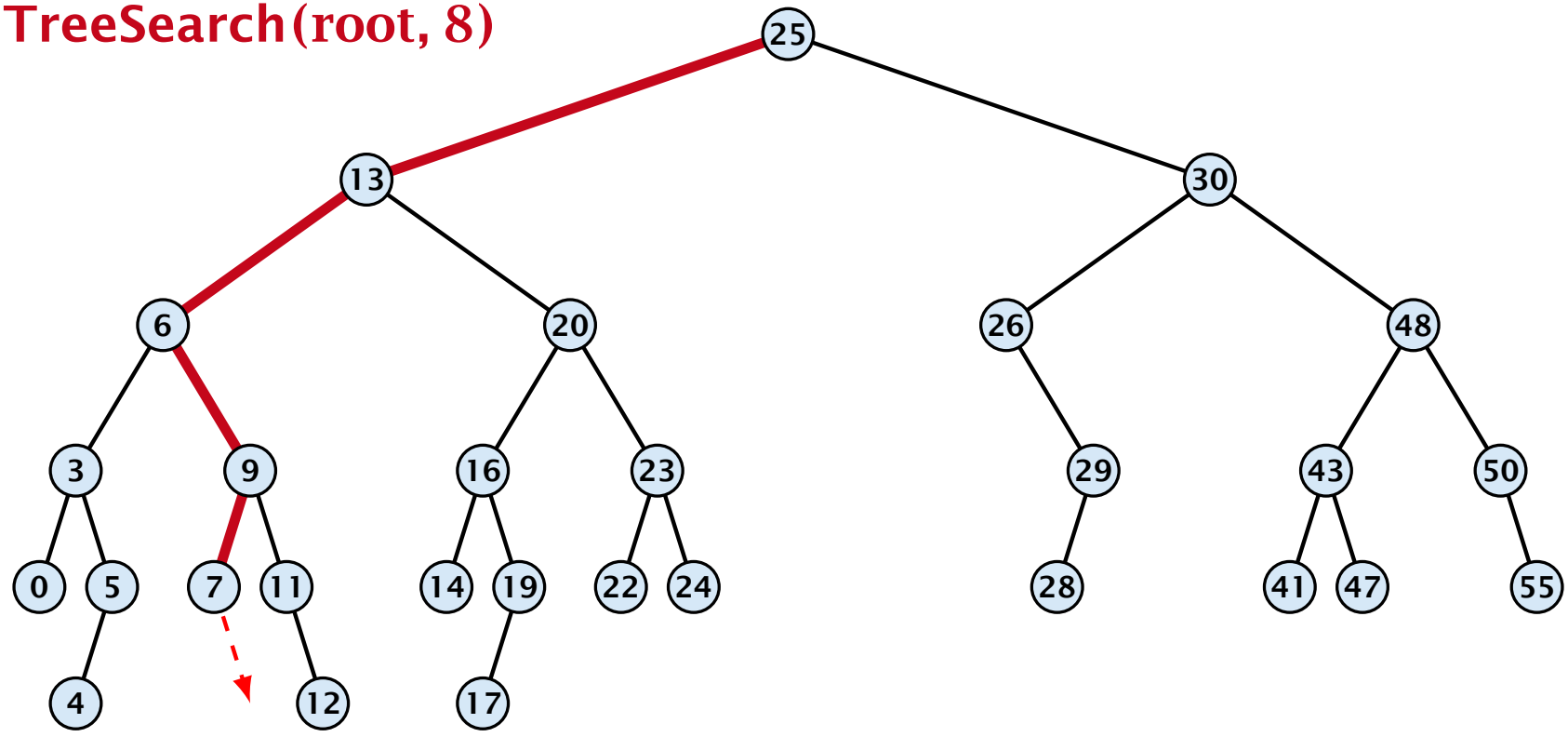

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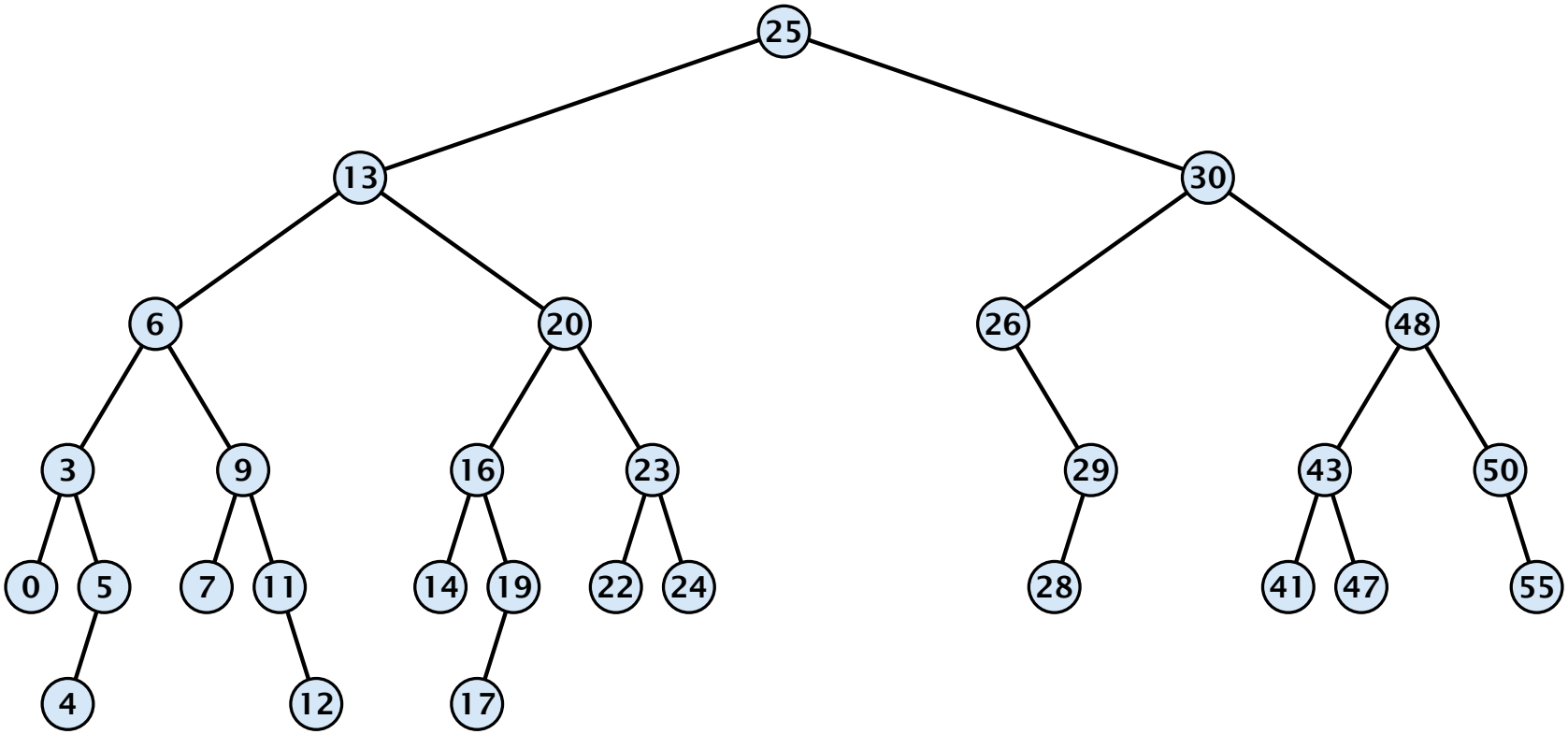
Binary Search Trees: Minimum



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1: **if** $x = \text{null}$ **or** $\text{left}[x] = \text{null}$ **return** x
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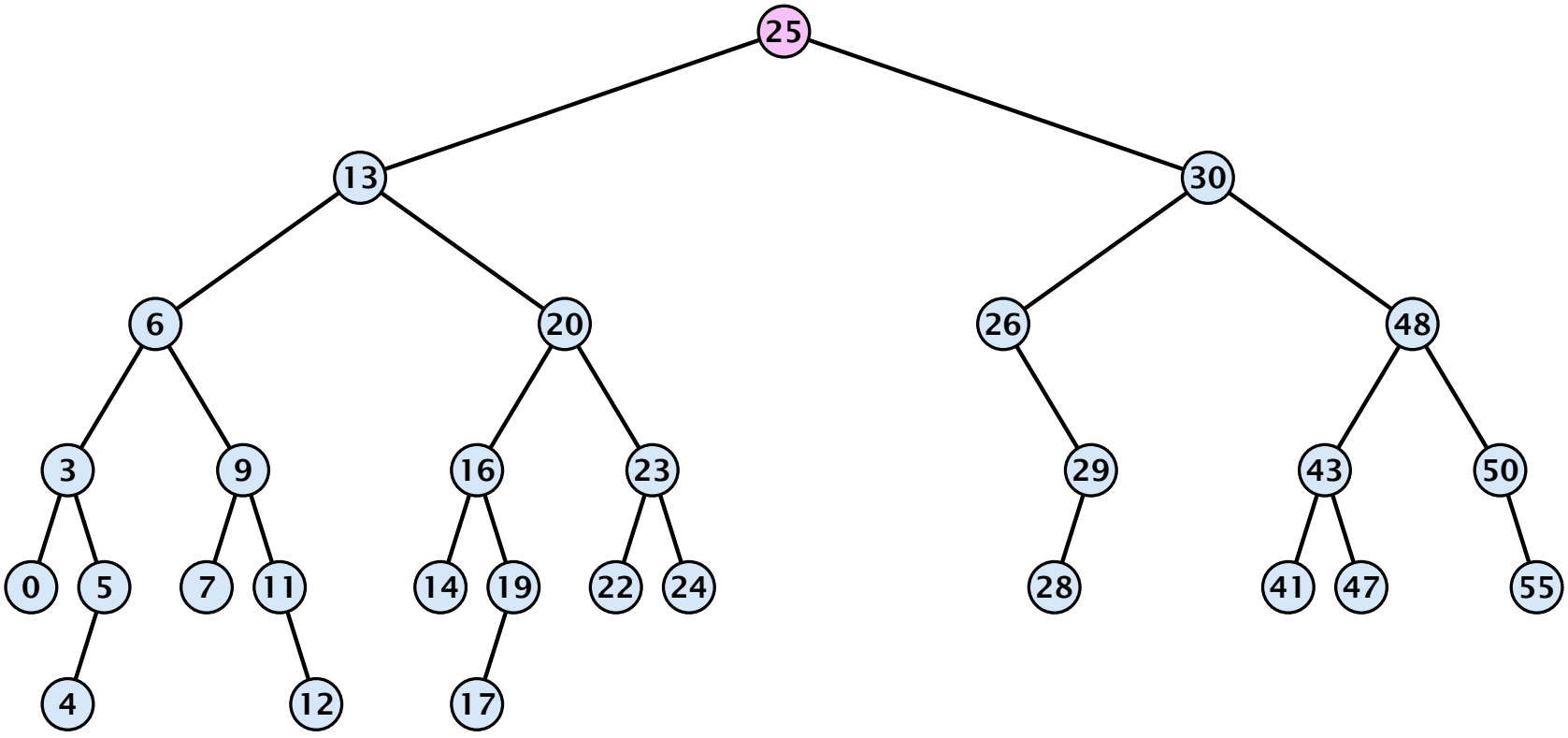
Binary Search Trees: Minimum



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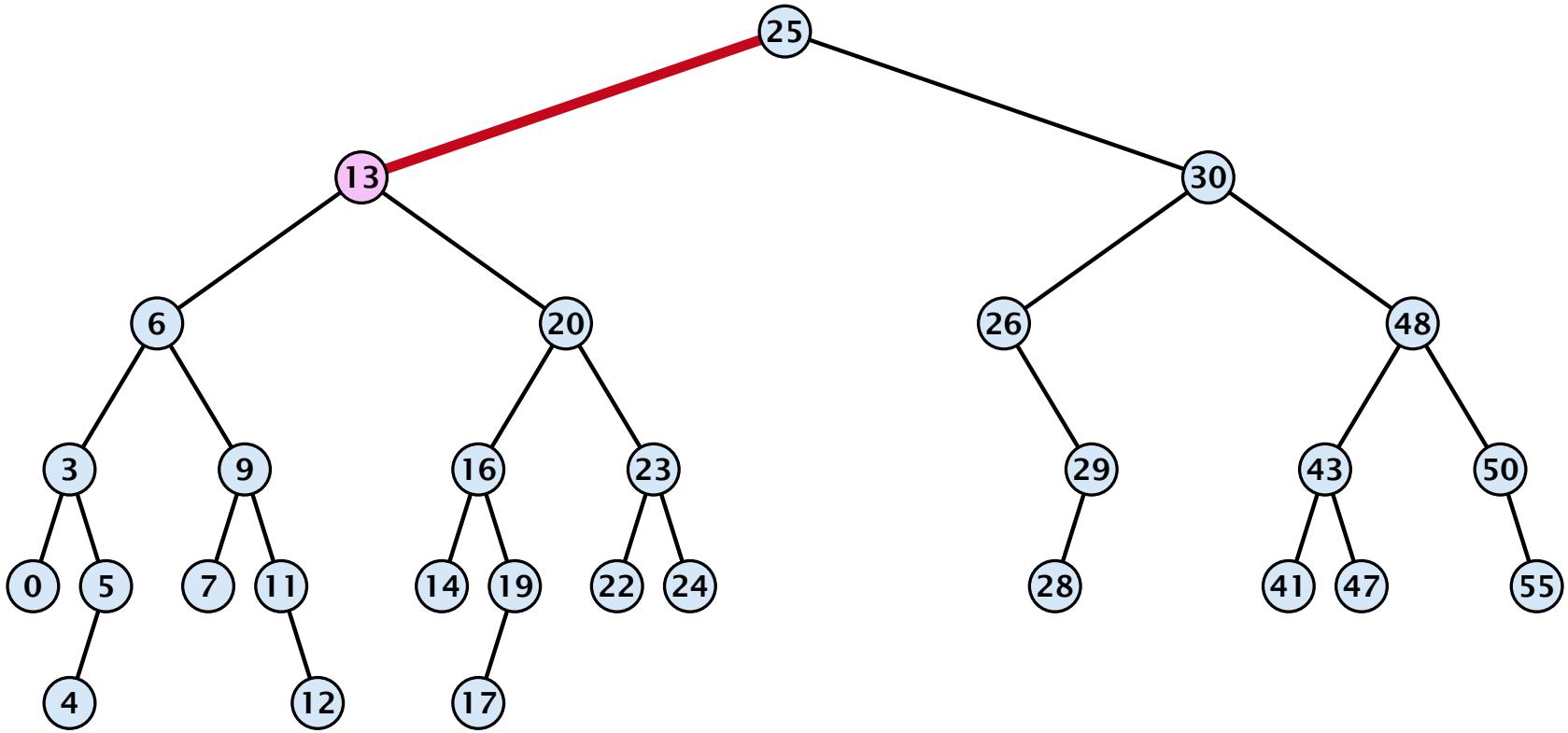

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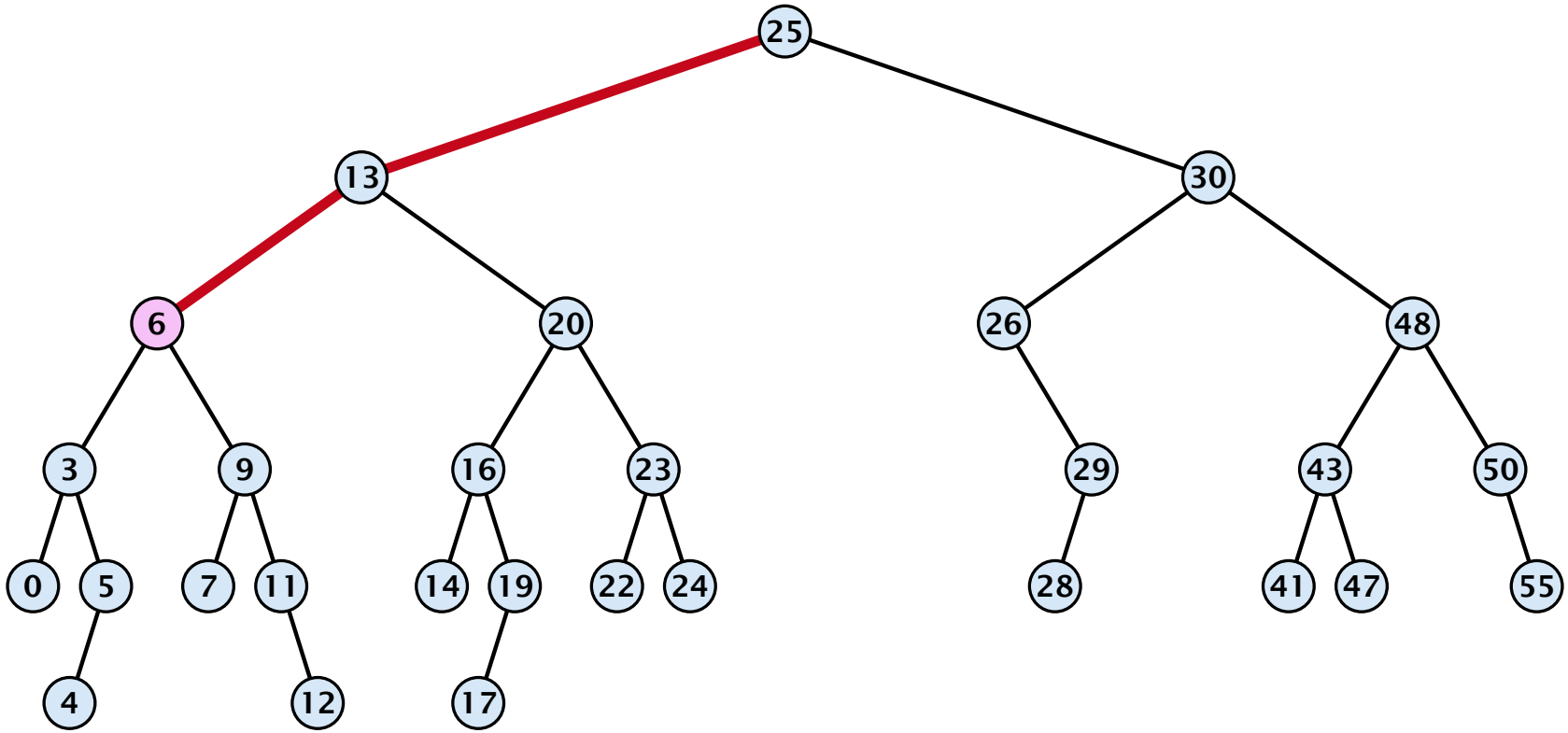

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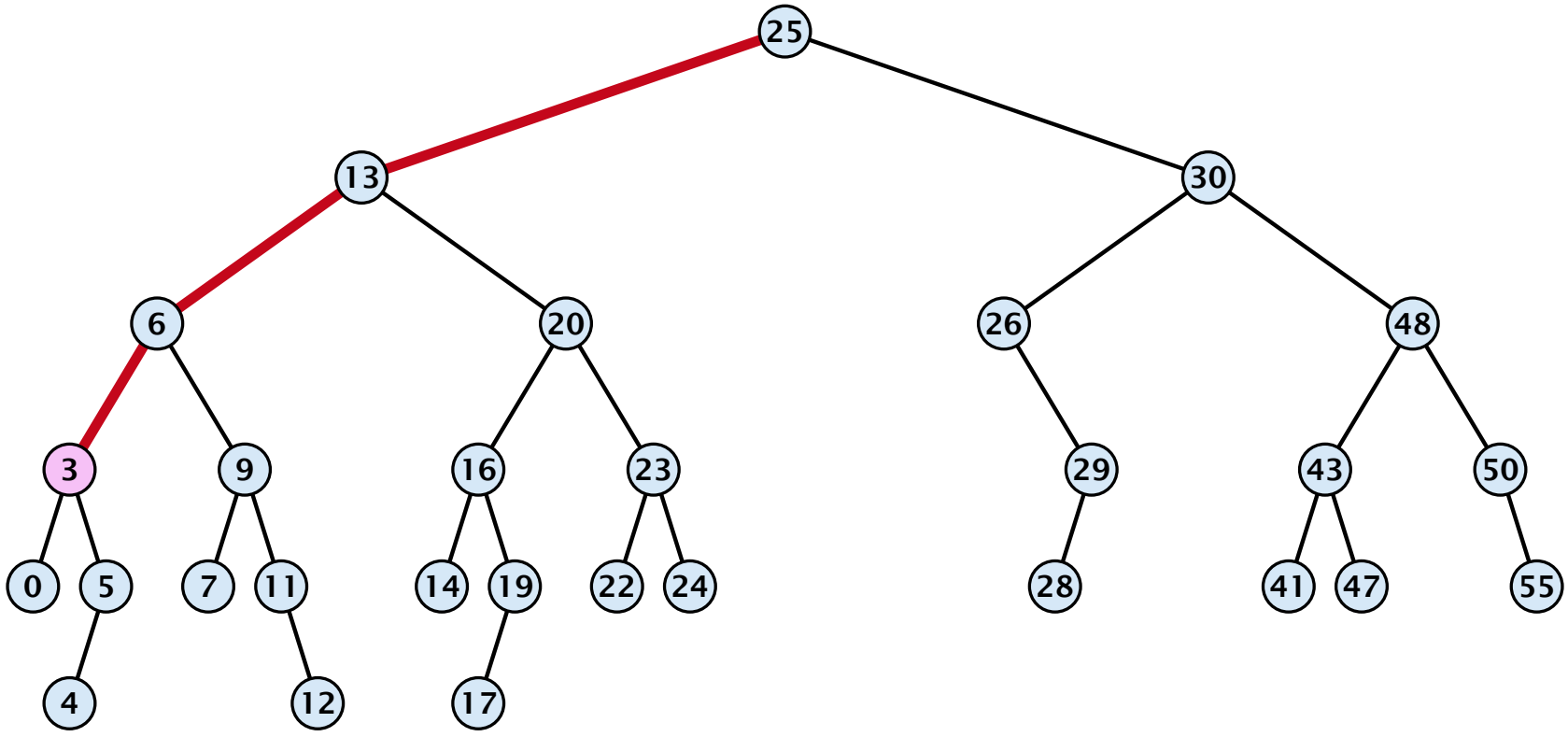

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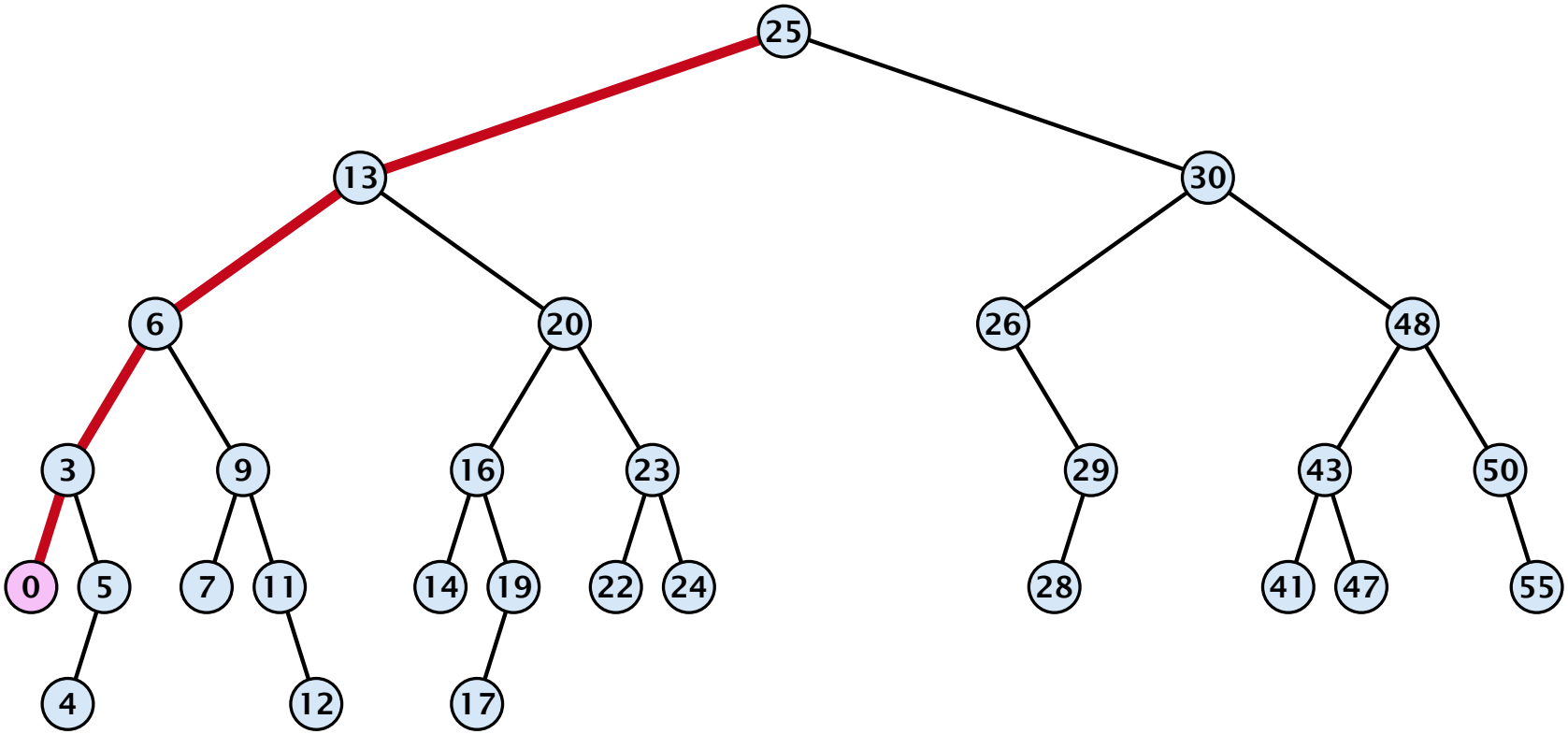

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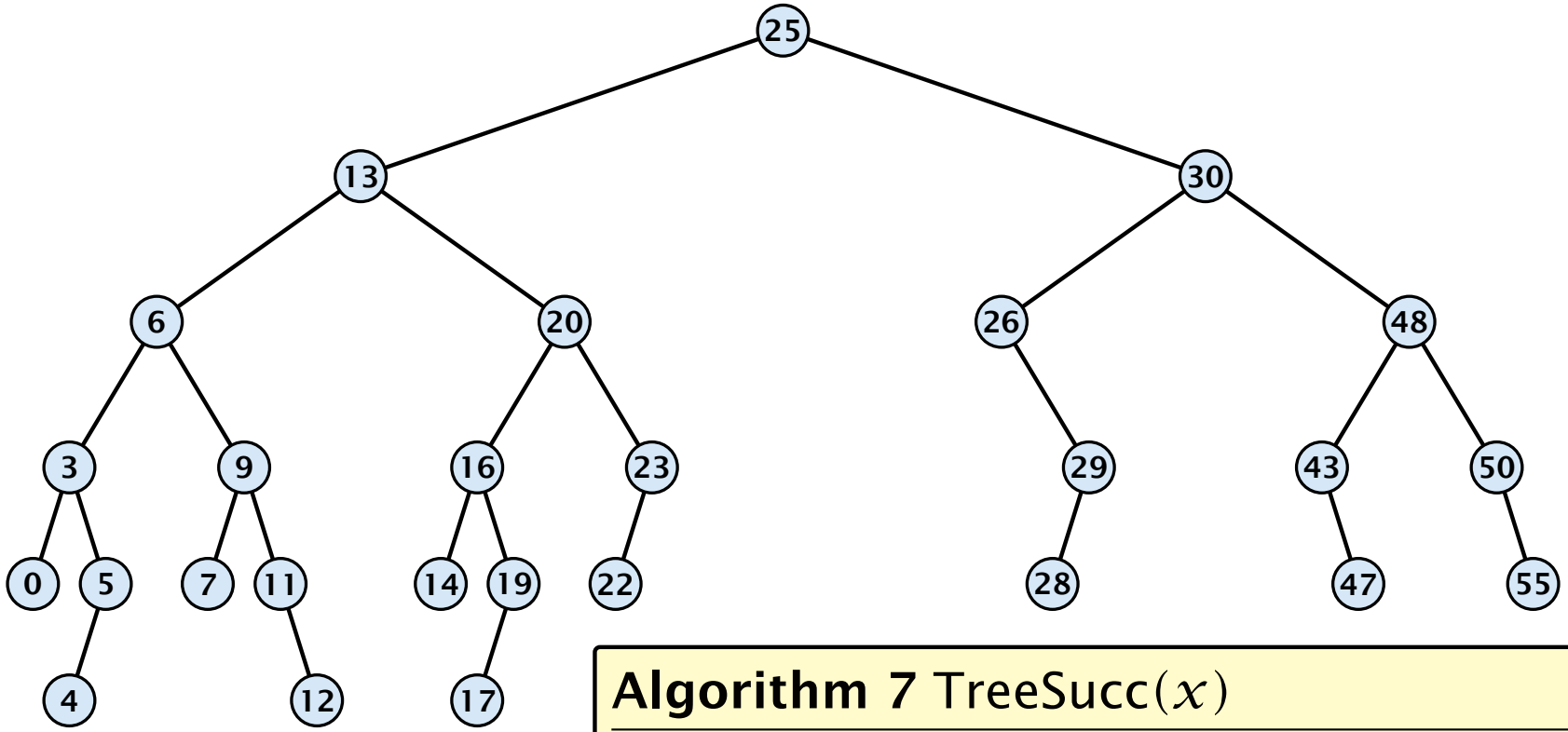
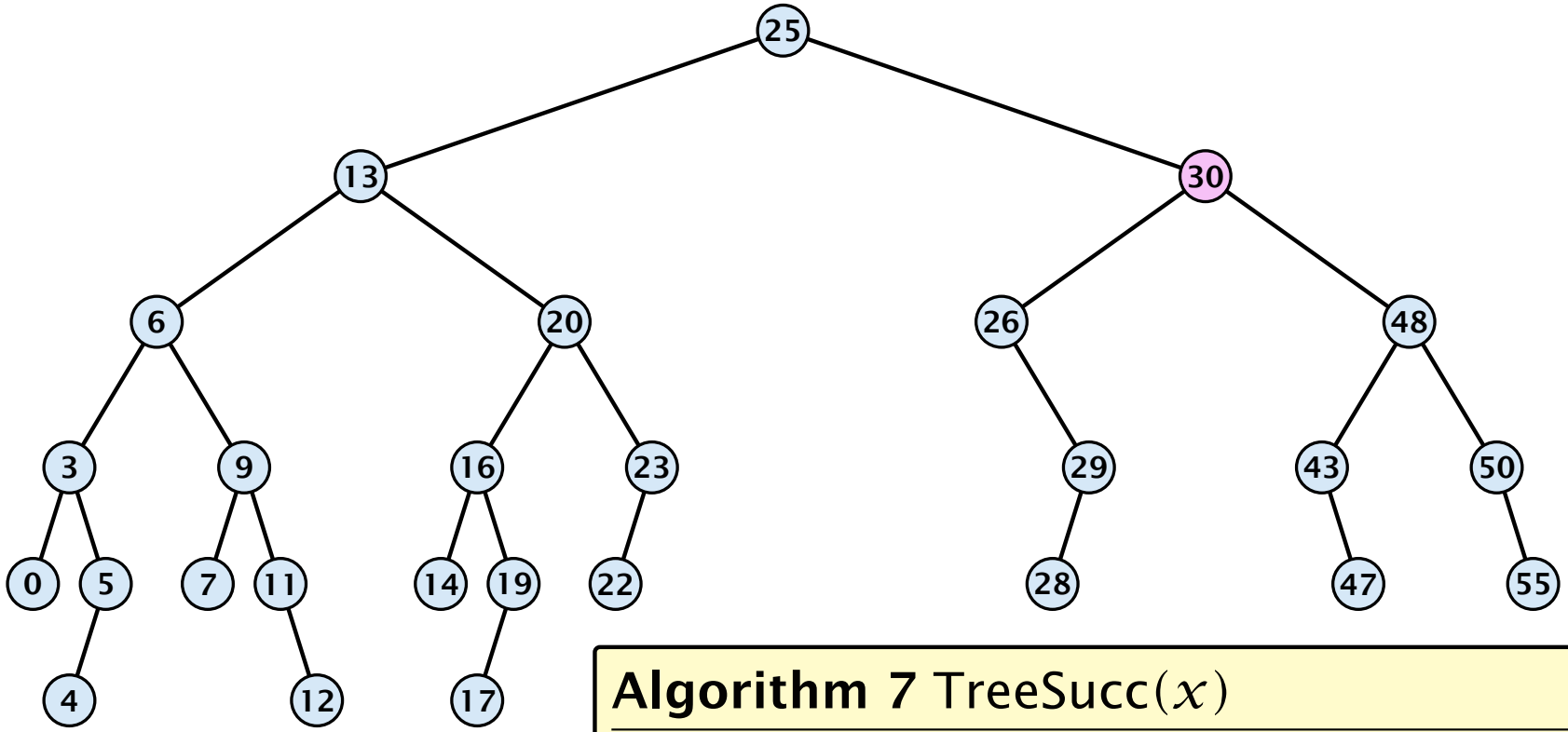
Binary Search Trees: Successor



Algorithm 7 TreeSucc(x)

```
1: if right[x] ≠ null return TreeMin(right[x])
2: y ← parent[x]
3: while y ≠ null and x = right[y] do
4:         x ← y; y ← parent[x]
5: return y;
```

Binary Search Trees: Successor



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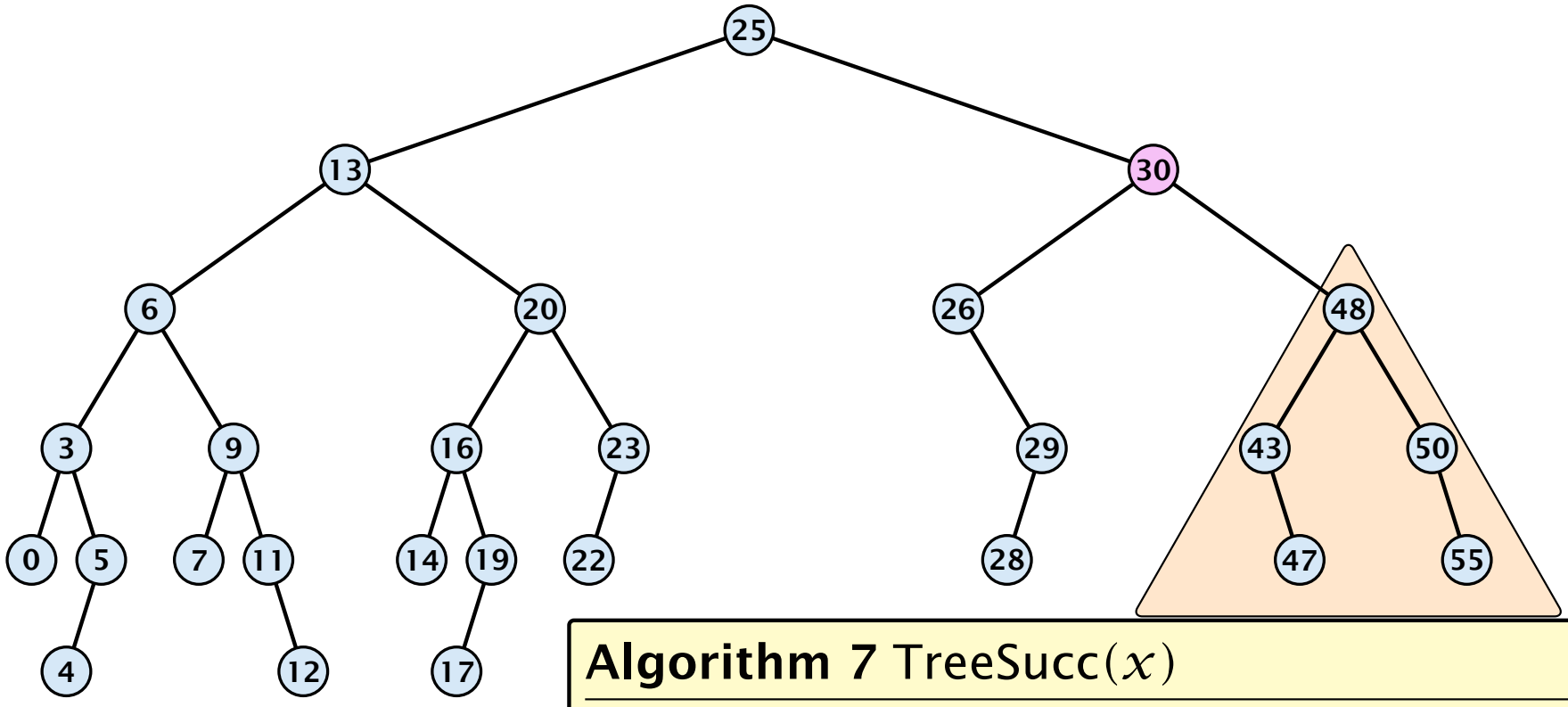
Binary Search Trees: Successor



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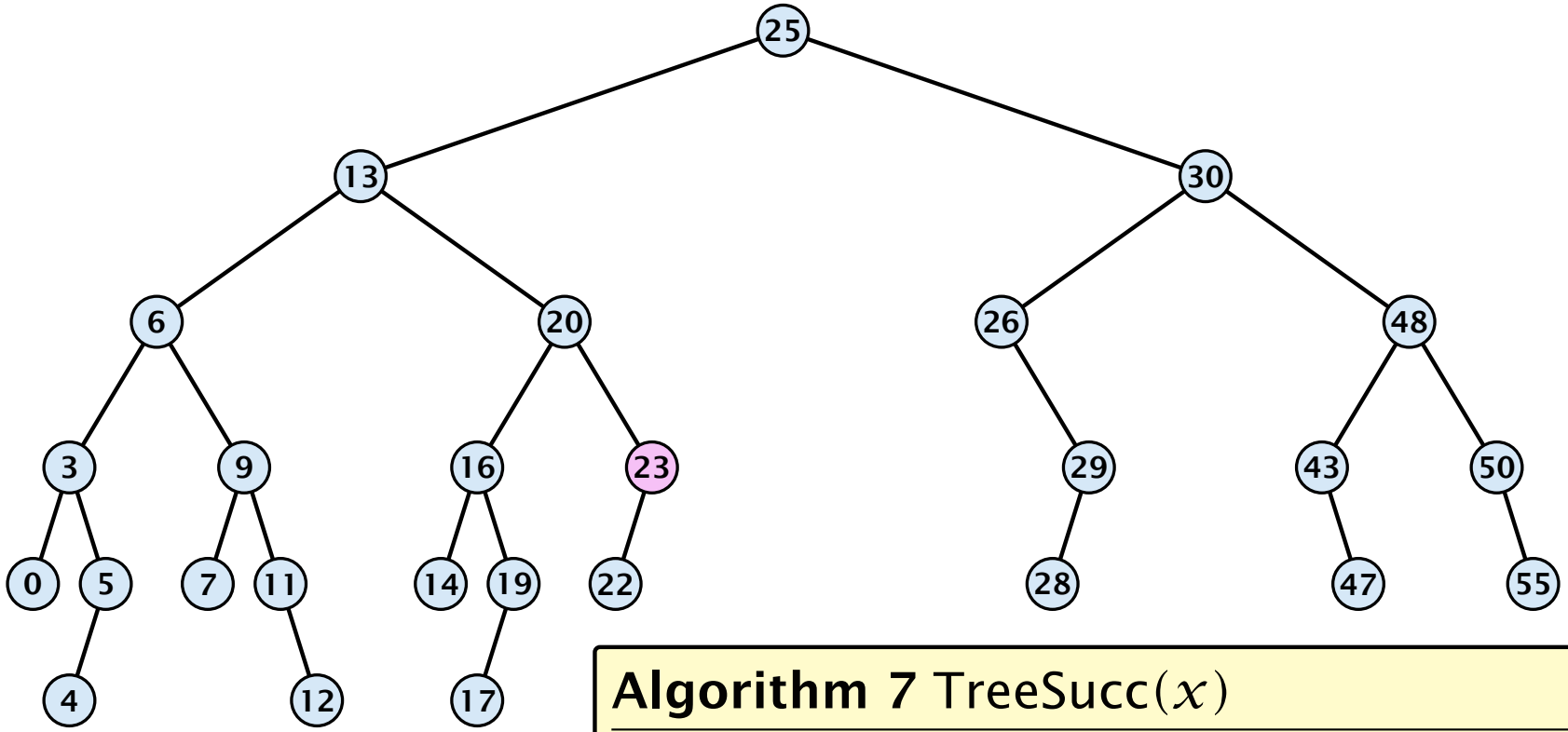

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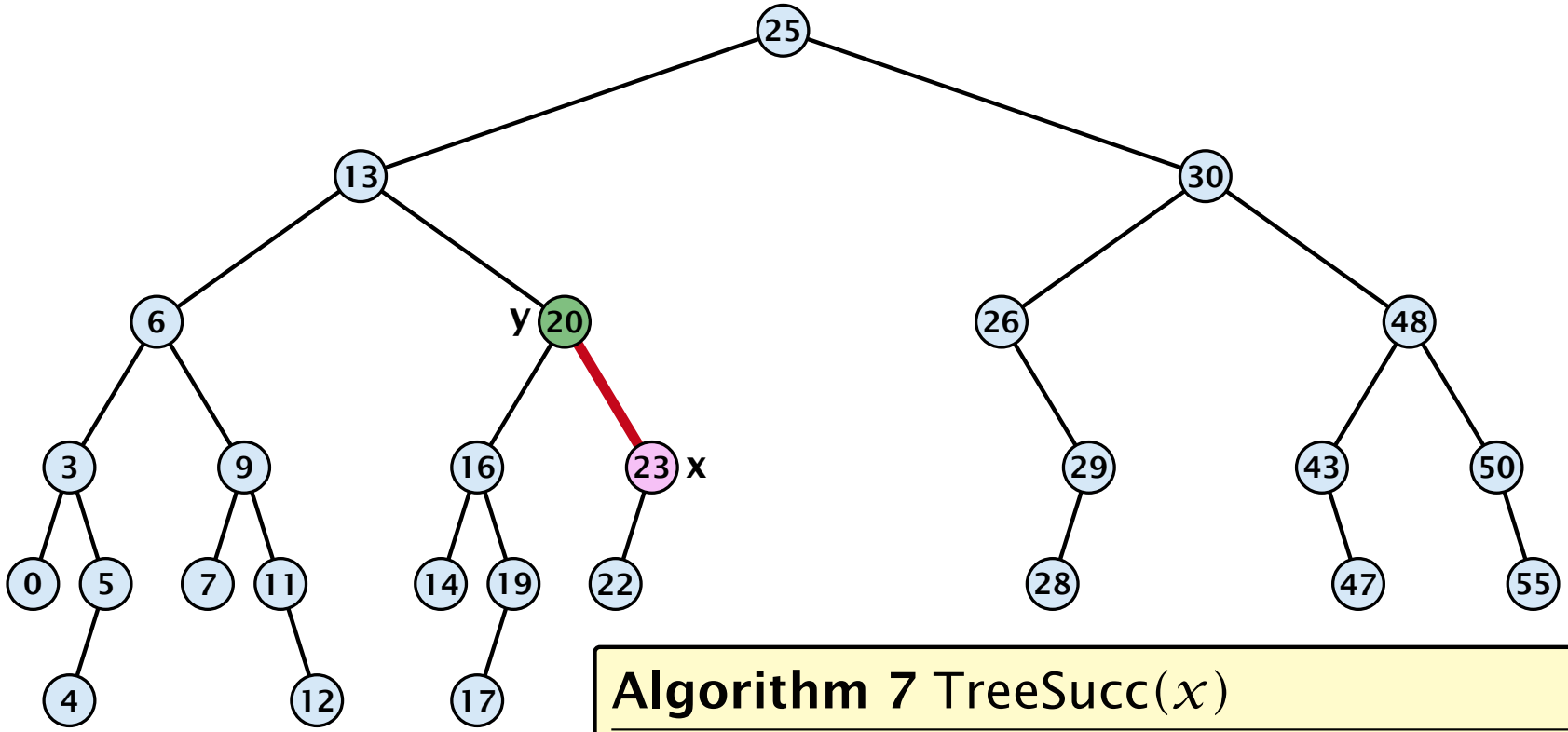

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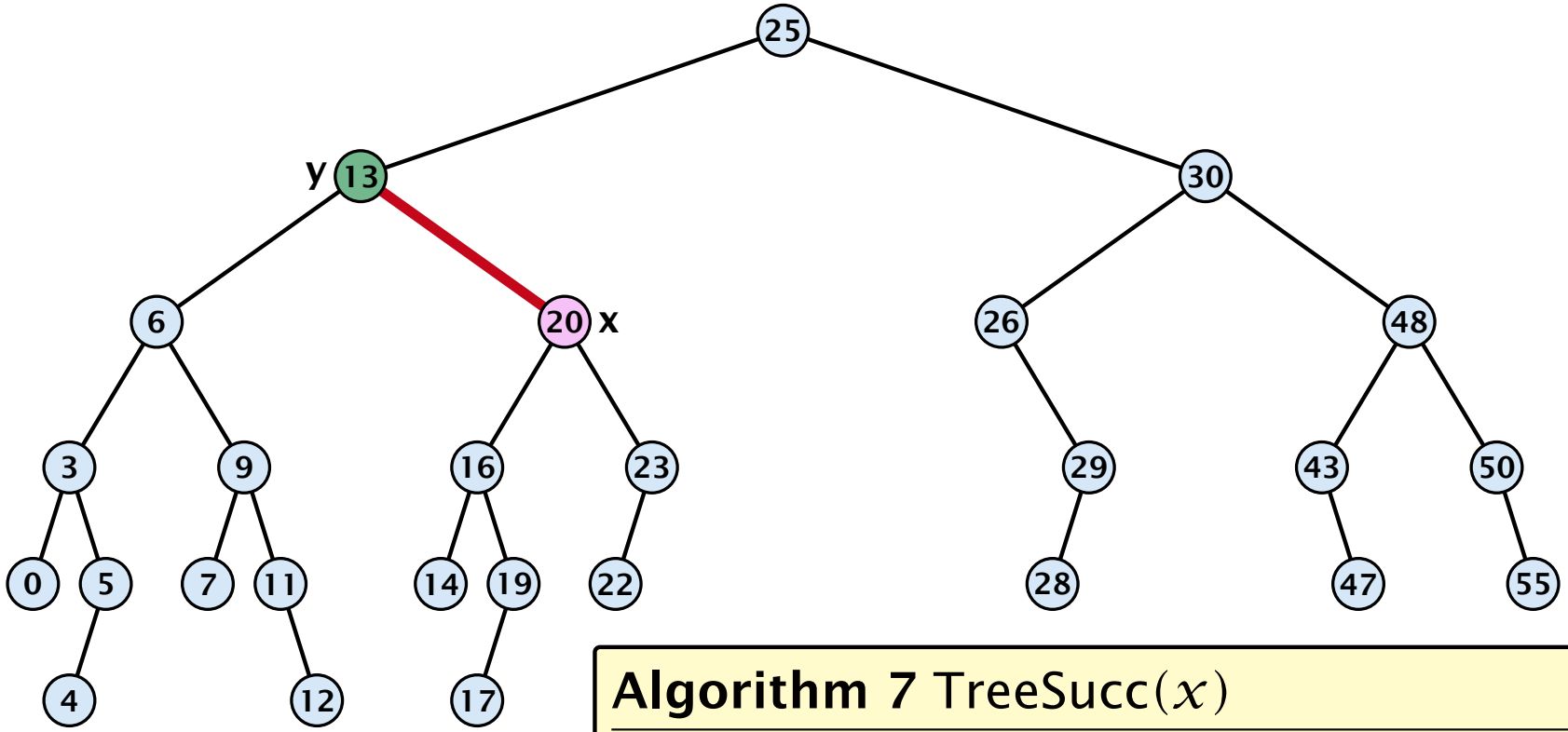

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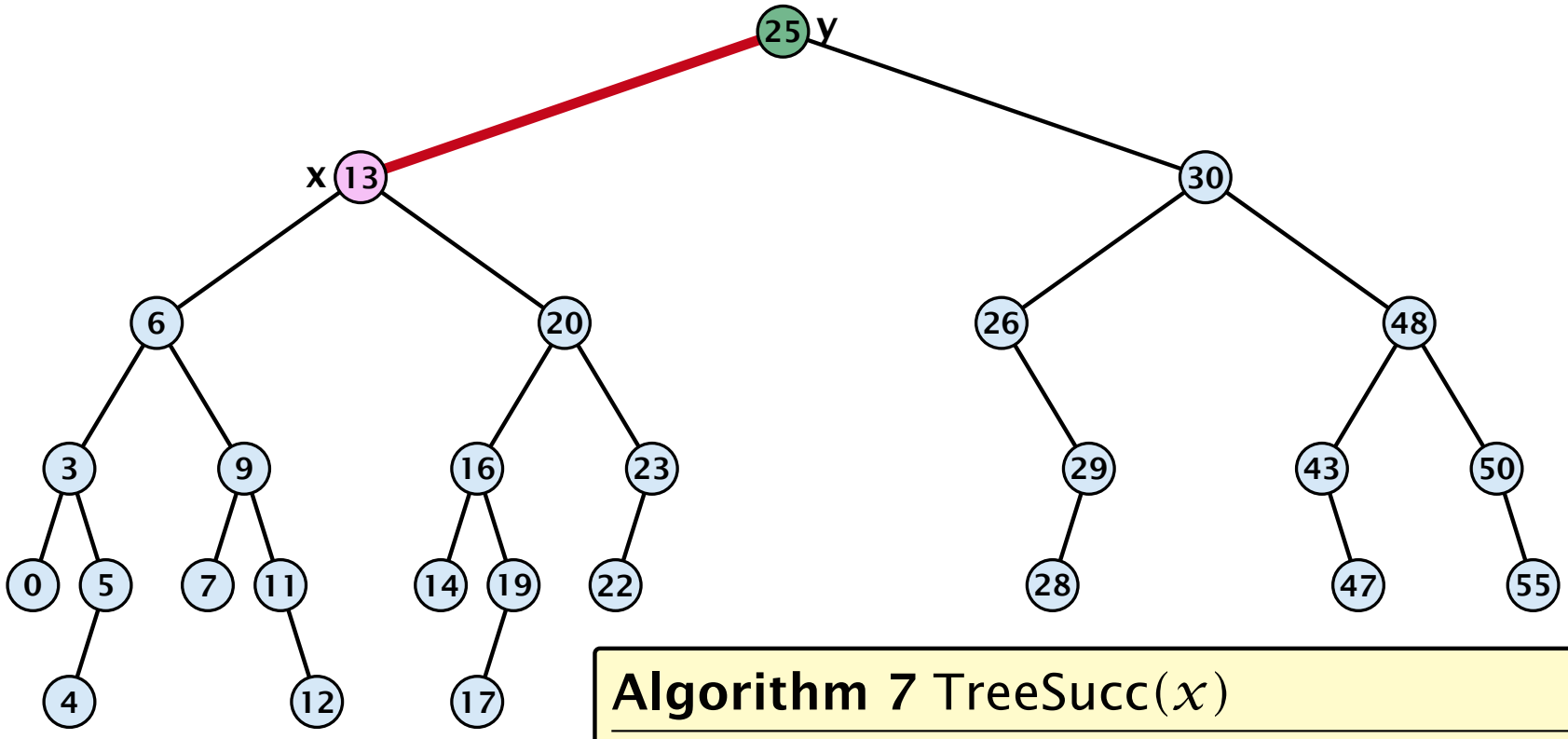

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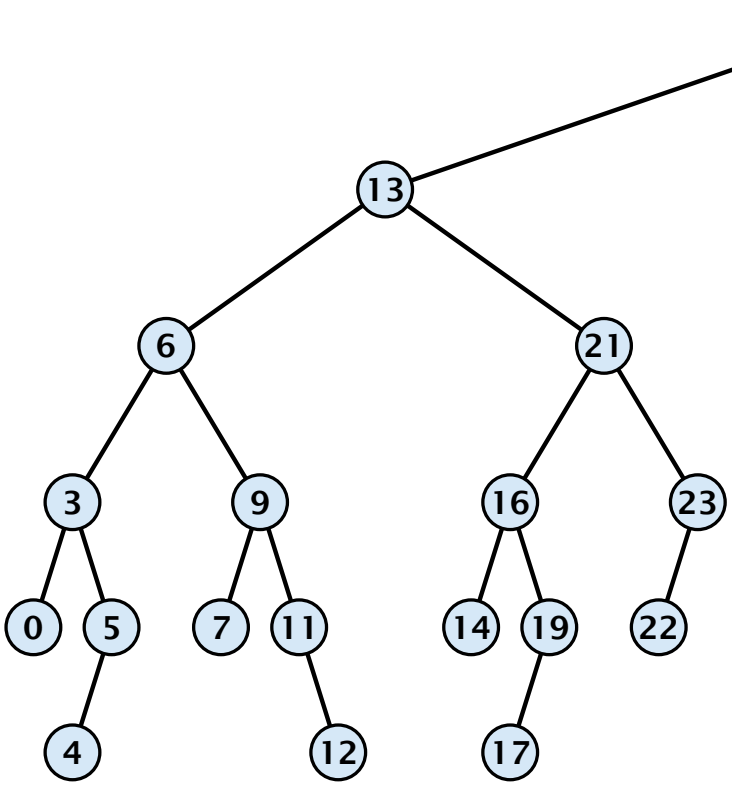

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Binary Search Trees: Insert

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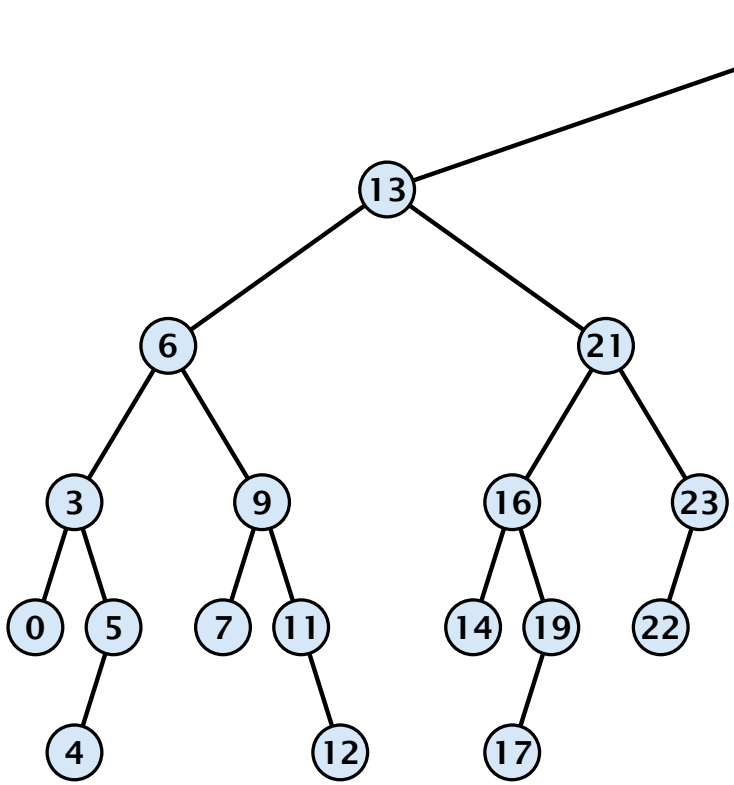

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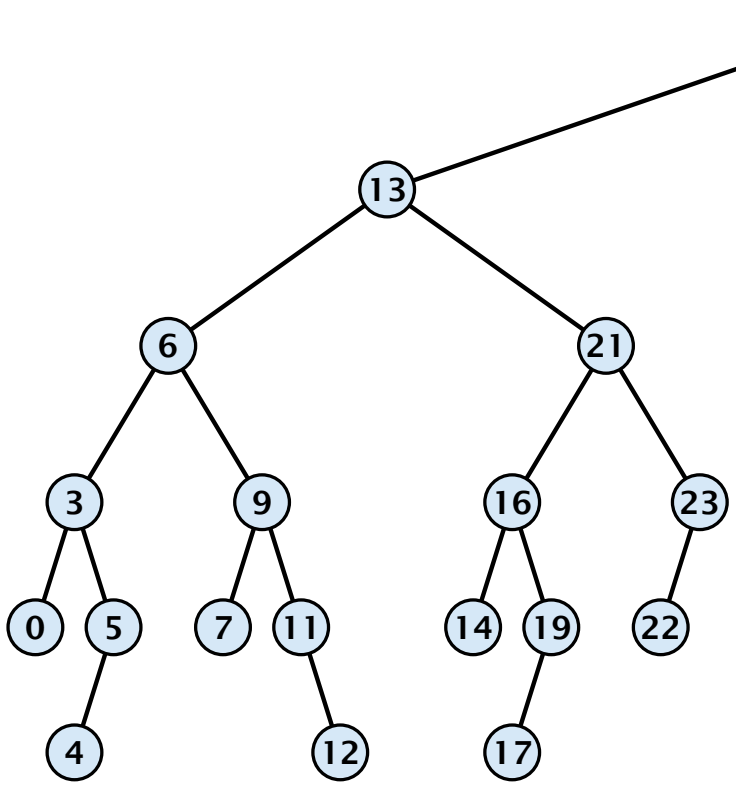

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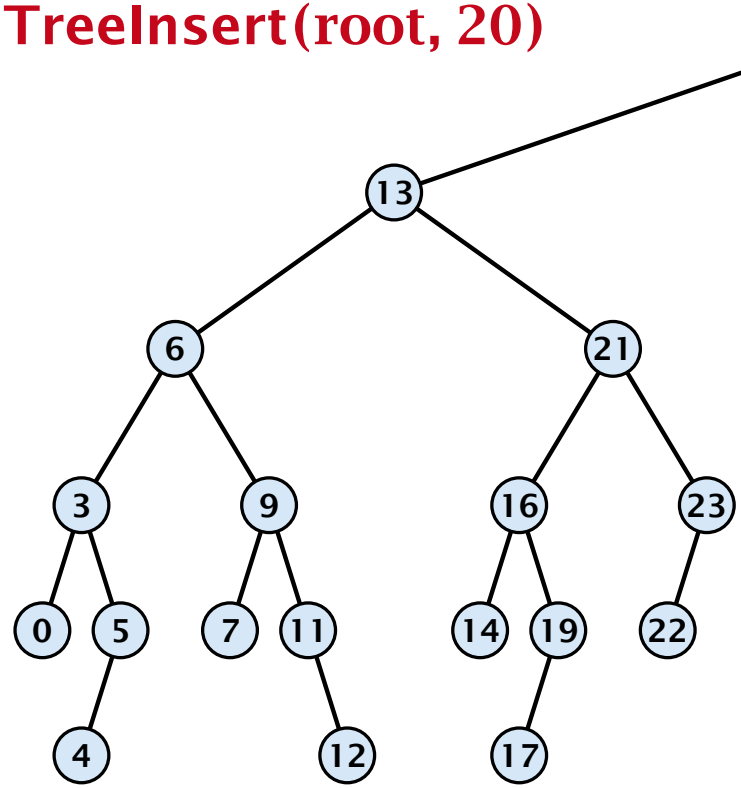

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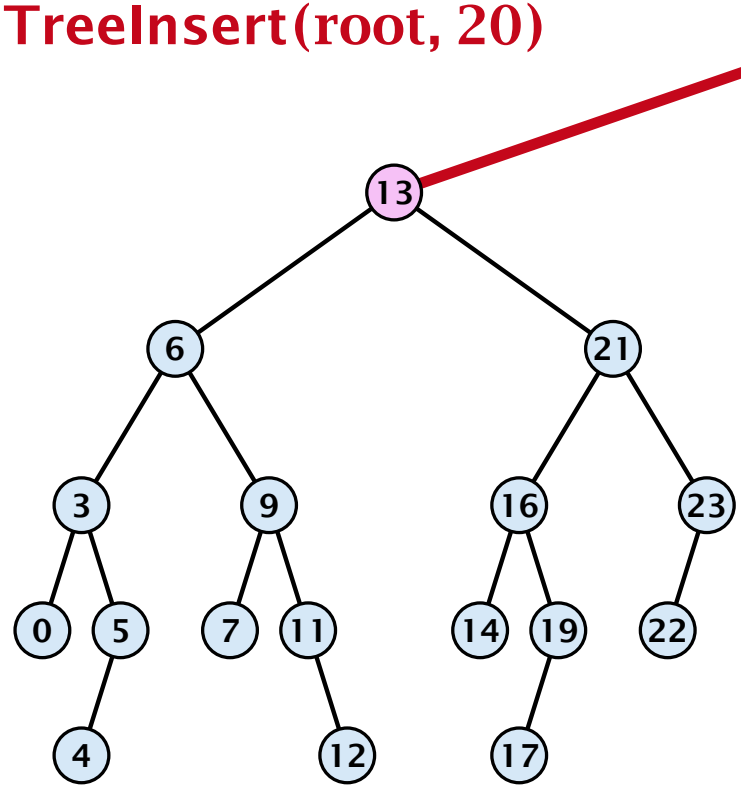

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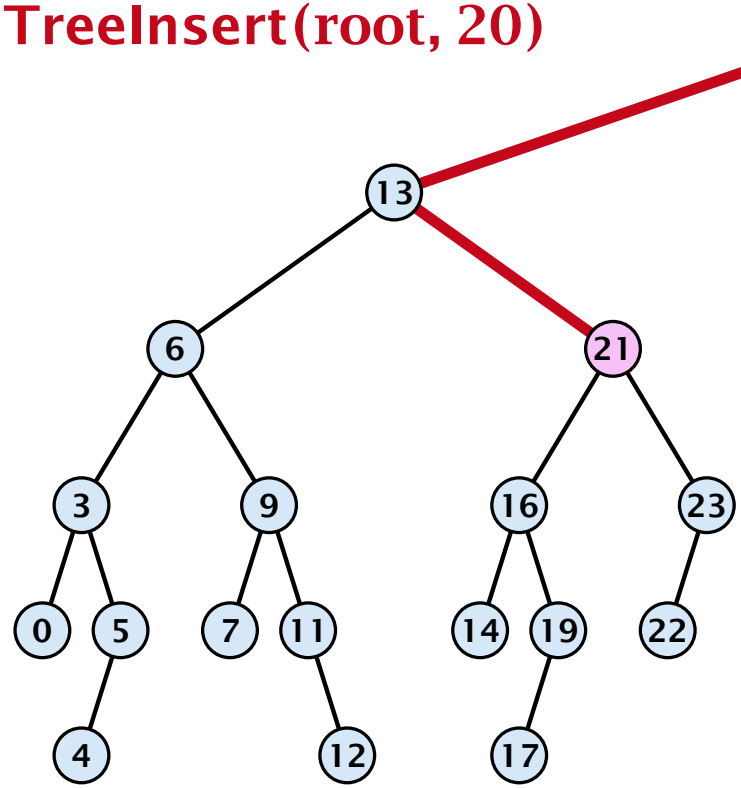

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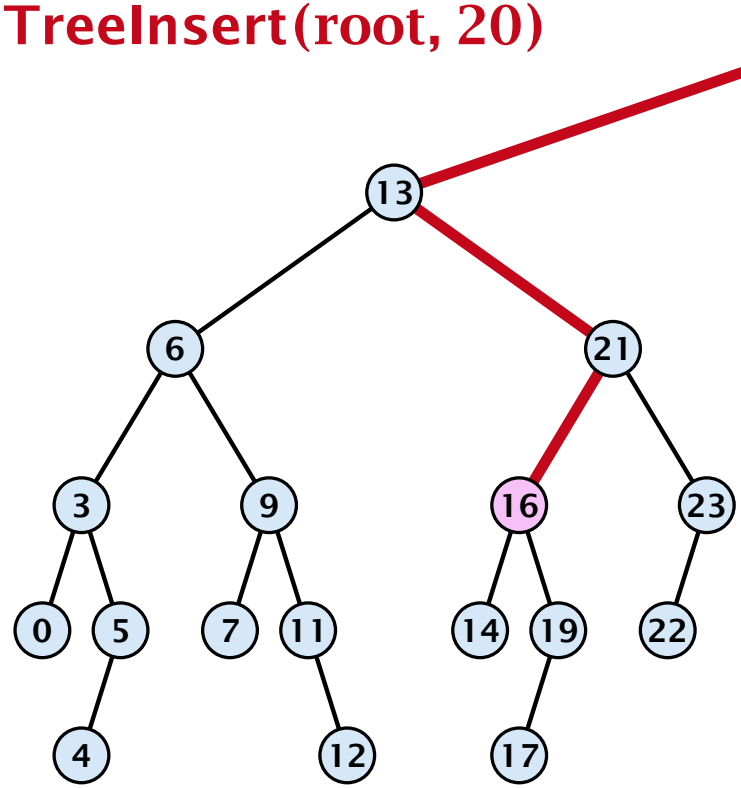

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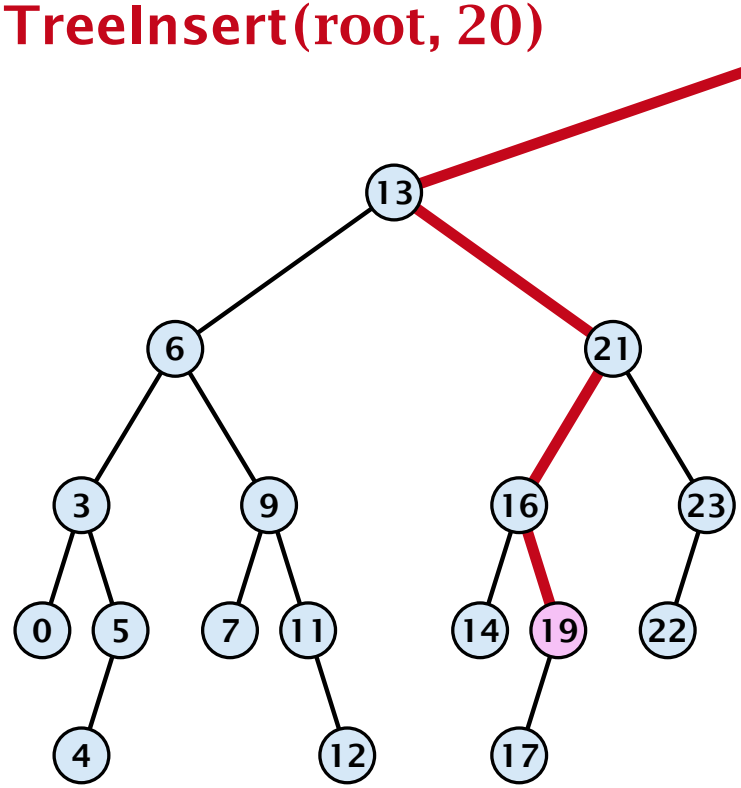

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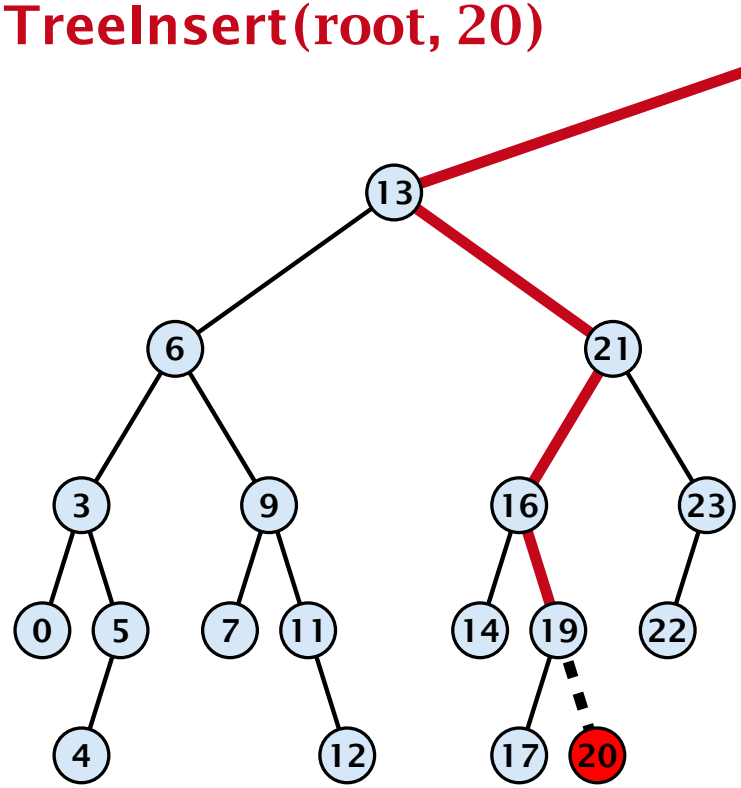

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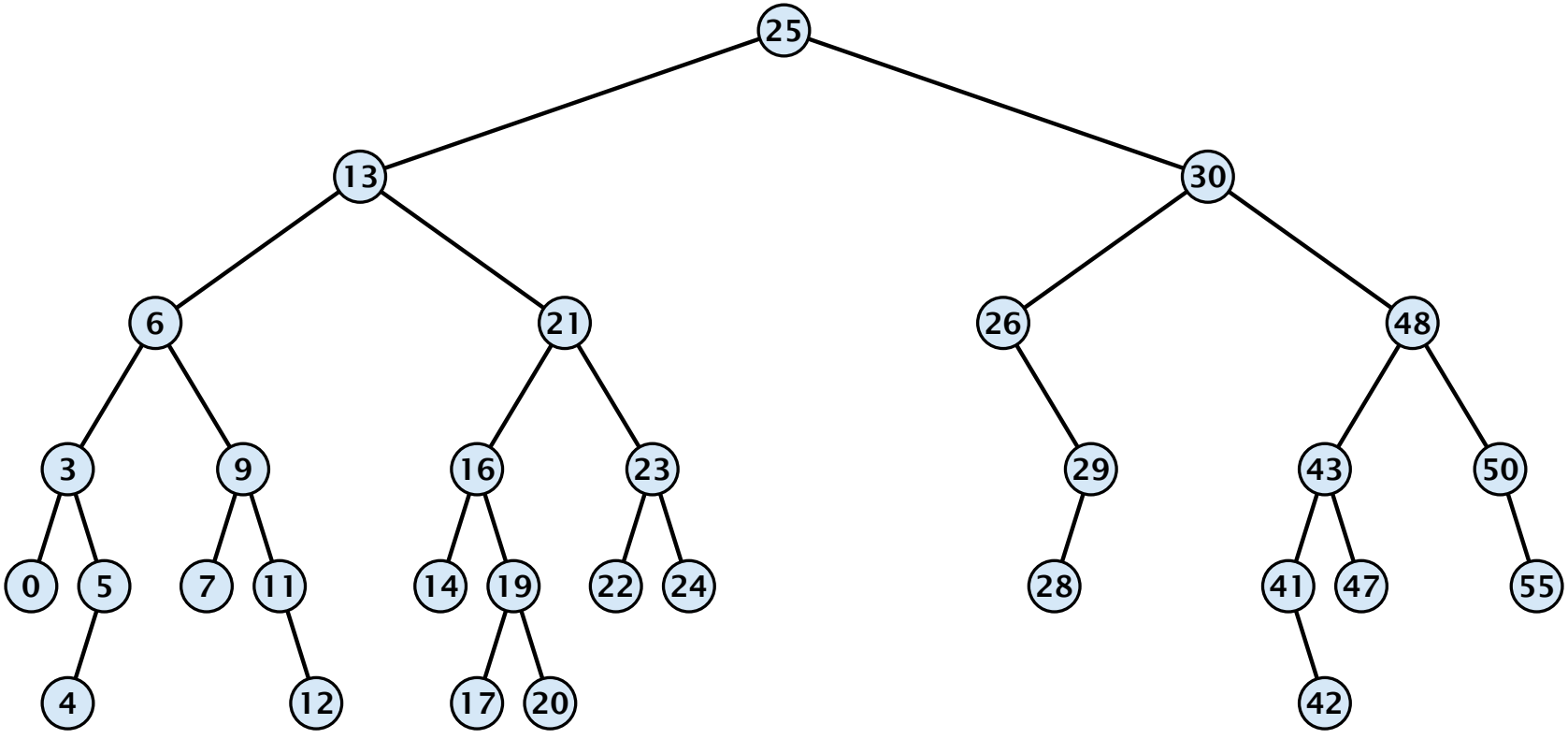

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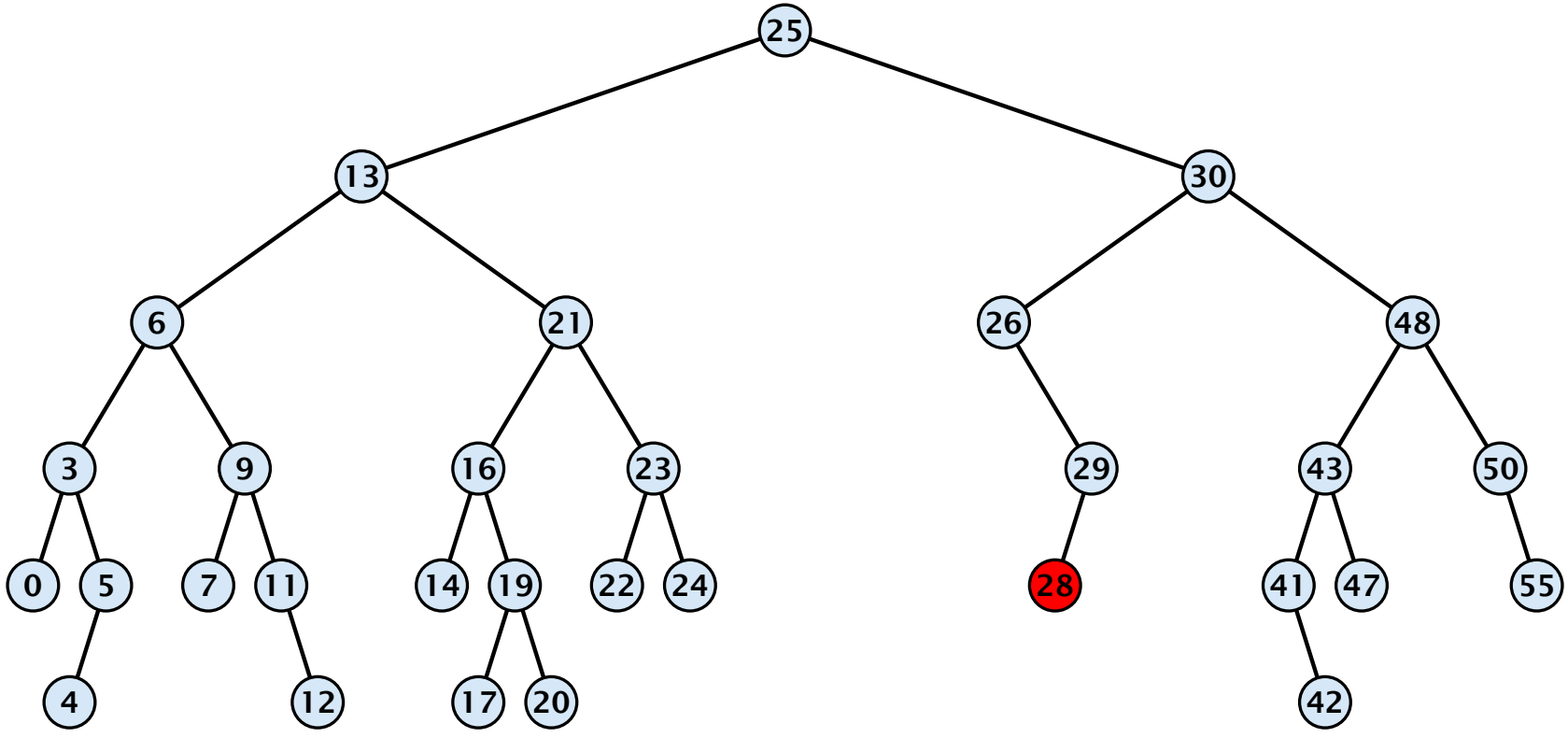

Case 1:

Element does not have any children

- Simply go to the parent and set the corresponding pointer to null.

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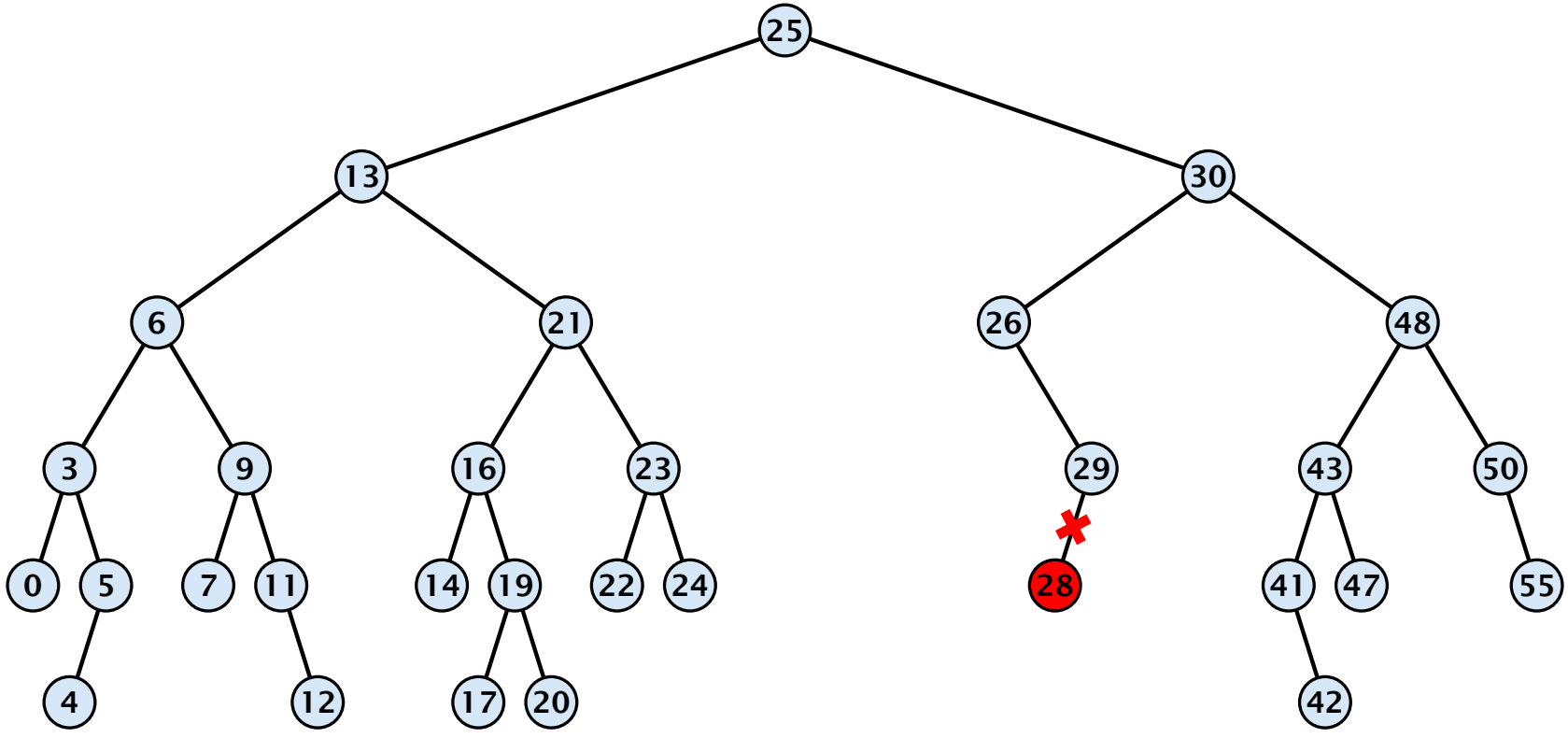

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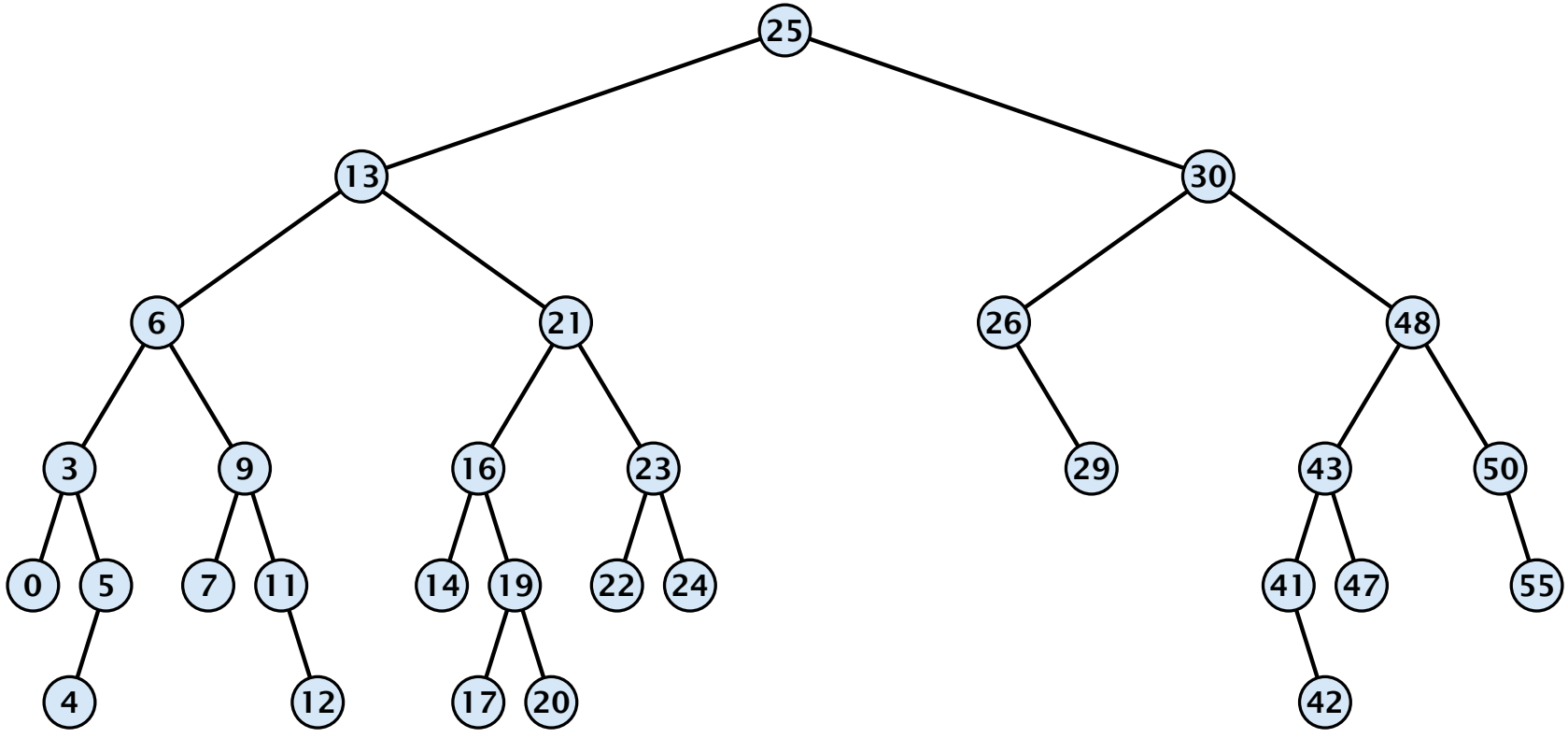

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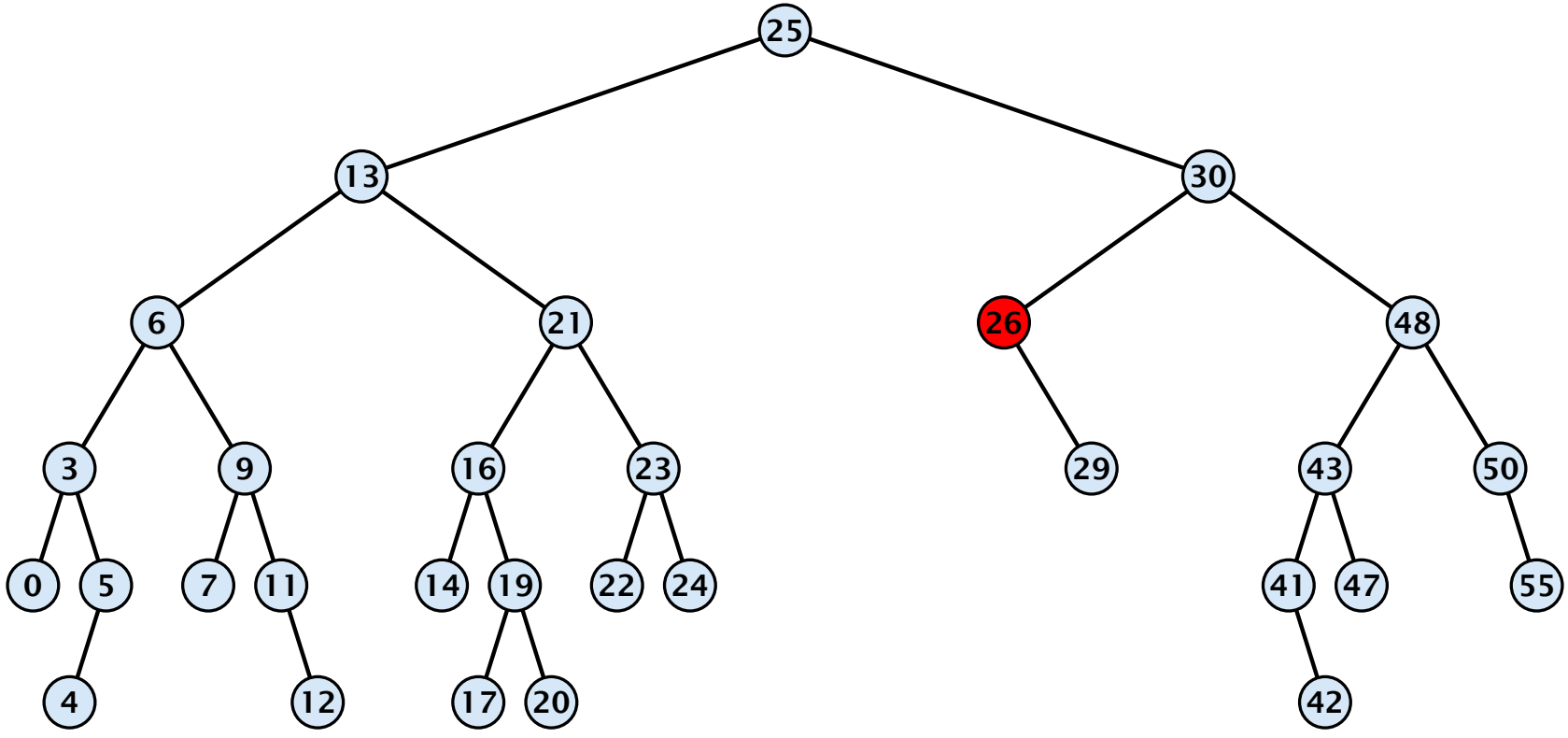
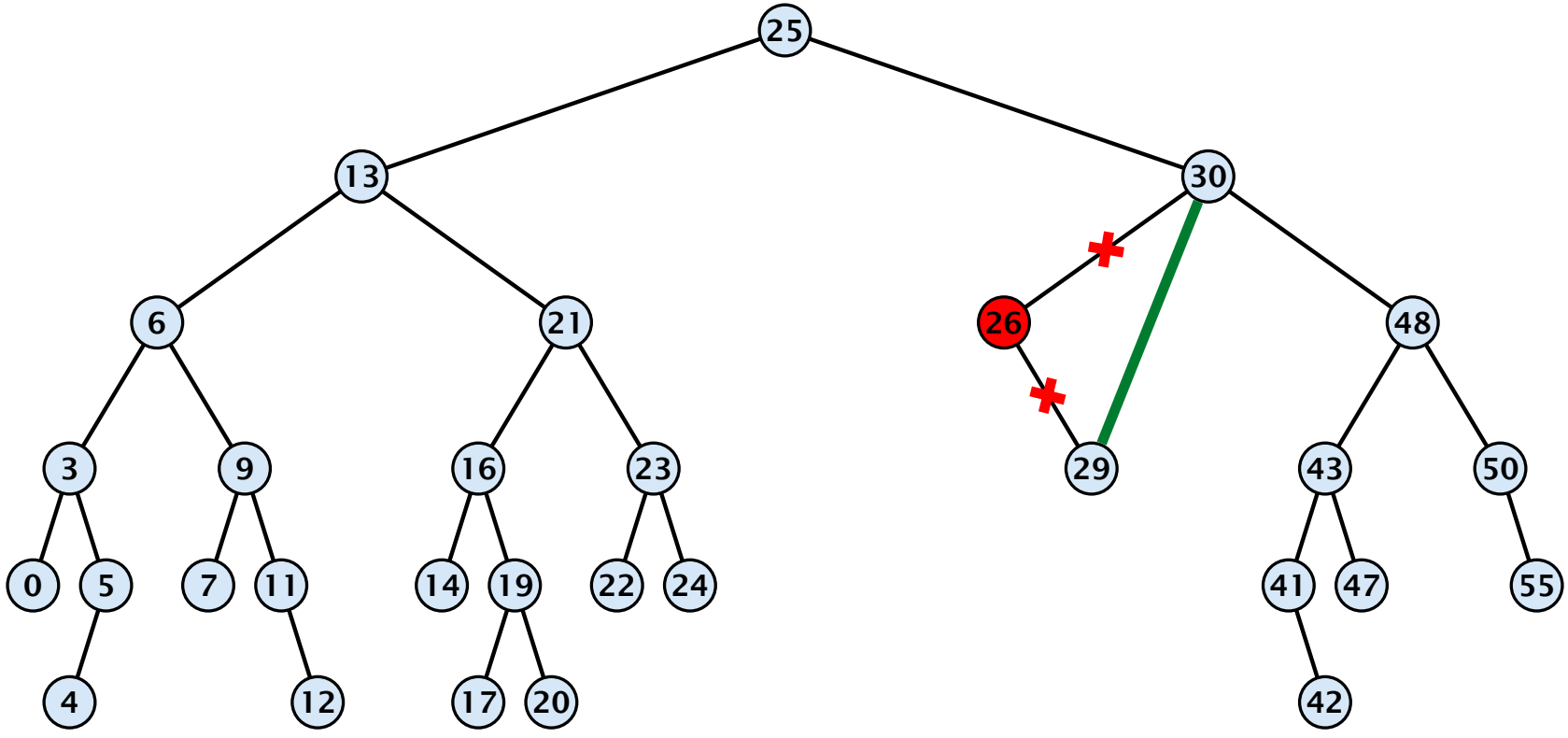


Case 2:

Element has exactly one child

- Splice the element out of the tree by connecting its parent to its successor.

Binary Search Trees: Delete



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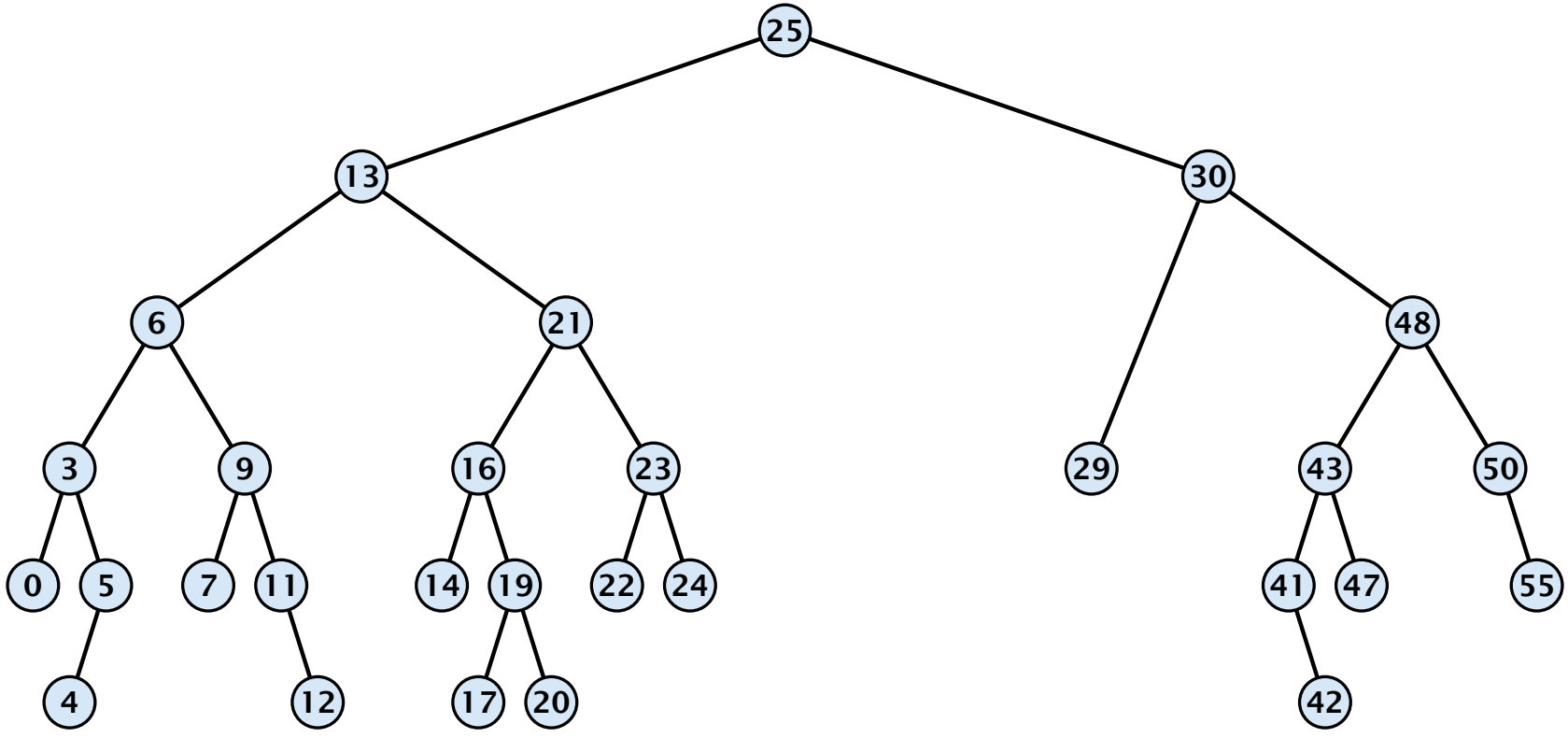

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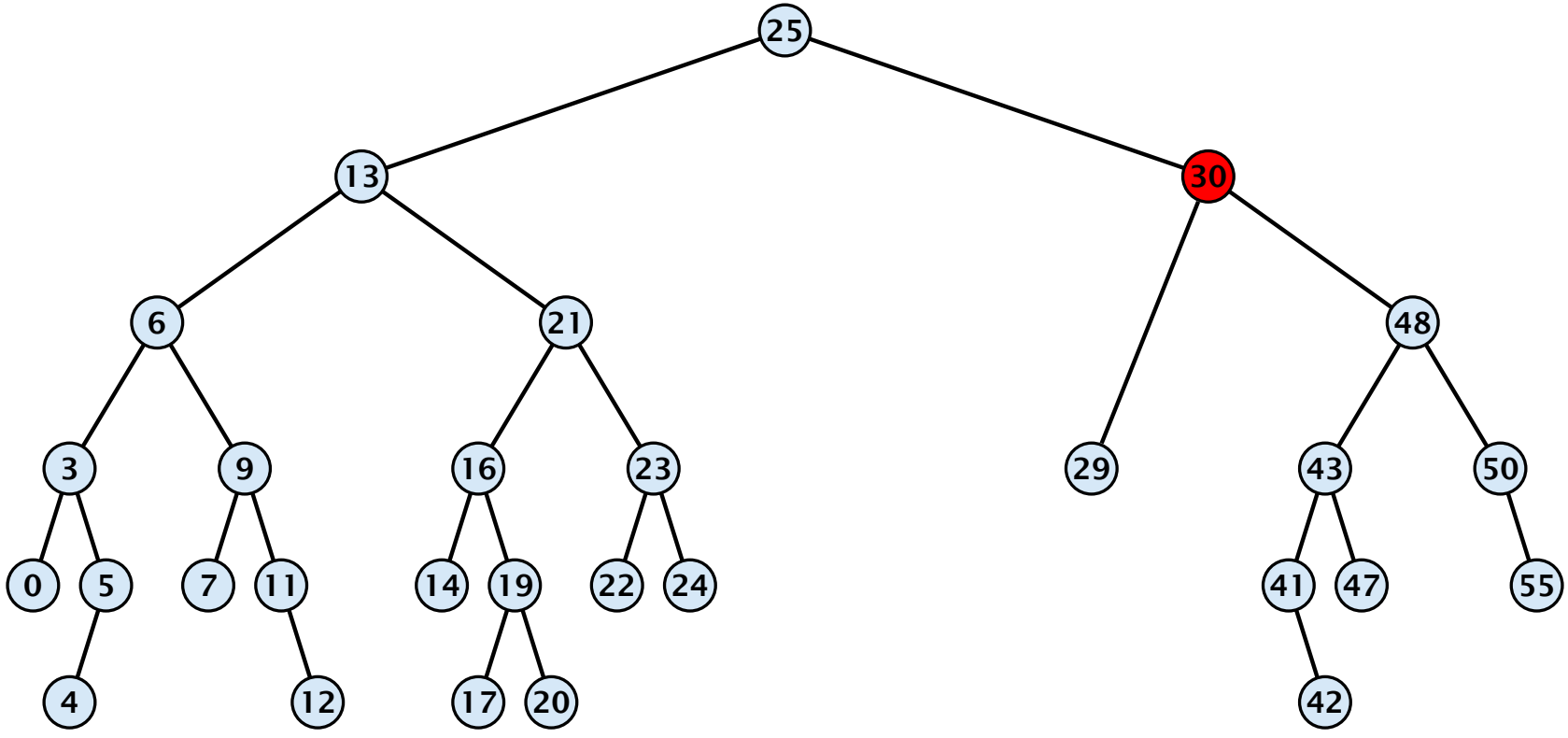

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Element has two children

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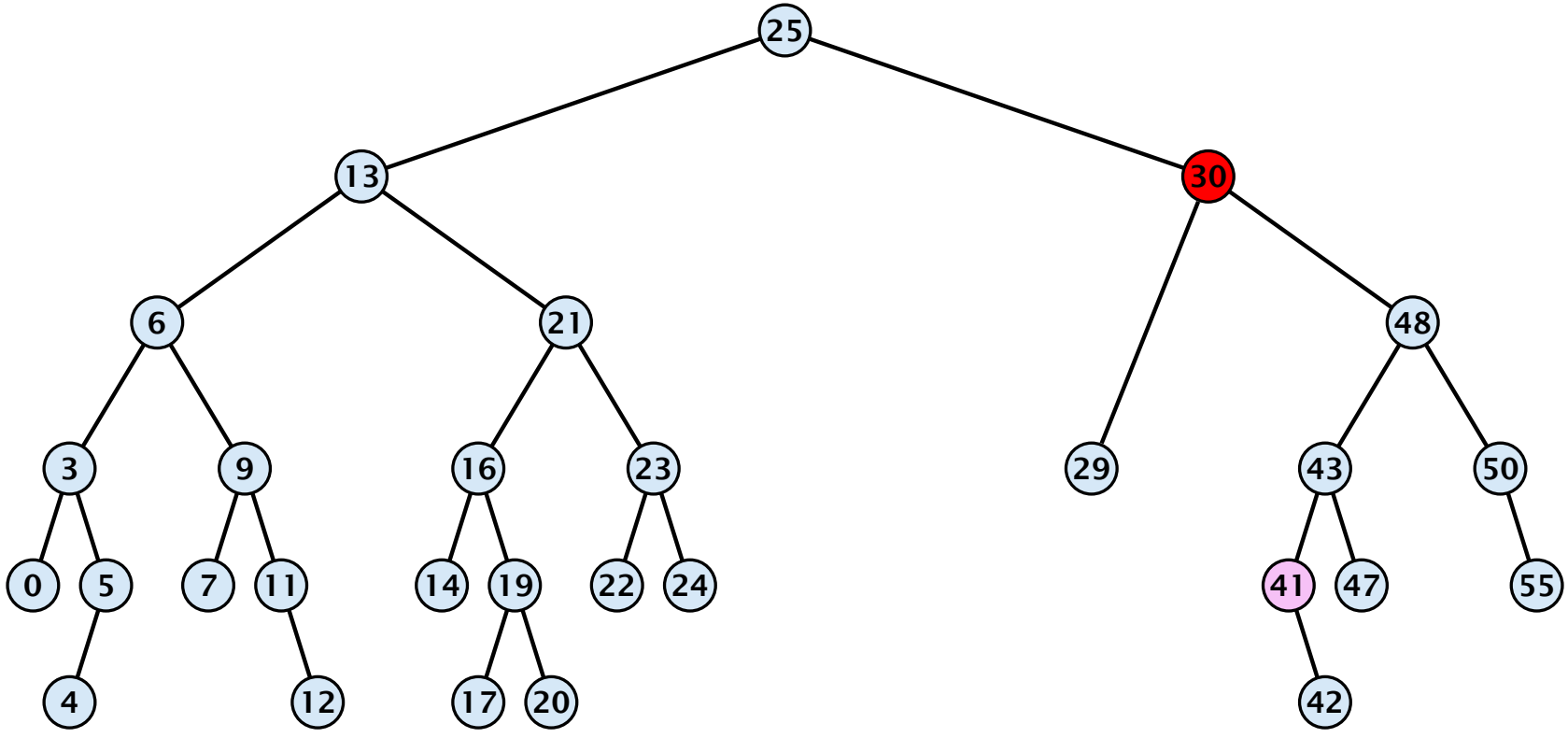

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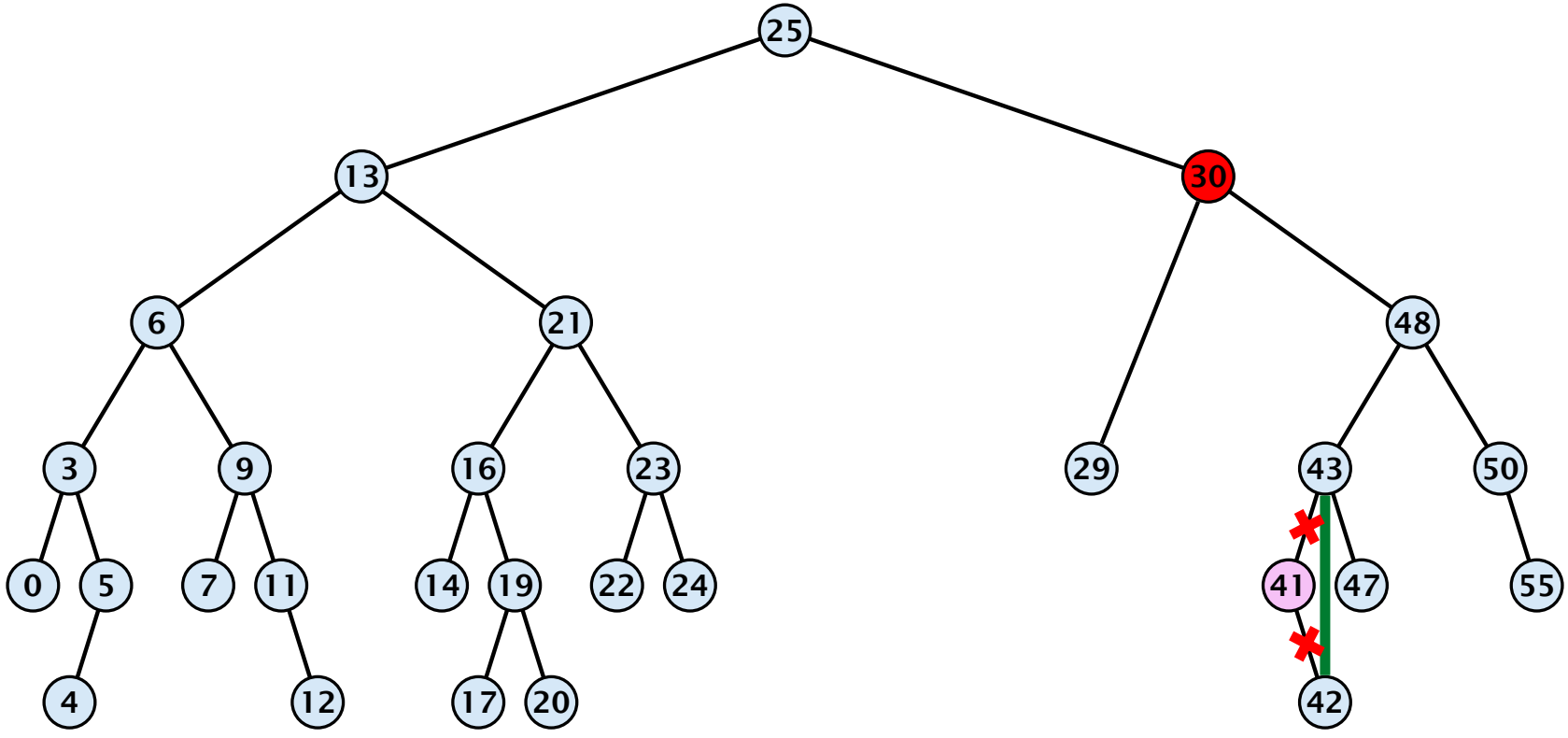

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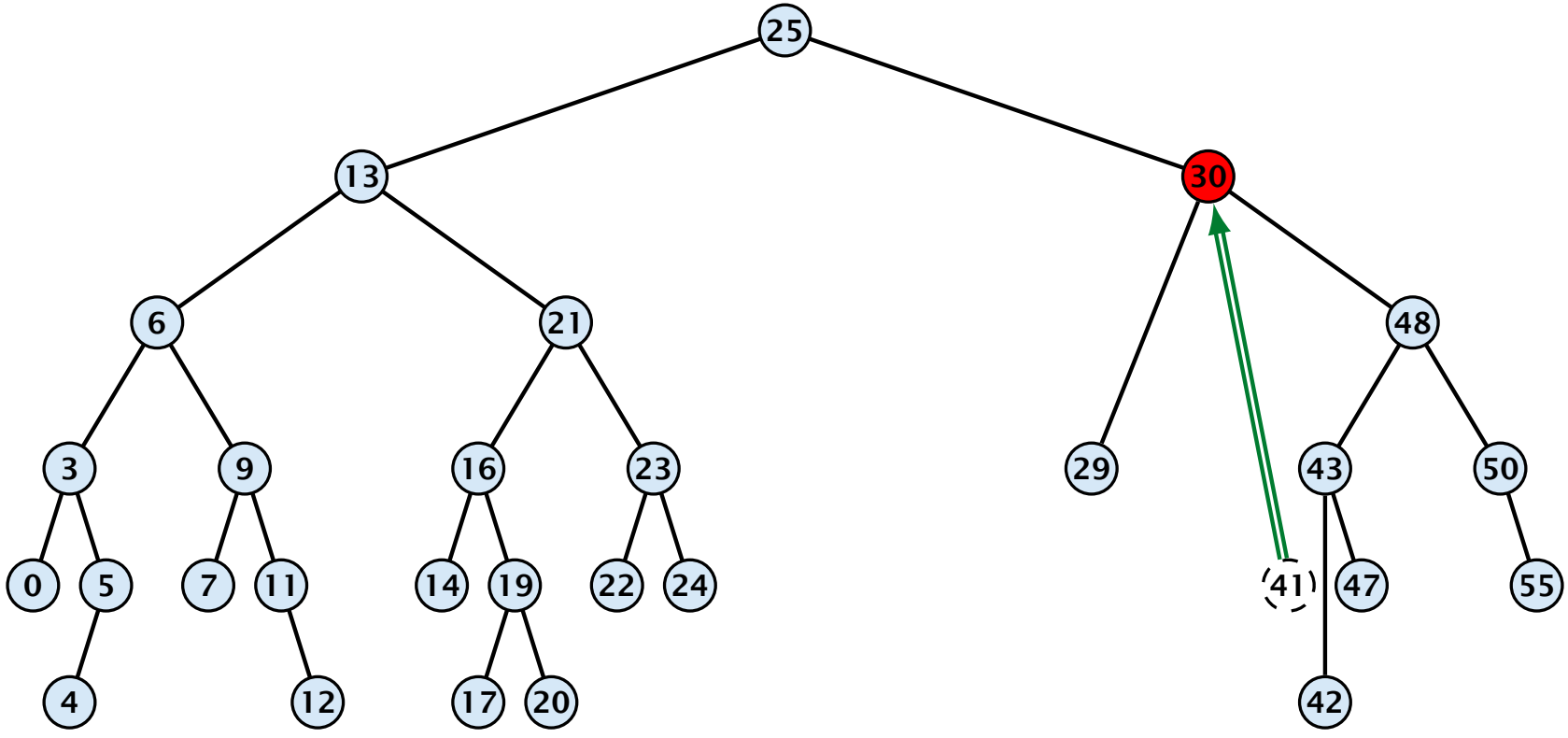

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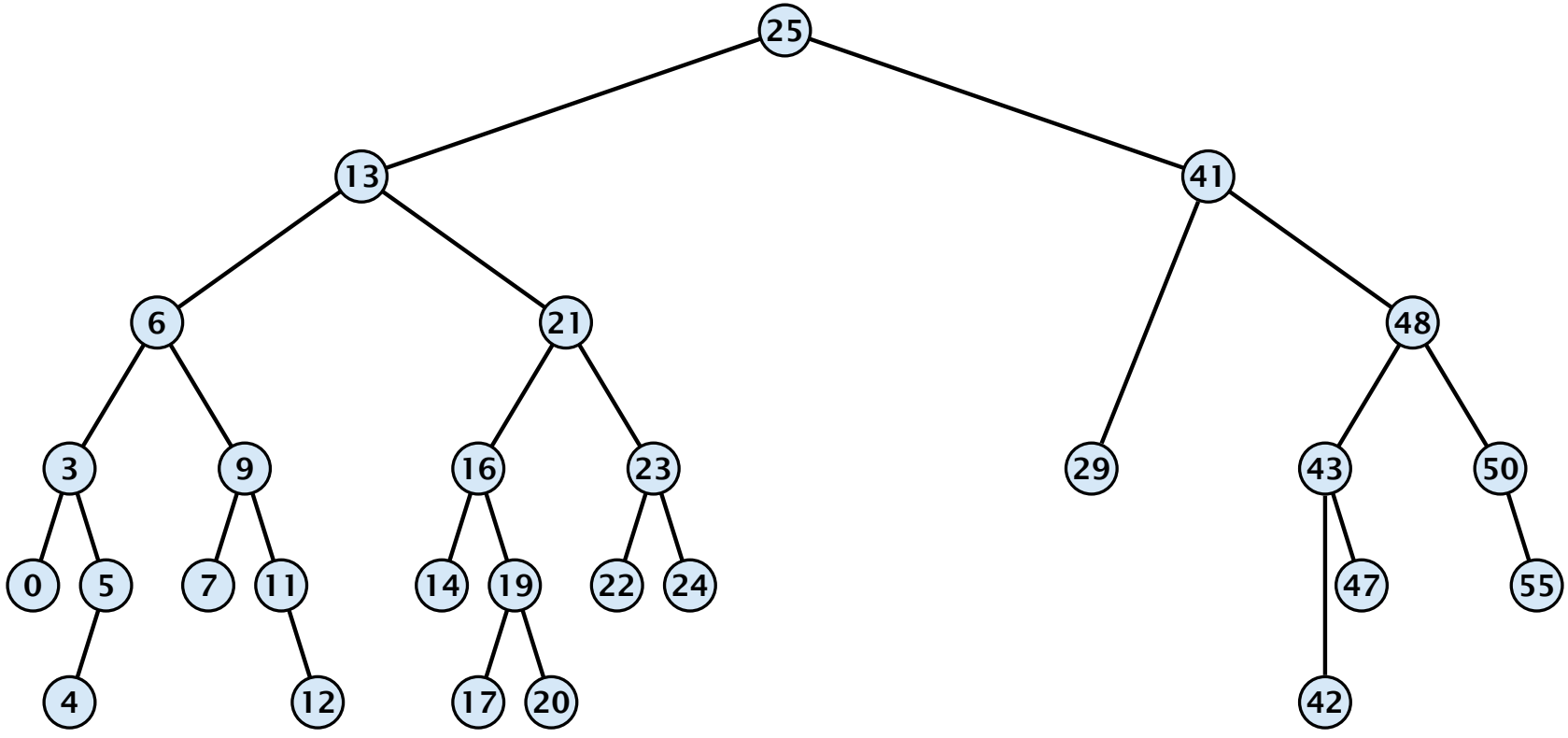

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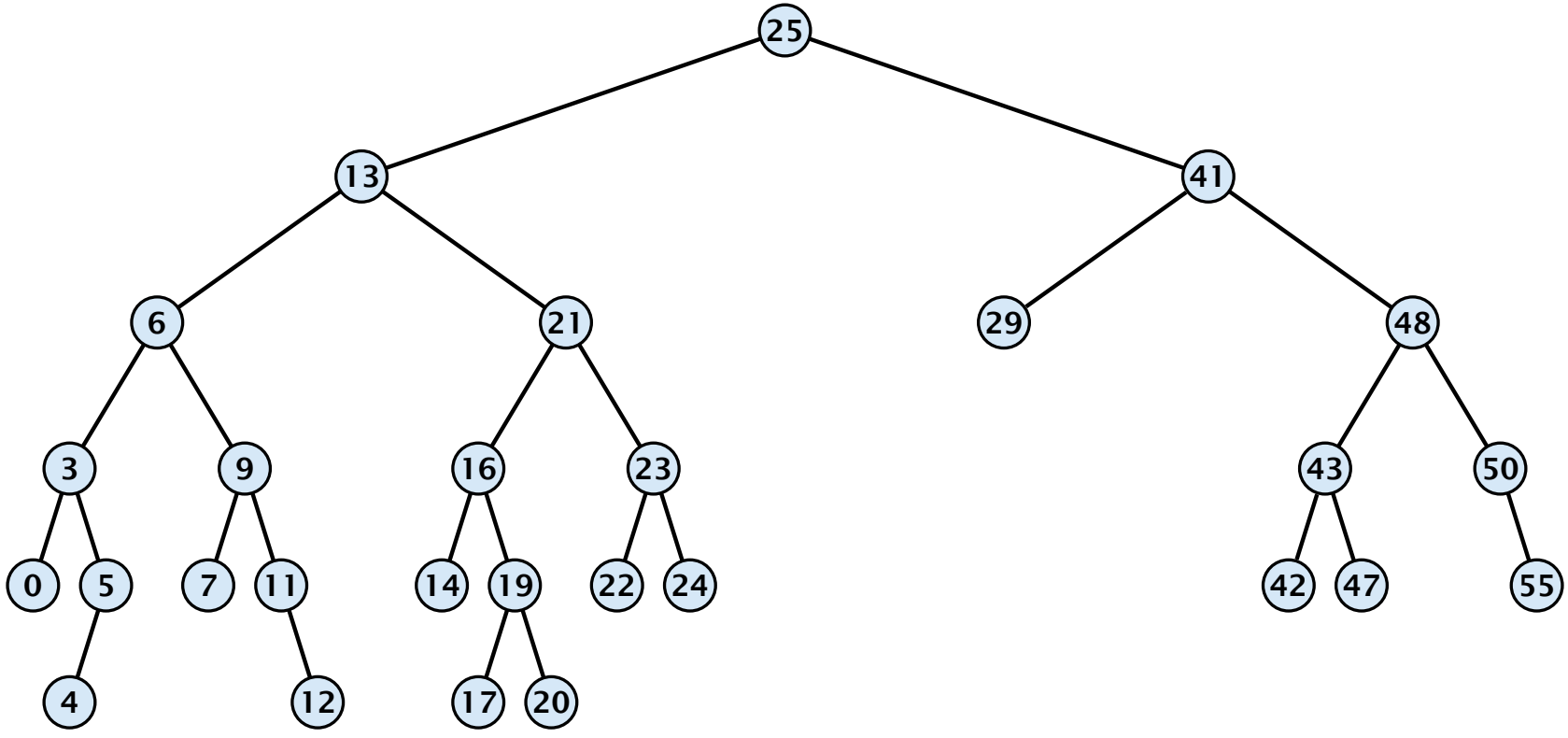

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Binary Search Trees: Delete

Algorithm 9 TreeDelete(z)

```
 1: if left[z] = null or right[z] = null
 2:       then y ← z else y ← TreeSucc(z);      select y to splice out
 3: if left[y] ≠ null
 4:       then x ← left[y] else x ← right[y]; x is child of y (or null)
 5: if x ≠ null then parent[x] ← parent[y];     parent[x] is correct
 6: if parent[y] = null then                    } fix pointer to x
 7:       root[T] ← x                           }
 8: else                                        }
 9:       if y = left[parent[y]] then           }
10:            left[parent[y]] ← x              }
11:       else                                  }
12:            right[parent[y]] ← x             }
13: if y ≠ z then copy y-data to z
```

Balanced Binary Search Trees

All operations on a binary search tree can be performed in time $\mathcal{O}(h)$, where h denotes the height of the tree.

However the height of the tree may become as large as $\Theta(n)$.

Balanced Binary Search Trees
With each insert- and delete-operation perform local adjustments to guarantee a height of $\mathcal{O}(\log n)$.

AVL-trees, Red-black trees, Scapegoat trees, 2-3 trees, B-trees, AA trees, Treaps

similar: SPLAY trees.

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7.2 Red Black Trees

Definition 11

A red black tree is a balanced binary search tree in which each internal node has two children. Each internal node has a color, such that

1. The root is black.
2. All leaf nodes are black.
3. For each node, all paths to descendant leaves contain the same number of black nodes.
4. If a node is red then both its children are black.

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A red black tree is a balanced binary search tree in which each internal node has two children. Each internal node has a color, such that

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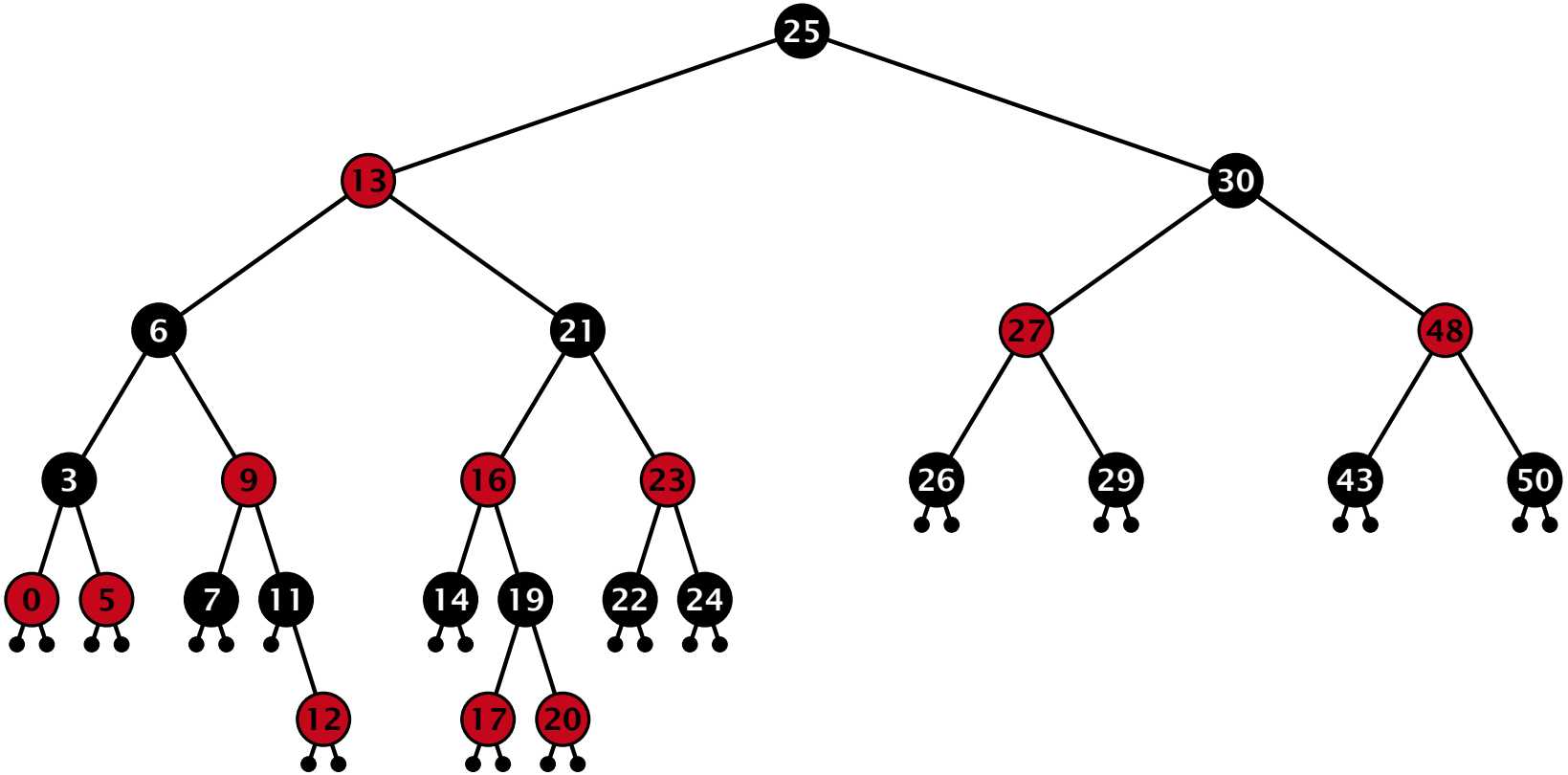
Red Black Trees: Example



7.2 Red Black Trees

Lemma 12

A red-black tree with n internal nodes has height at most $\mathcal{O}(\log n)$.

Definition 13

The black height $\mathrm{bh}(v)$ of a node v in a red black tree is the number of black nodes on a path from v to a leaf vertex (not counting v).

We first show:

Lemma 14

A sub-tree of black height $\mathrm{bh}(v)$ in a red black tree contains at least $2^{\mathrm{bh}(v)} - 1$ internal vertices.

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Proof of Lemma 14.

Induction on the height of v.

base case ($\text{height}(v) = 0$)

- If $\text{height}(v) = 0$ (maximum distance btw. v and a node in the sub-tree rooted at v) is 0 then v is a leaf.
- The black height of v is 0.
- The sub-tree rooted at v contains 0 internal vertices.

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- The black height of v is 1.
- The sub-tree rooted at v contains $2^1 - 1 = 0$ inner vertices.

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7.2 Red Black Trees

Proof (cont.)

induction step

- Suppose v is a node with $bh(v) = k > 1$.
- v has two children with strictly smaller height.
- These children (w_1 and w_2) either have black height k or $k-1$ (depending on their colors).
- By induction hypothesis both sub-trees contain at least $2^{bh(v)-1}-1$ internal vertices.
- Then T_v contains at least $2 \cdot (2^{bh(v)-1}-1) + 1 = 2^{bh(v)} - 1$ internal vertices.

Proof (cont.)

induction step

- Supose v is a node with $\text{height}(v) > 0$.
- v has two children with strictly smaller height.
- These children (c_1, c_2) either have $\text{bh}(c_i) = \text{bh}(v)$ or $\text{bh}(c_i) = \text{bh}(v) - 1$.
- By induction hypothesis both sub-trees contain at least $2^{\text{bh}(v)-1} - 1$ internal vertices.
- Then T_v contains at least $2(2^{\text{bh}(v)-1} - 1) + 1 \geq 2^{\text{bh}(v)} - 1$ vertices.

□

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Proof of Lemma 12.

Let h denote the height of the red-black tree, and let P denote a path from the root to the furthest leaf.

At least half of the node on P must be black, since a red node must be followed by a black node.

Hence, the black height of the root is at least $h/2$.

The tree contains at least $2^{h/2} - 1$ internal vertices. Hence, $2^{h/2} - 1 \leq n$.

Hence, $h \leq 2\log(n+1) = \mathcal{O}(\log n)$. □

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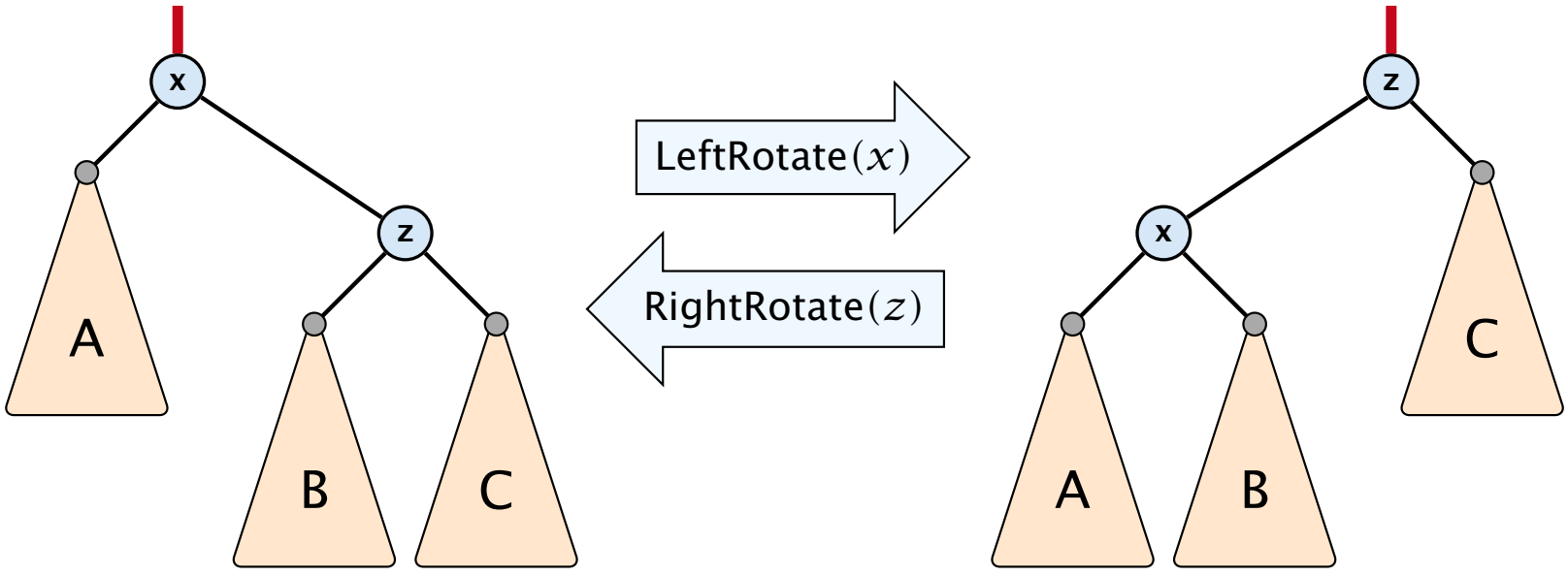
7.2 Red Black Trees

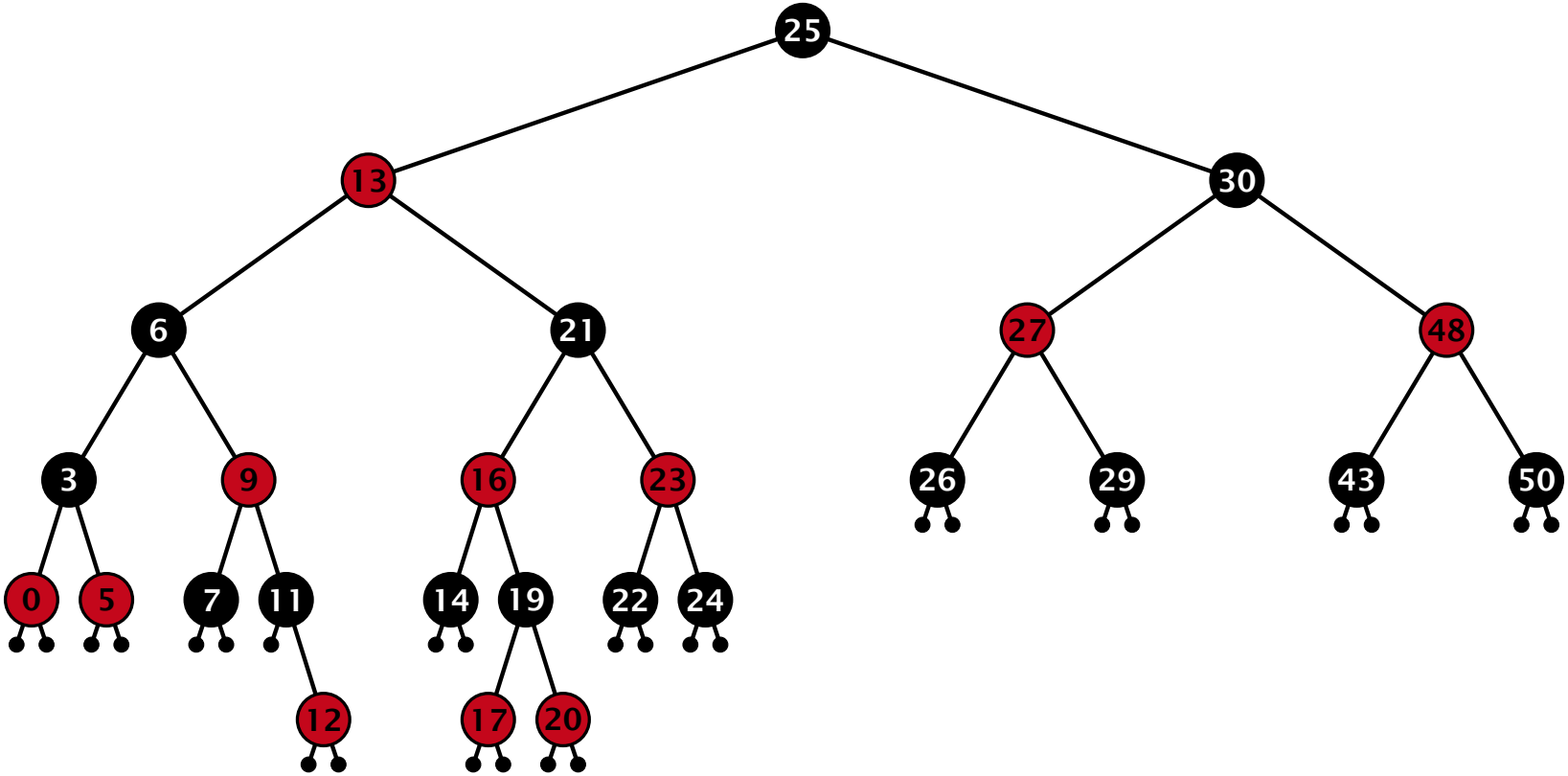
We need to adapt the insert and delete operations so that the red black properties are maintained.

Rotations

The properties will be maintained through rotations:



Red Black Trees: Insert

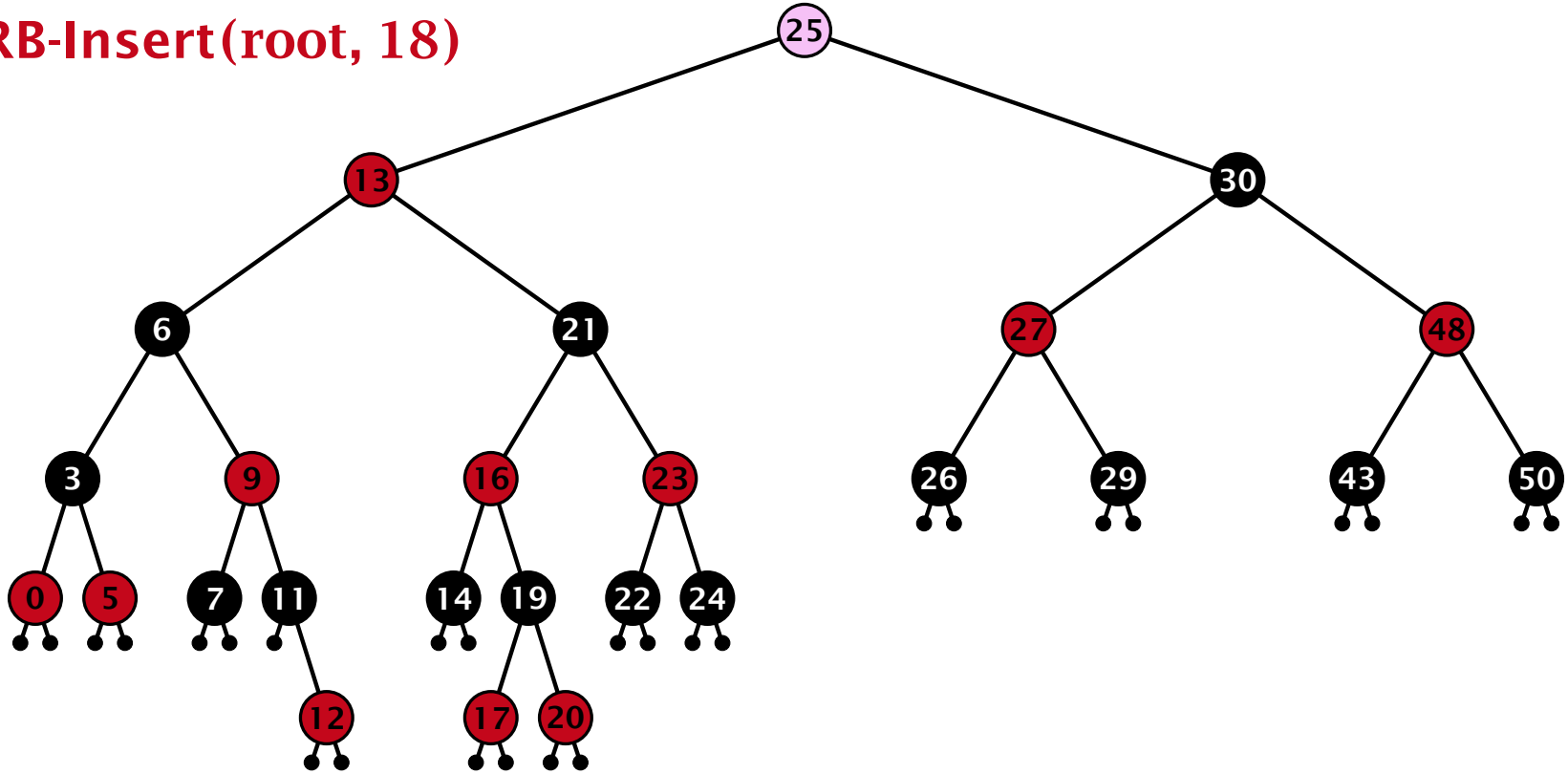


Insert:

- first make a normal insert into a binary search tree
- then fix red-black properties

Red Black Trees: Insert

RB-Insert(root, 18)



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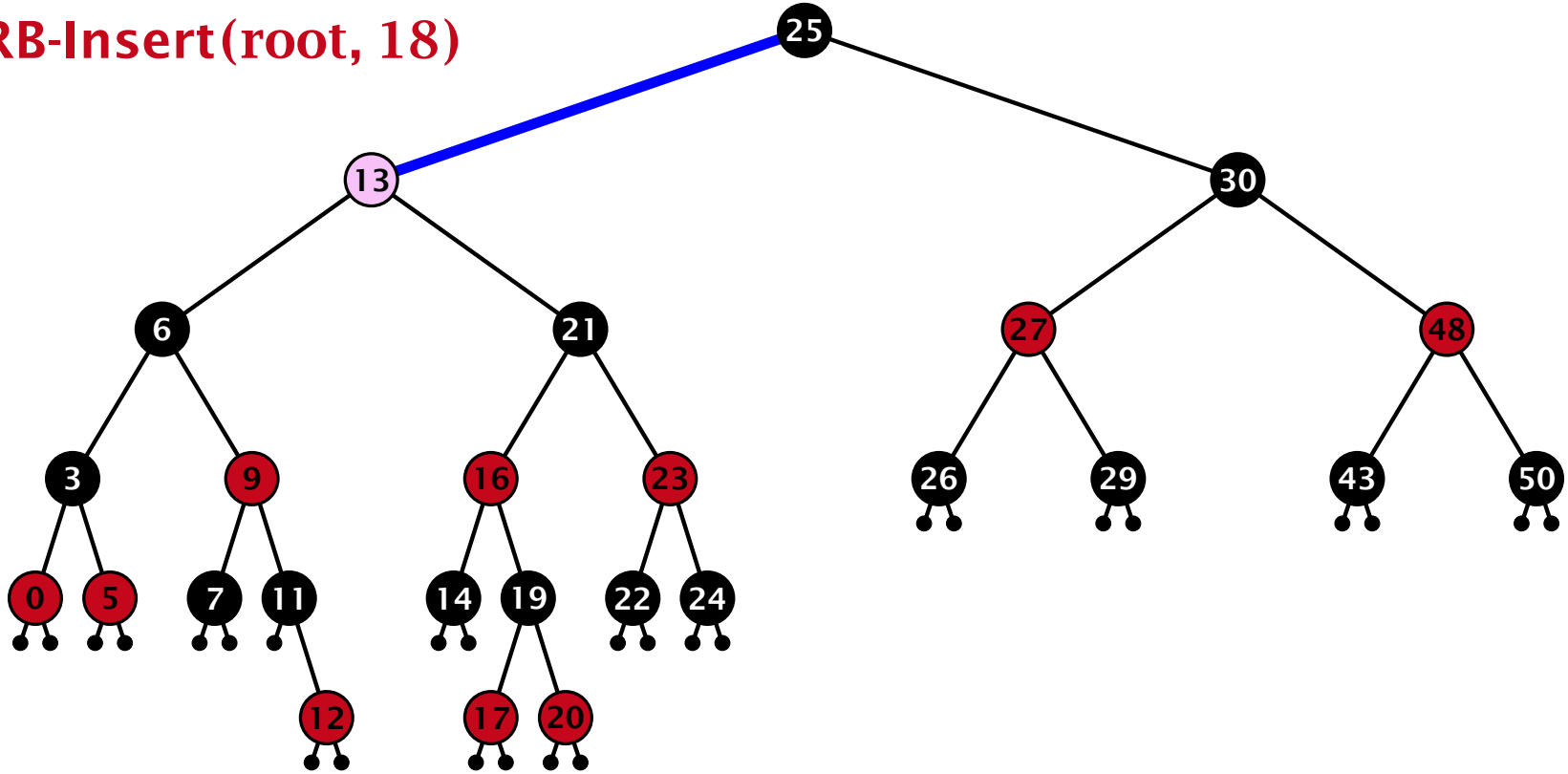

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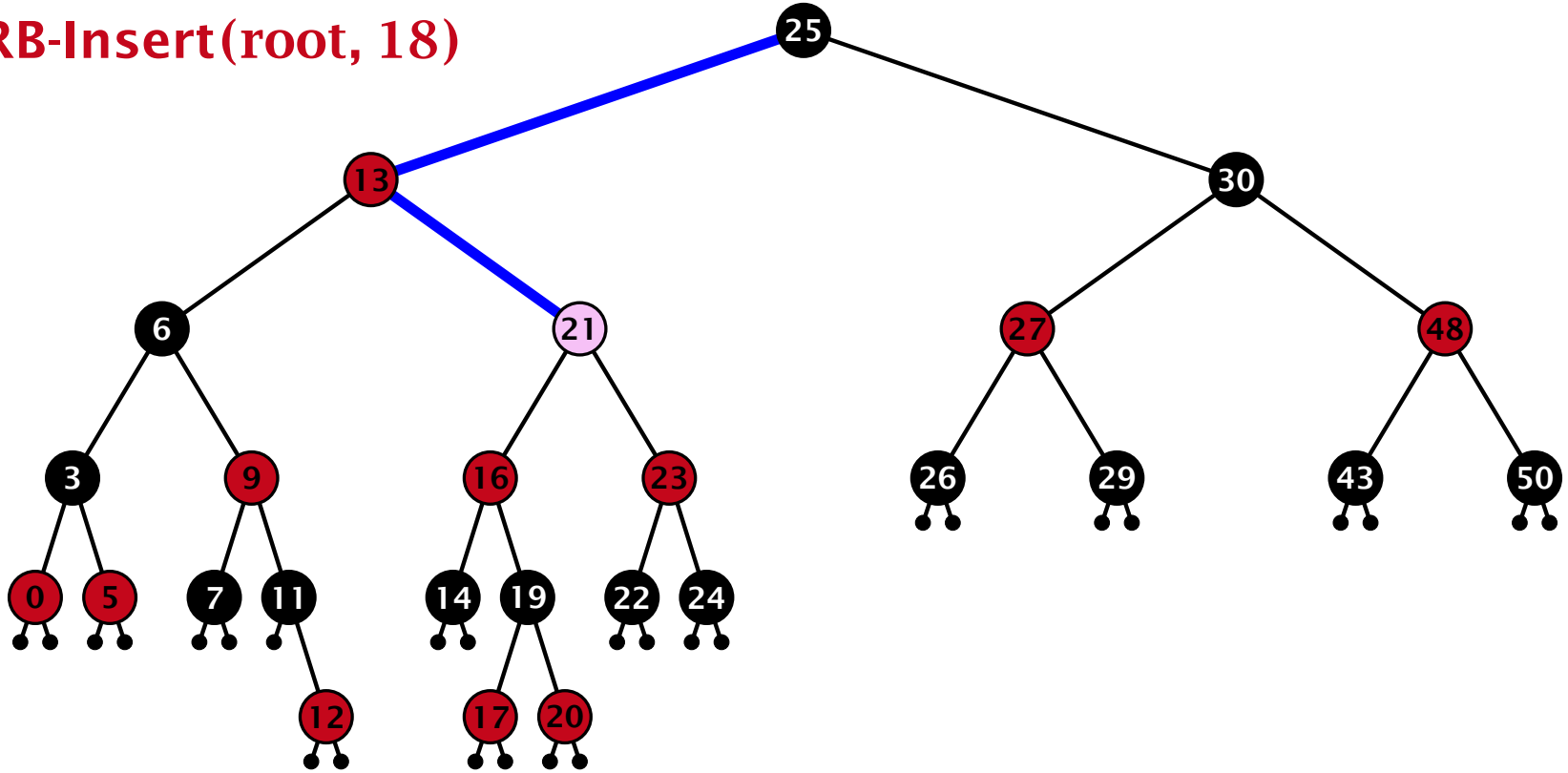

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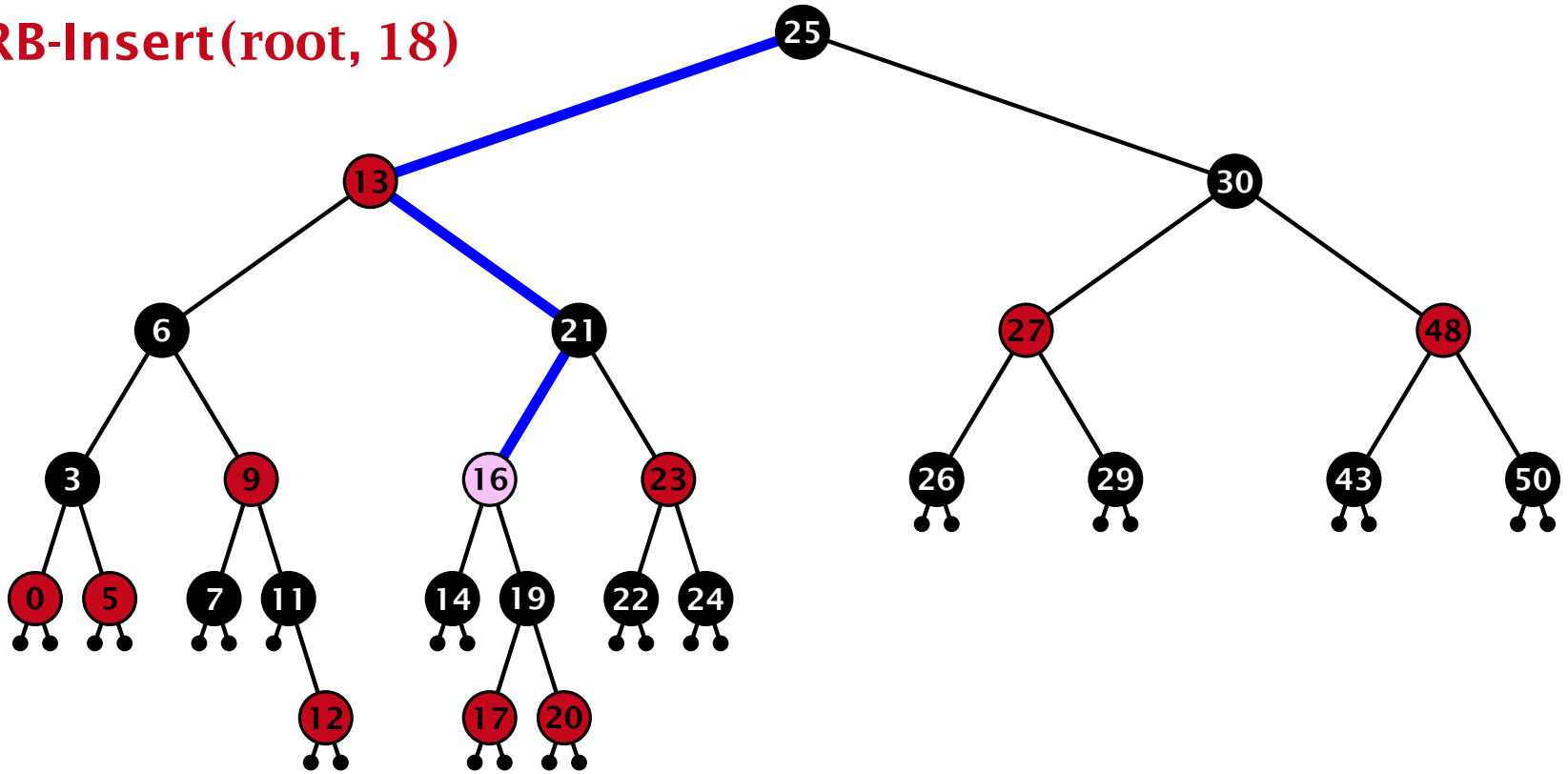

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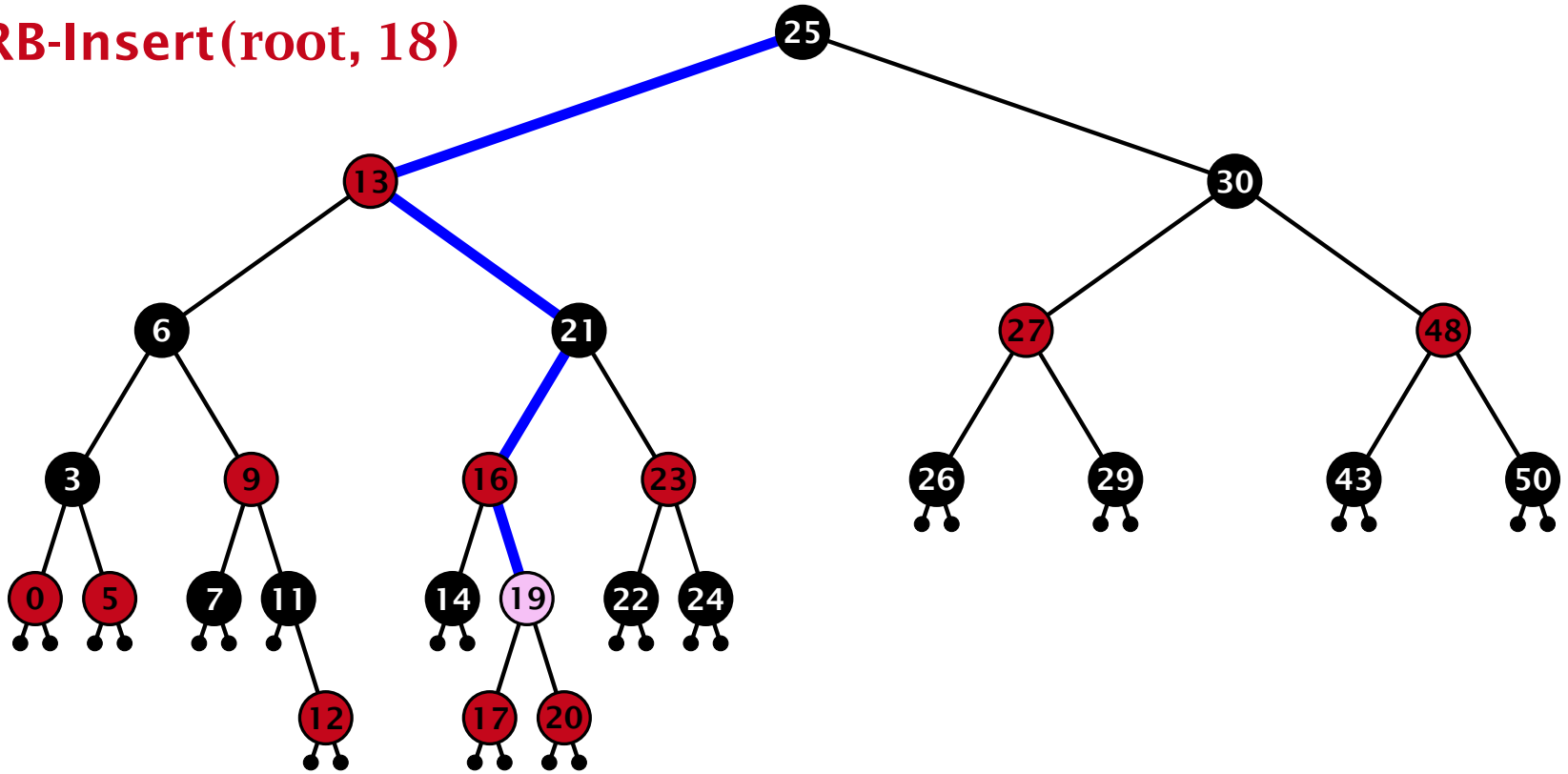

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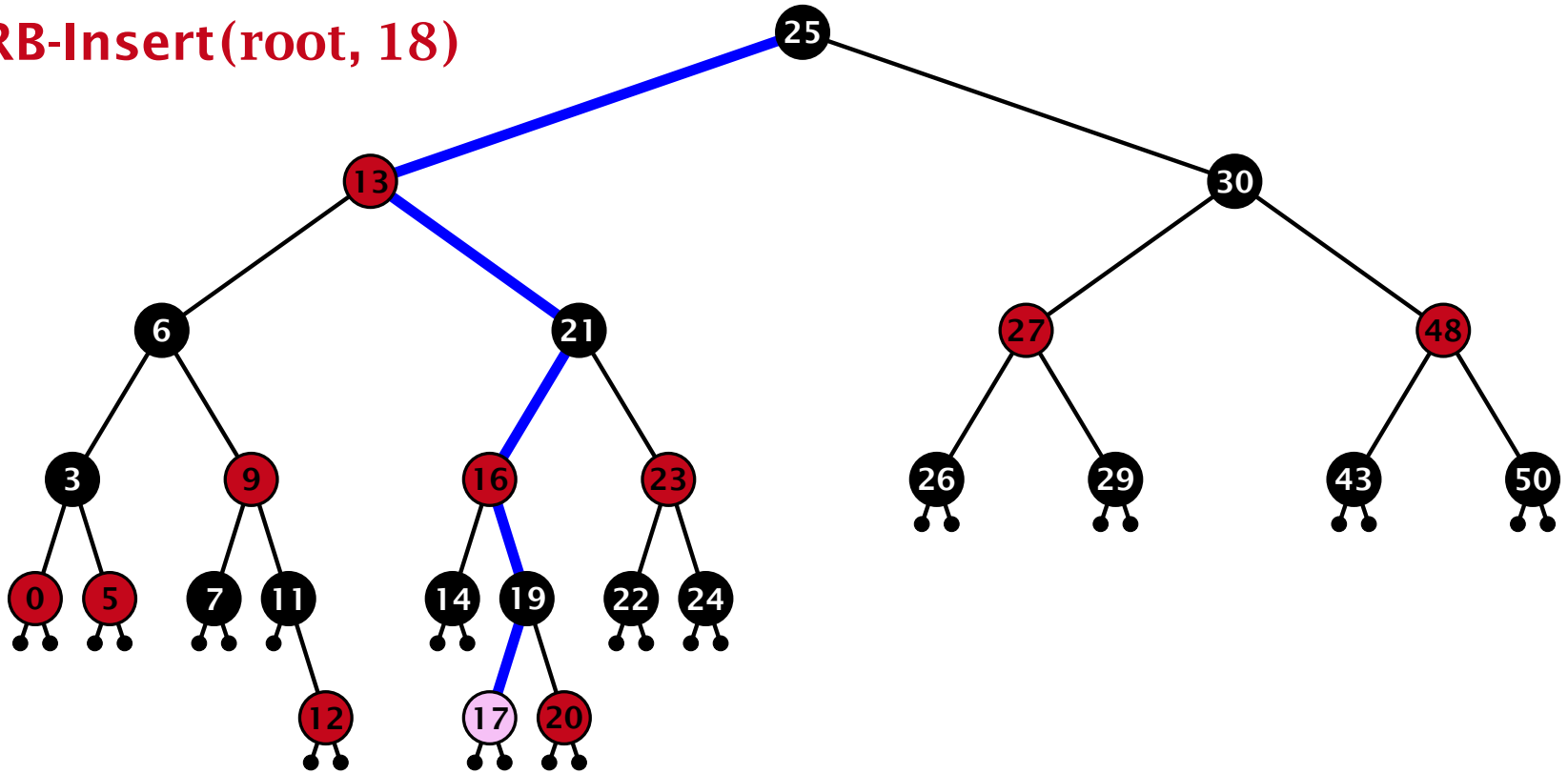

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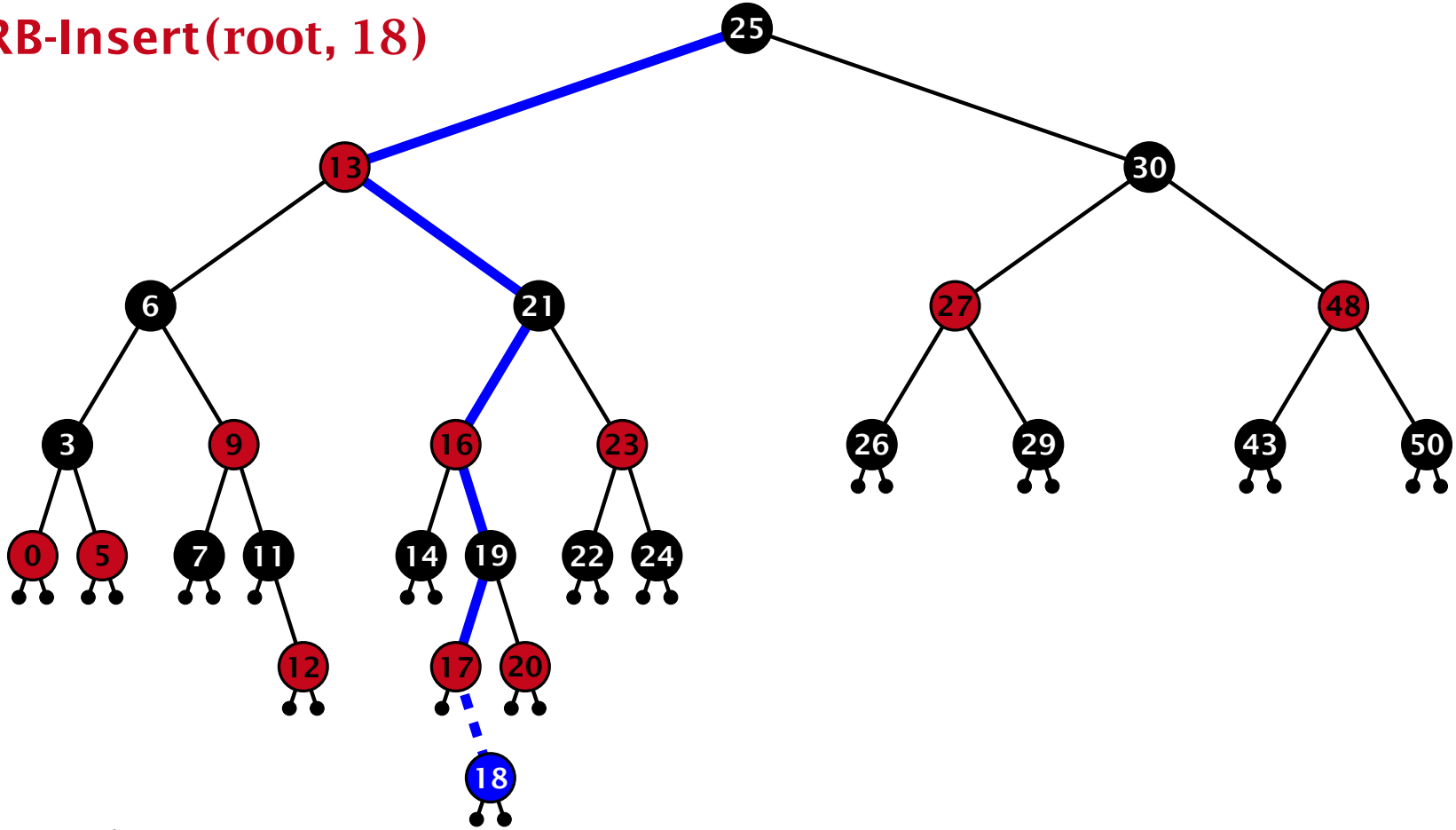

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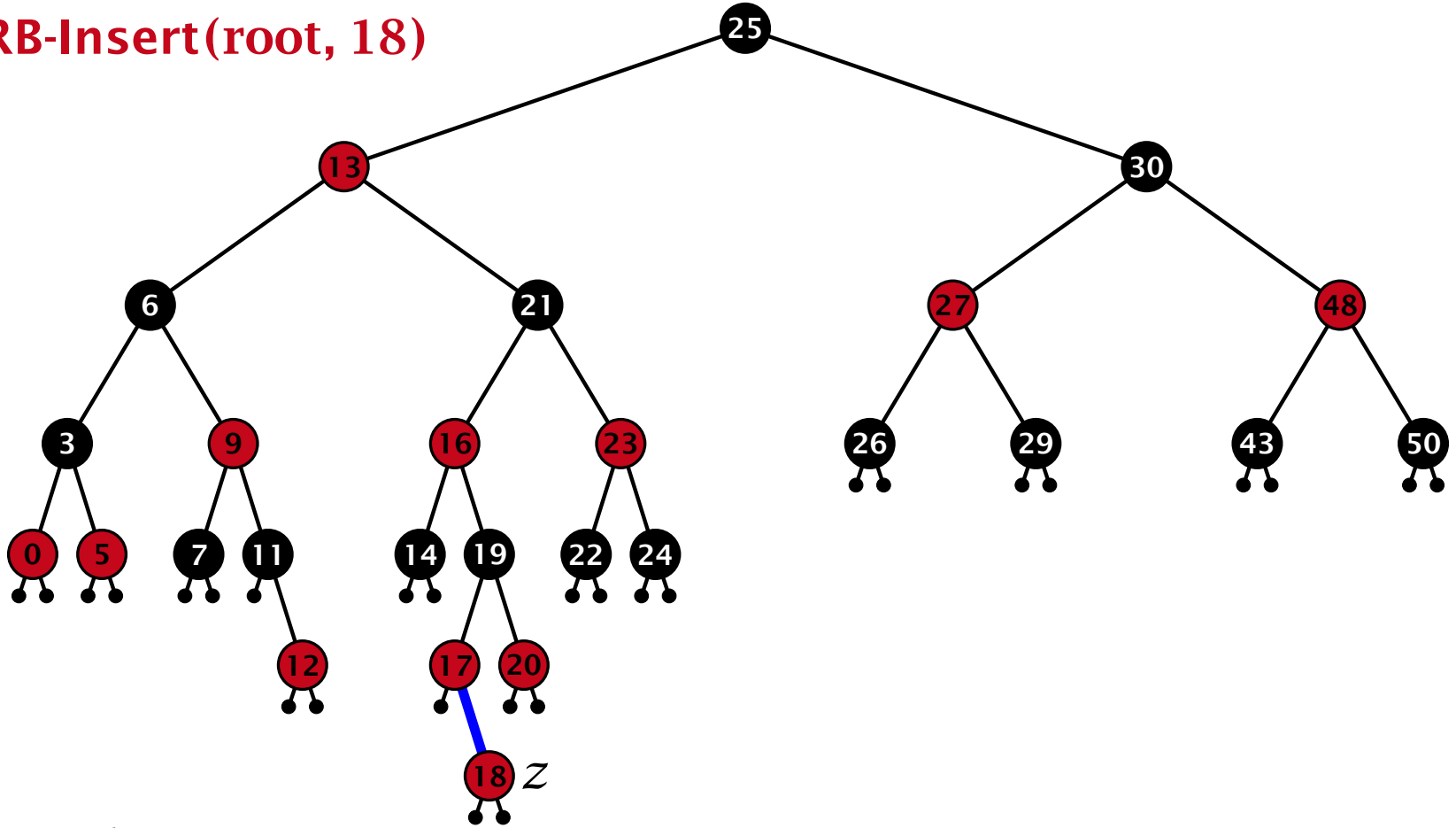

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Red Black Trees: Insert

Invariant of the fix-up algorithm:

- z is a red node
- the black-height property is fulfilled at every node
- the only violation of red-black properties occurs at z and $\text{parent}[z]$
  - either both of them are red
    (most important case)
  - or the parent does not exist
    (violation since root must be black)

If z has a parent but no grand-parent we could simply color the parent/root black; however this case never happens.

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Red Black Trees: Insert

Algorithm 10 InsertFix(z)

```
 1: while parent[z] ≠ null and col[parent[z]] = red do
 2:     if parent[z] = left[gp[z]] then
 3:         uncle ← right[grandparent[z]]
 4:         if col[uncle] = red then
 5:             col[p[z]] ← black; col[u] ← black;
 6:             col[gp[z]] ← red; z ← grandparent[z];
 7:         else
 8:             if z = right[parent[z]] then
 9:                 z ← p[z]; LeftRotate(z);
10:             col[p[z]] ← black; col[gp[z]] ← red;
11:             RightRotate(gp[z]);
12:     else same as then-clause but right and left exchanged
13: col(root[T]) ← black;
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Red Black Trees: Insert

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 2:     if parent[z] = left[gp[z]] then   z in left subtree of grandparent
 3:         uncle ← right[grandparent[z]]
 4:         if col[uncle] = red then
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 6:             col[gp[z]] ← red; z ← grandparent[z];
 7:         else                                   Case 2: uncle black
 8:             if z = right[parent[z]] then
 9:                 z ← p[z]; LeftRotate(z);
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 8:             if z = right[parent[z]] then
 9:                 z ← p[z]; LeftRotate(z);
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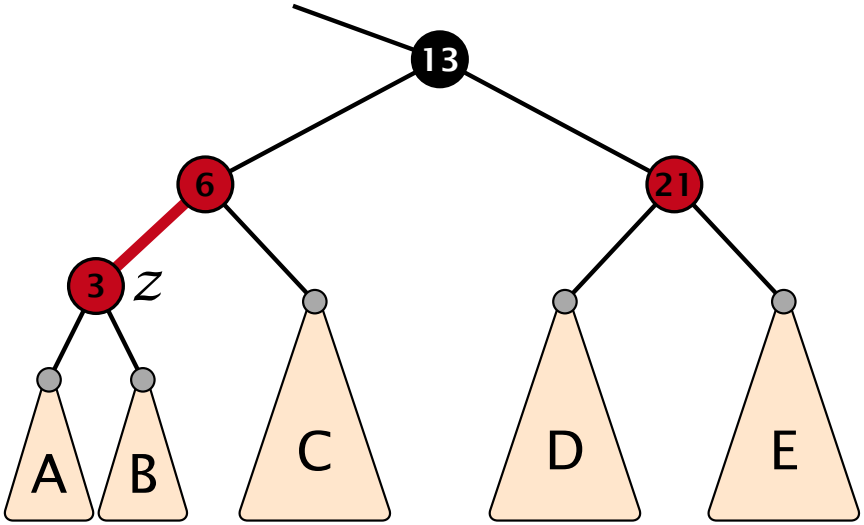
11:             RightRotate(gp[z]);
12:     else same as then-clause but right and left exchanged
13: col(root[T]) ← black;
```

Case 1: Red Uncle



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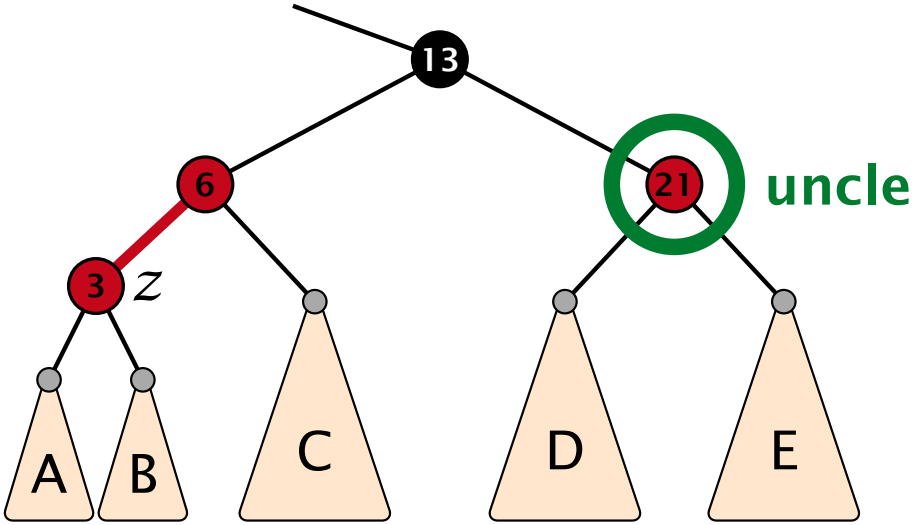

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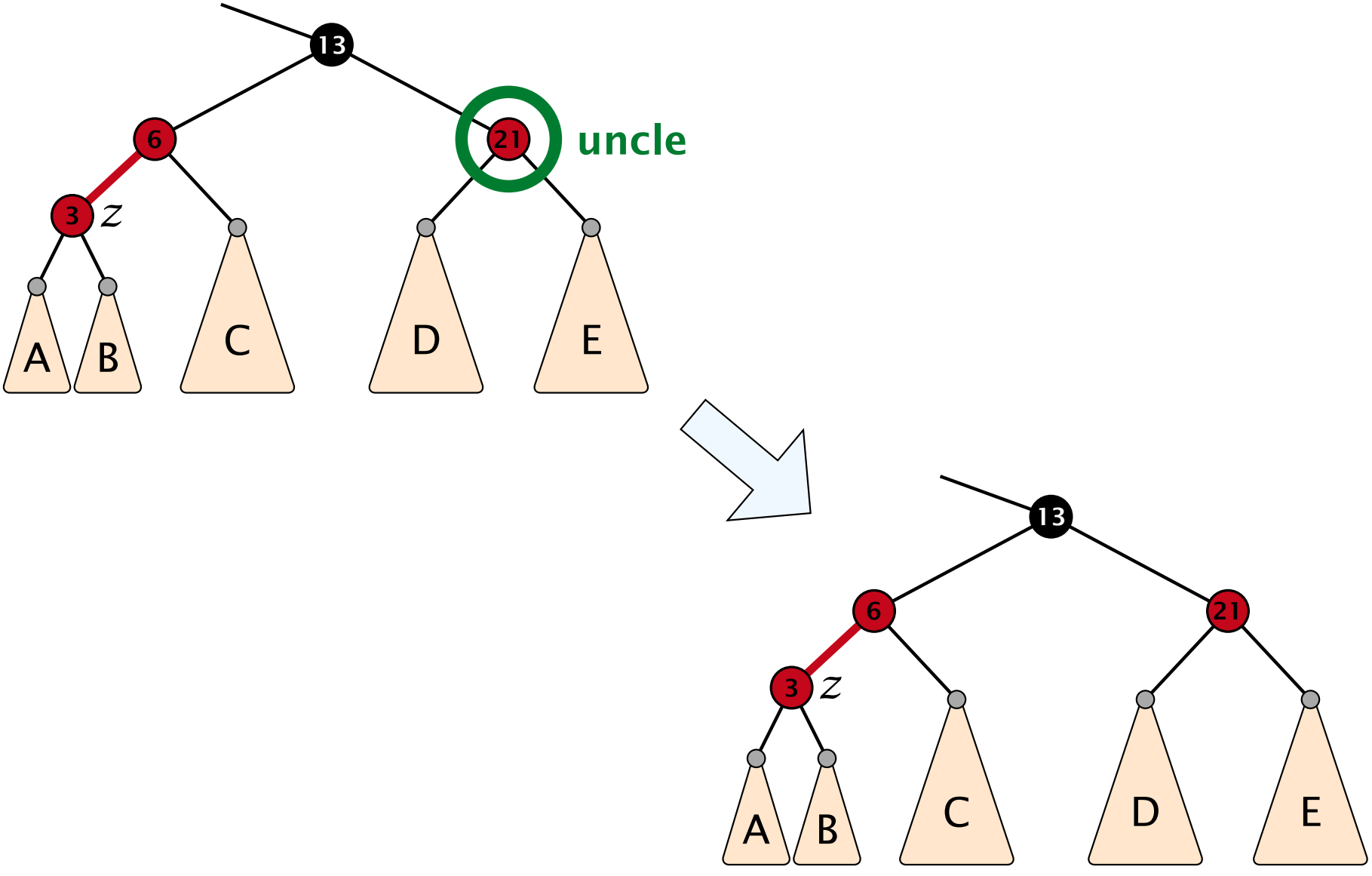

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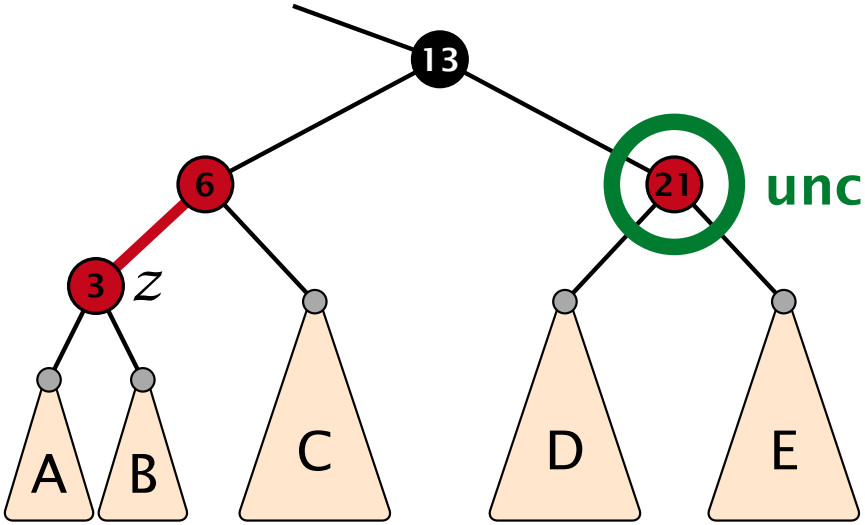
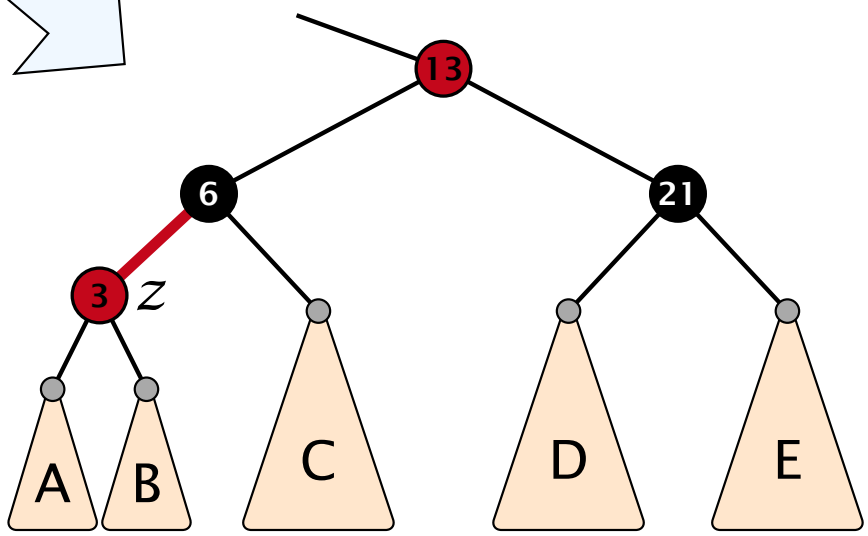


1. recolour



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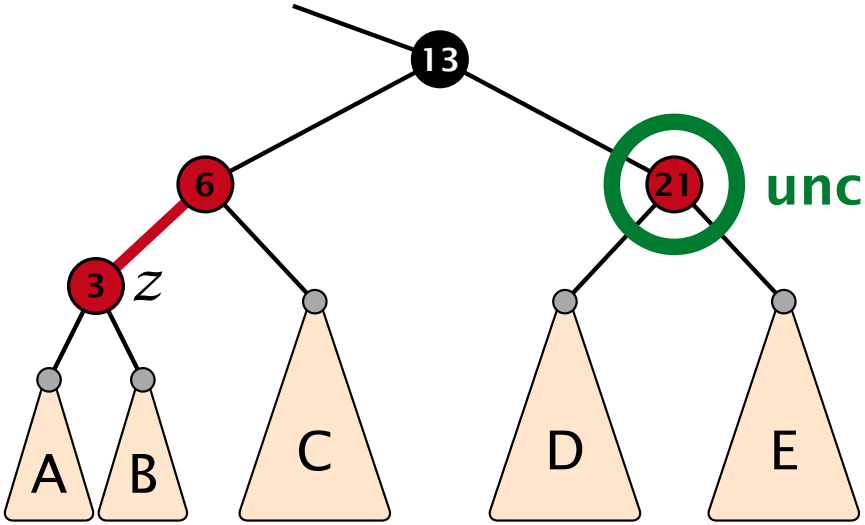

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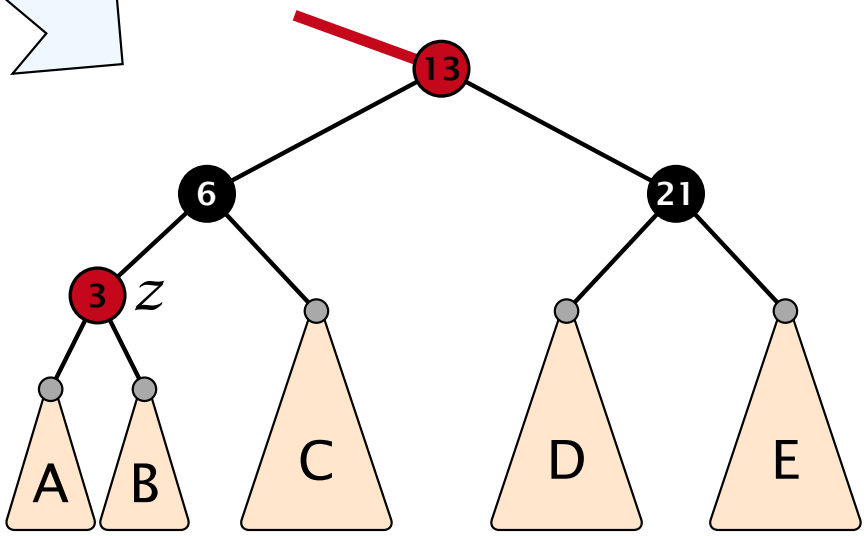

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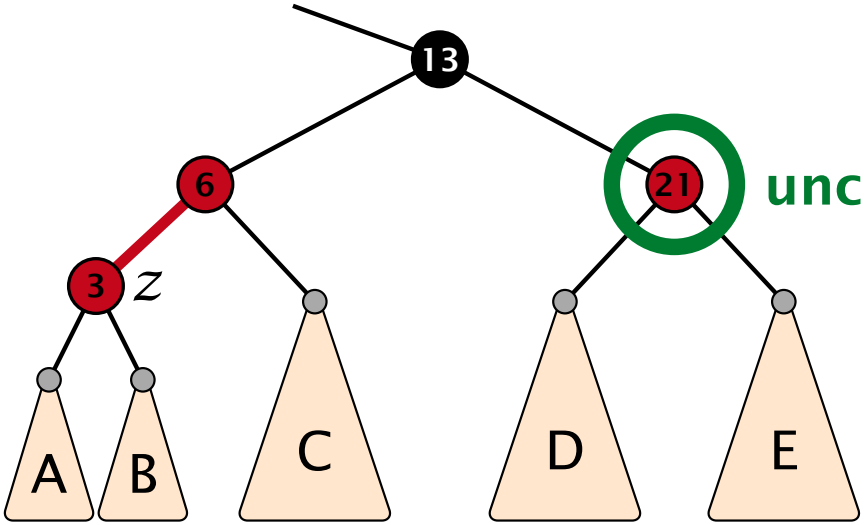
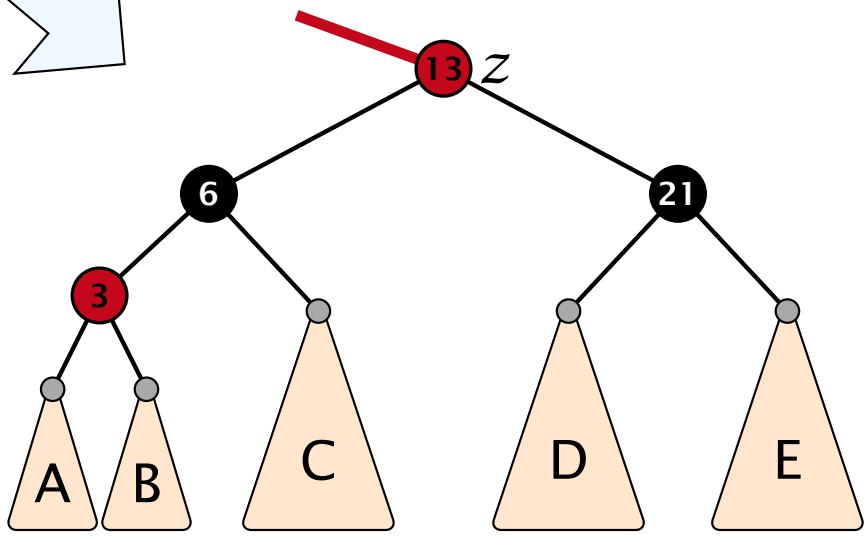


1. recolour
2. move z to grand-parent



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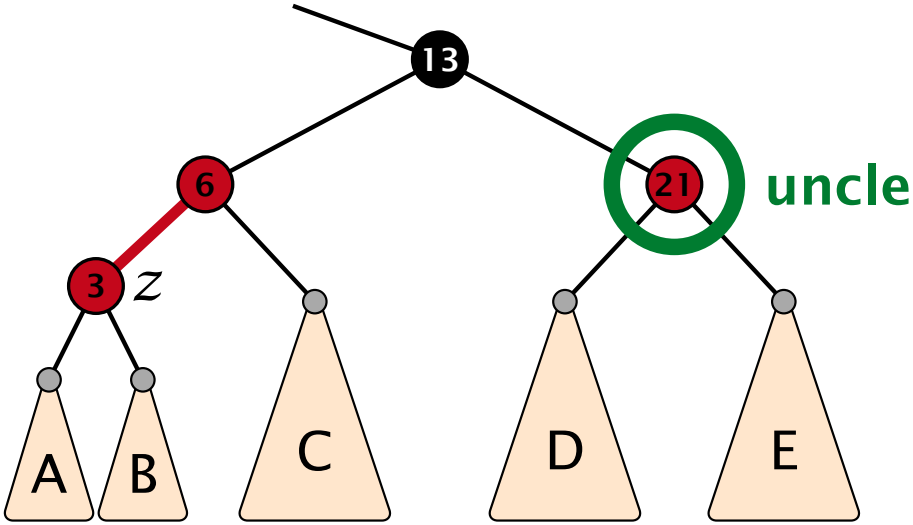
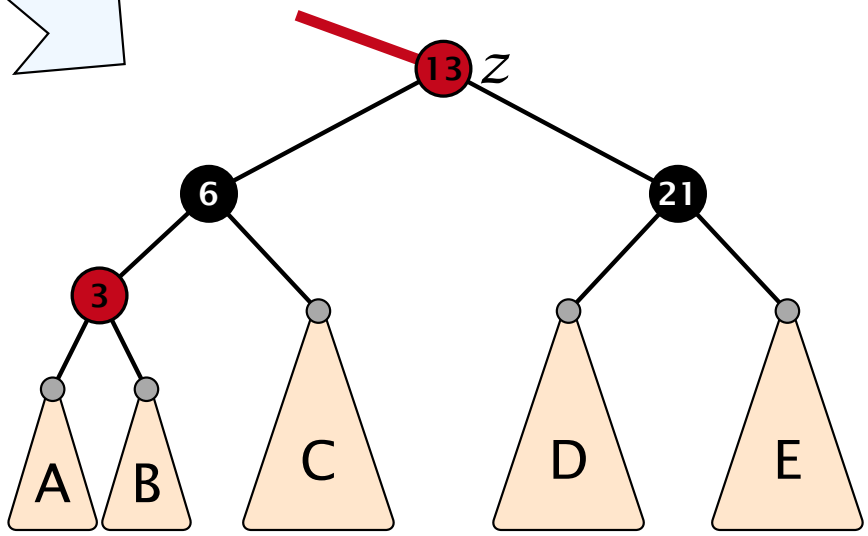


1. recolour
2. move z to grand-parent
3. invariant is fulfilled for new z



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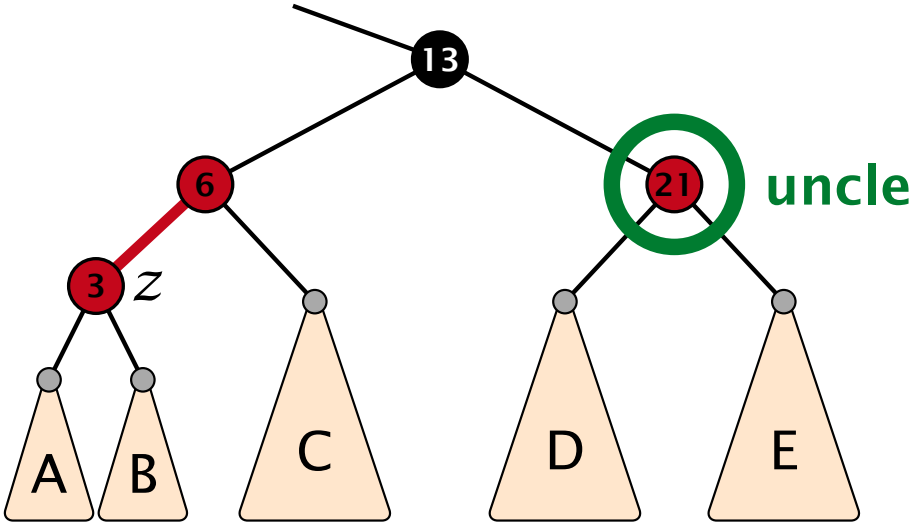

1. recolour
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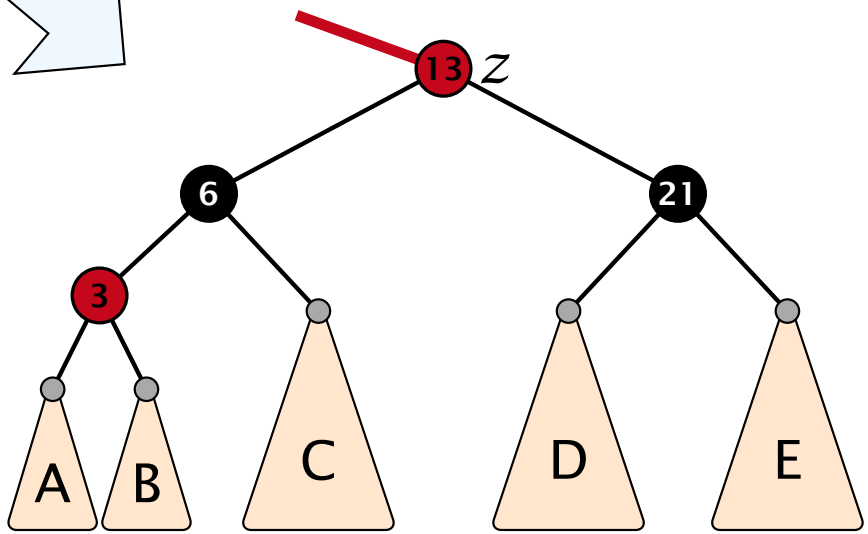
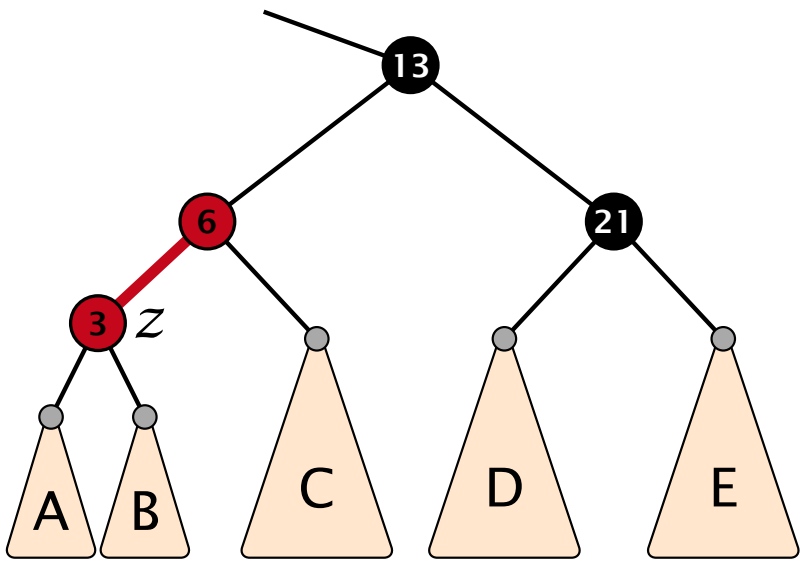
3. invariant is fulfilled for new z
4. you made progress



Case 2b: Black uncle and z is left child

1. rotate around grandparent
2. re-colour to ensure that
   black height property holds
3. you have a red black tree



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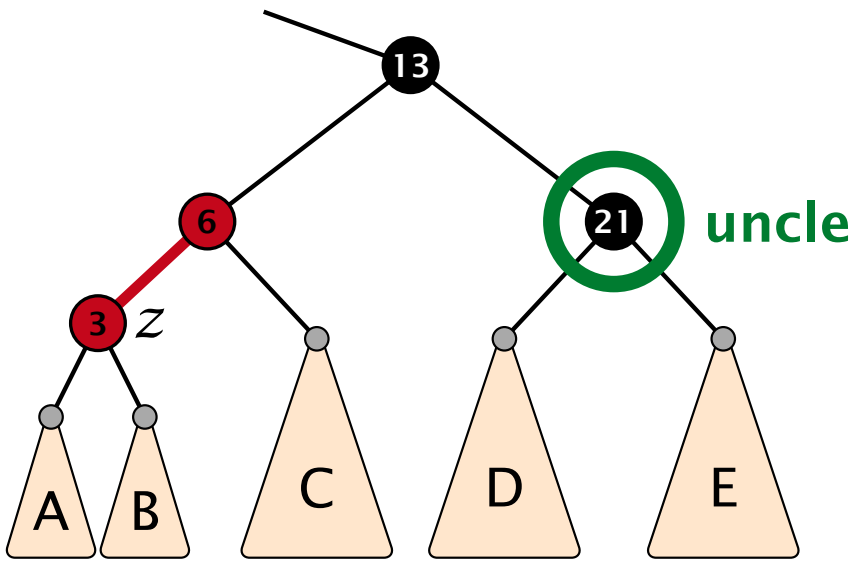

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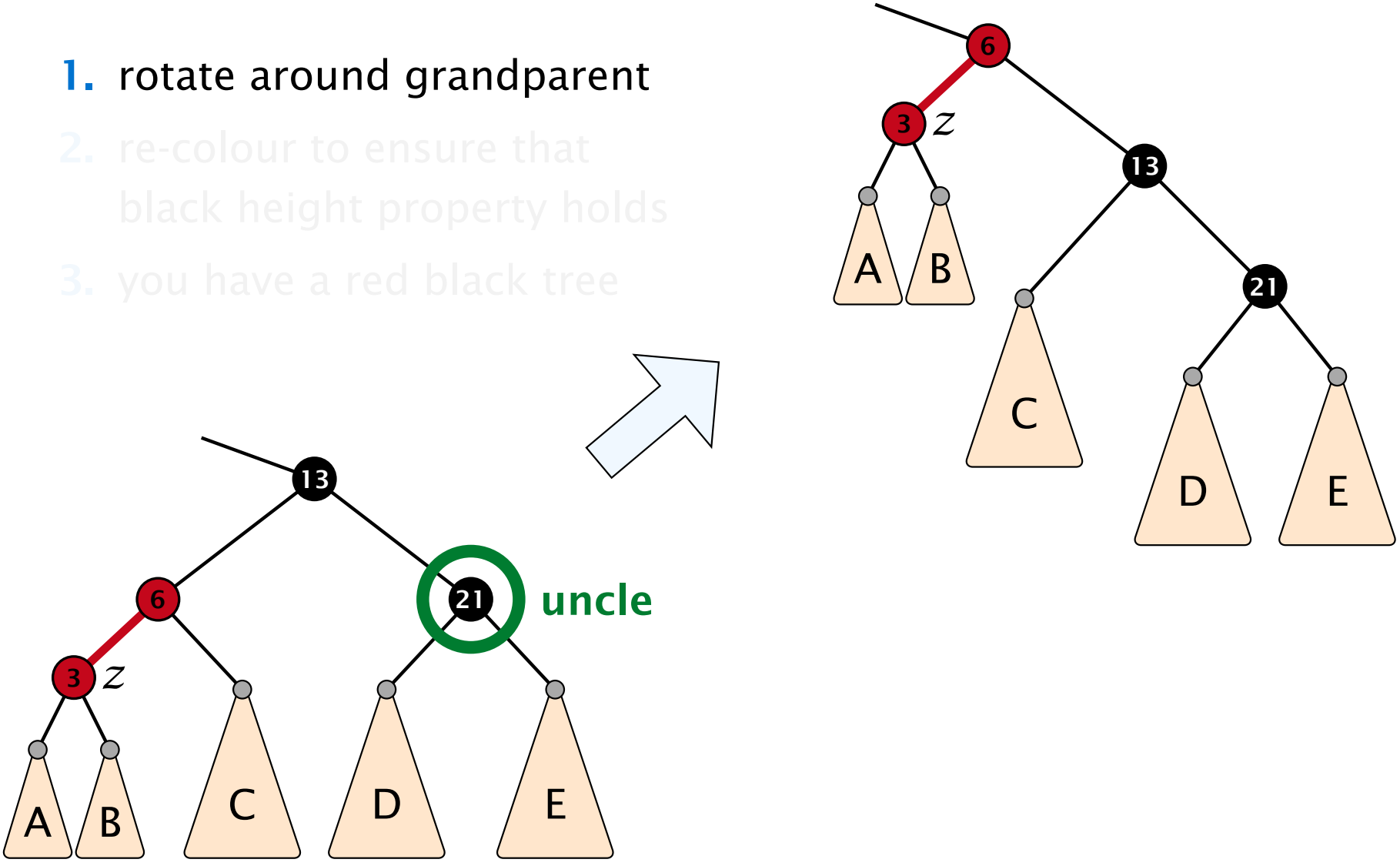

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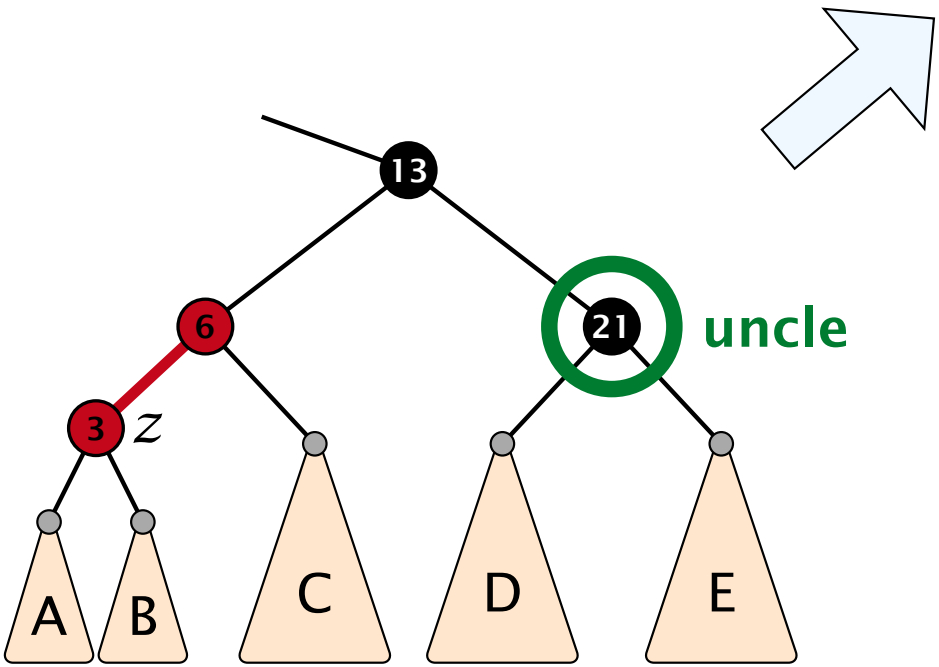

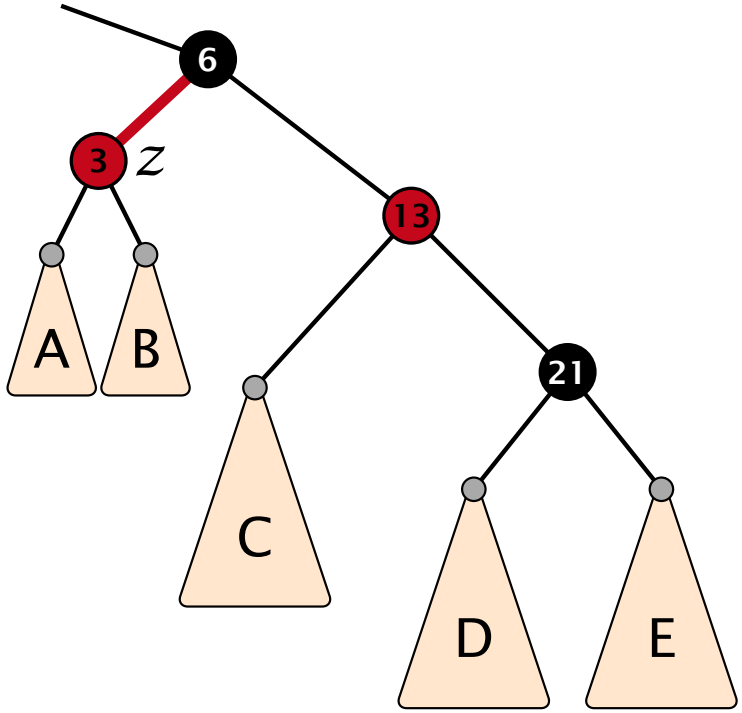

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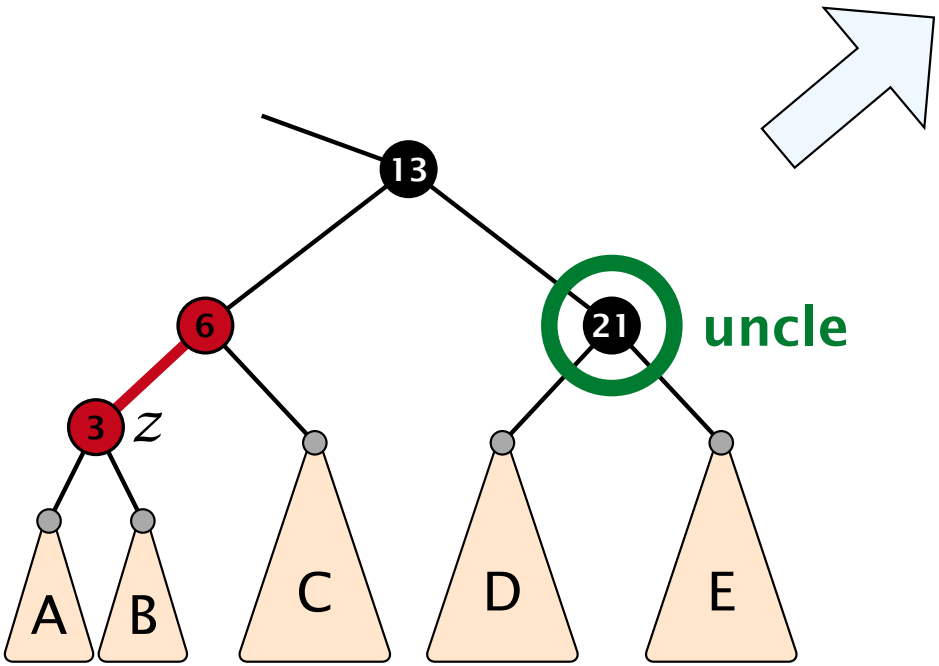
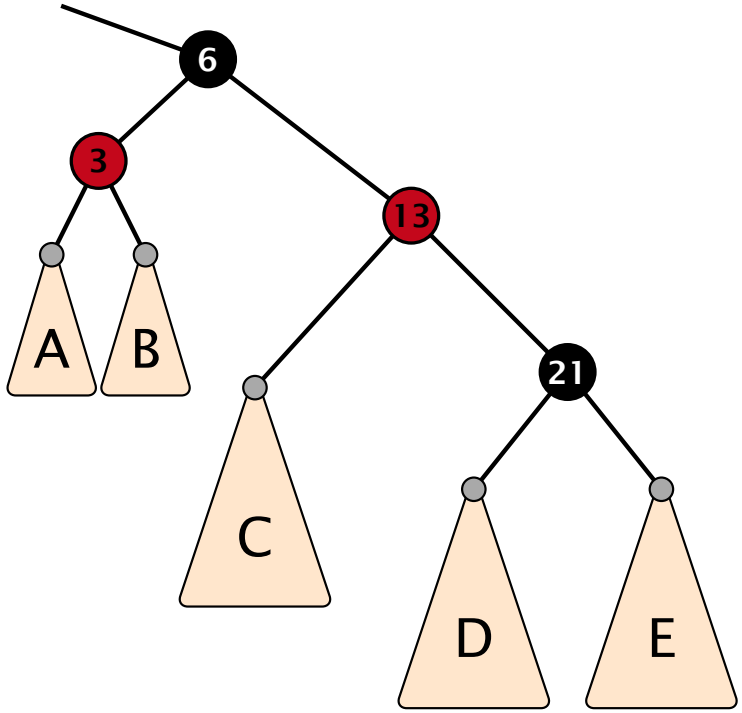
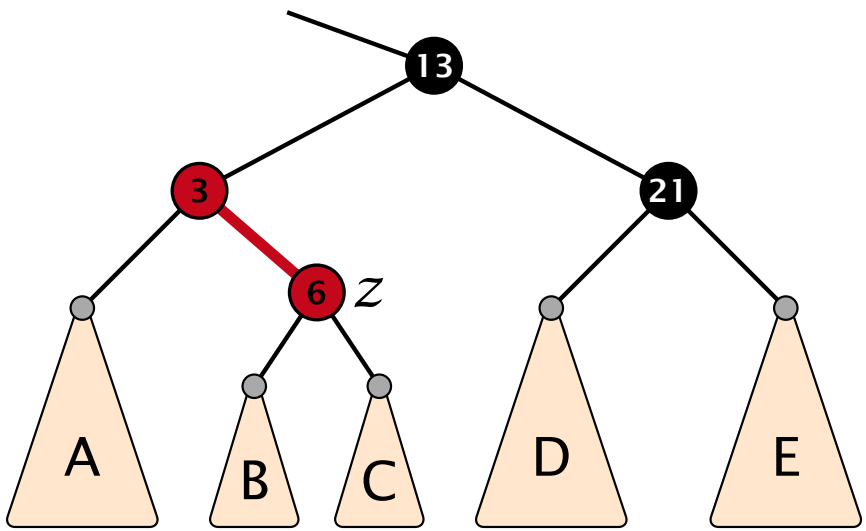




Case 2a: Black uncle and z is right child

1. rotate around parent
2. move z downwards
3. you have Case 2b.



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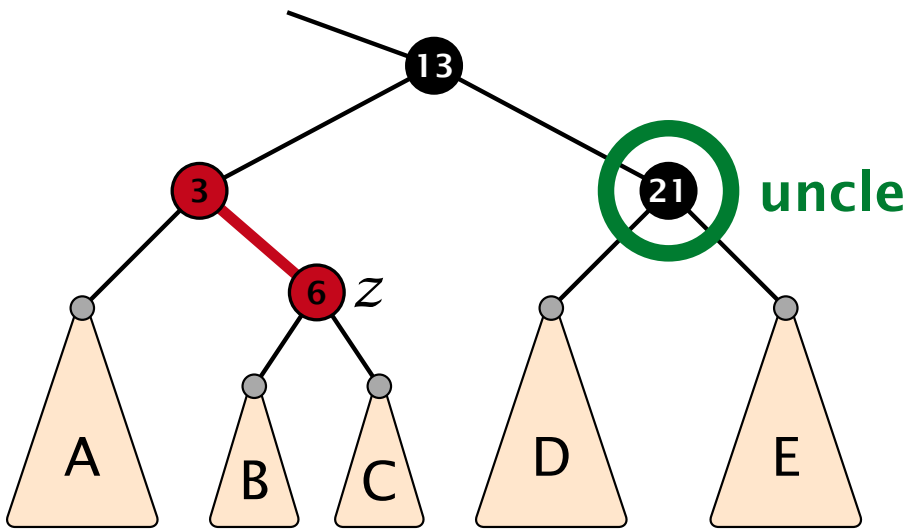

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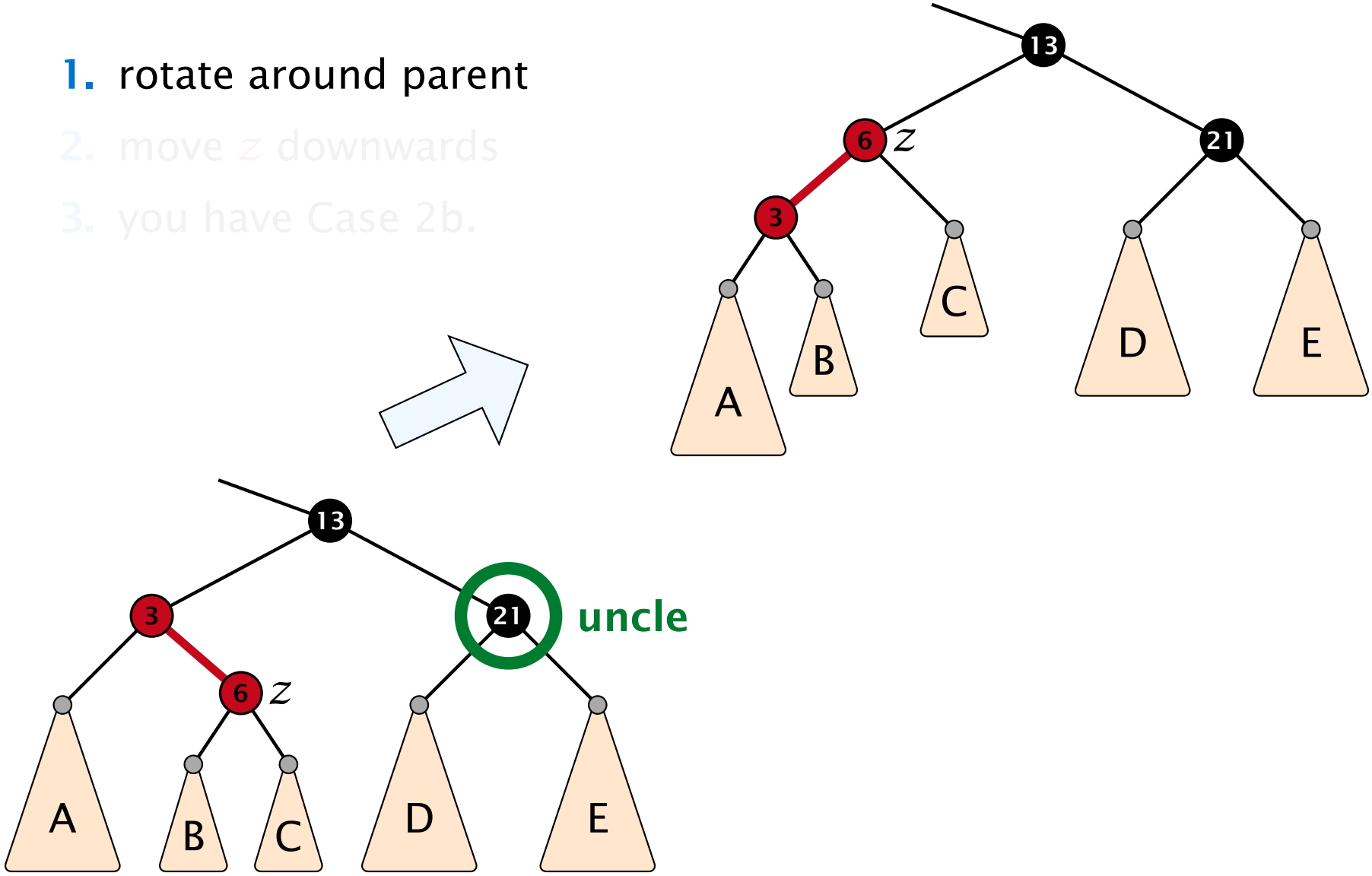

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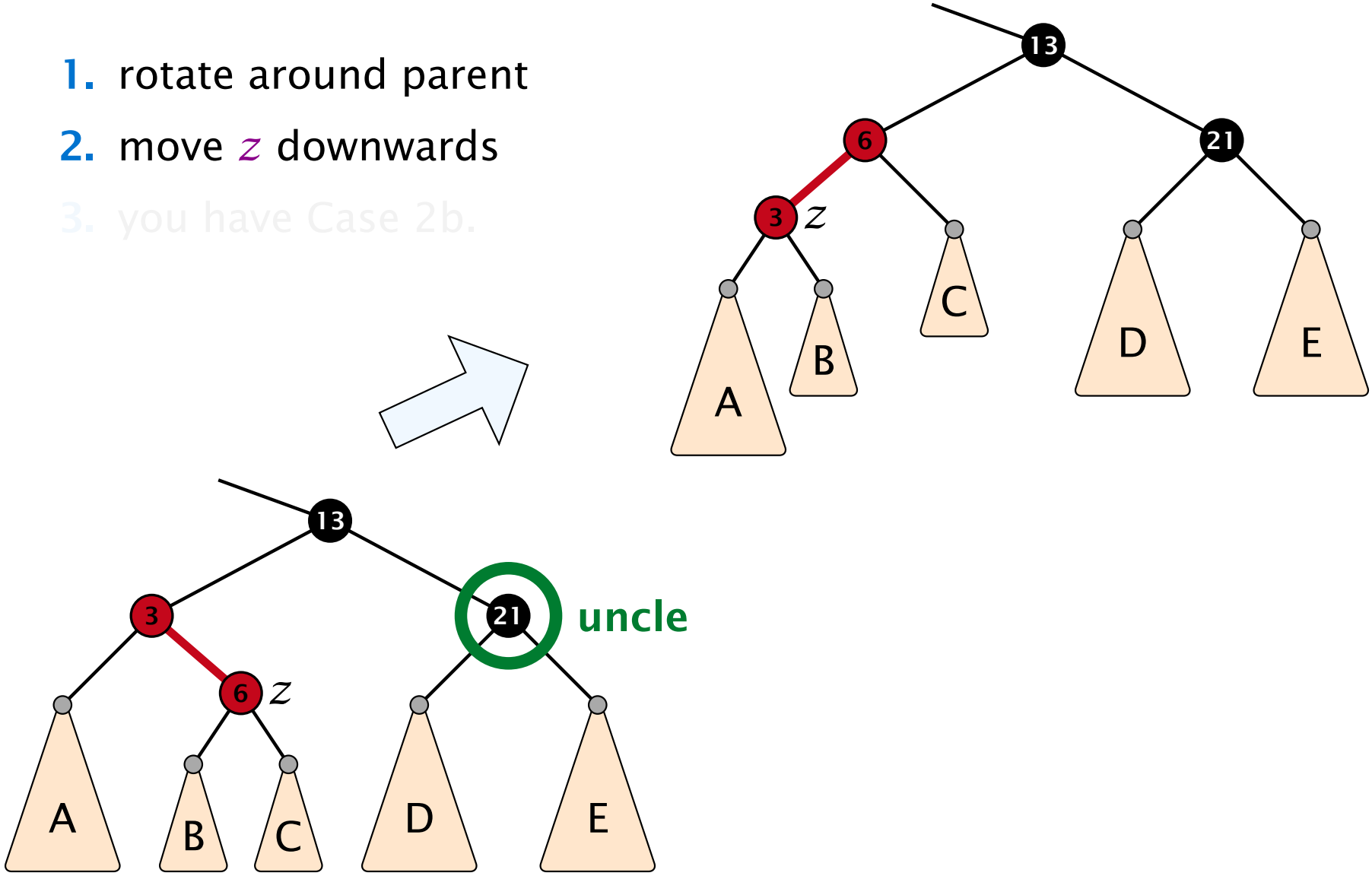

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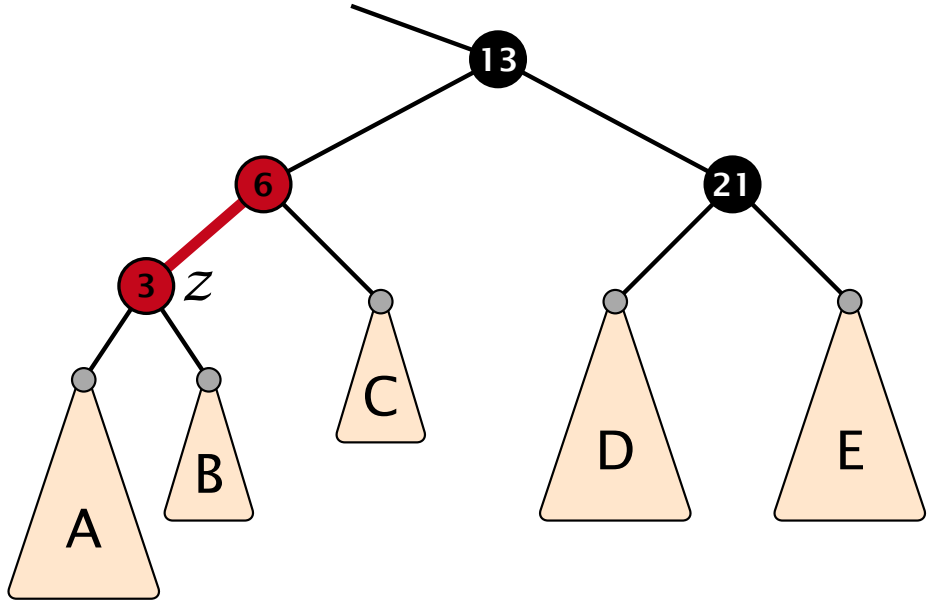

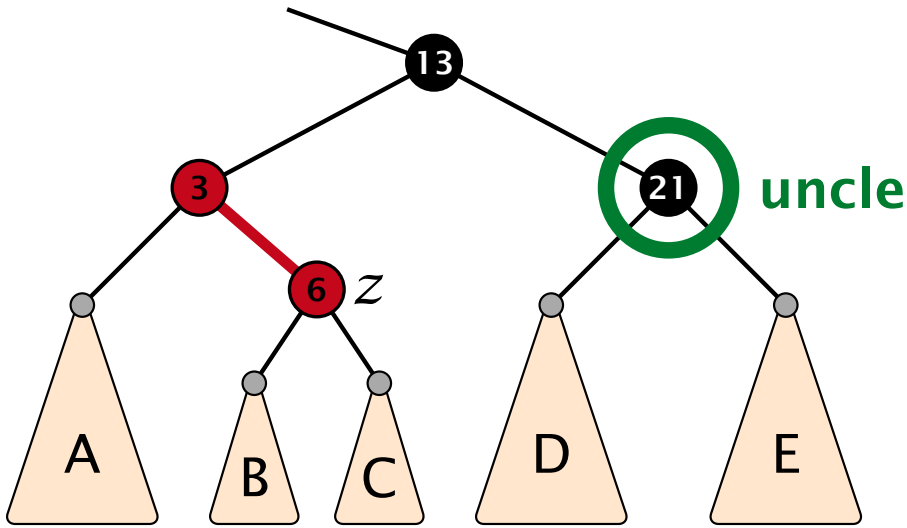

Red Black Trees: Insert

Running time:

- Only Case 1 may repeat; but only $h/2$ many steps, where h is the height of the tree.
- Case 2a $\rightarrow$ Case 2b $\rightarrow$ red-black tree
- Case 2b $\rightarrow$ red-black tree

Performing Case 1 at most $\mathcal{O}(\log n)$ times and every other case at most once, we get a red-black tree. Hence $\mathcal{O}(\log n)$ re-colorings and at most 2 rotations.

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Red Black Trees: Delete

First do a standard delete.

If the spliced out node x was red everything is fine.

If it was black there may be the following problems.

- Parent and child of x were red; two adjacent red vertices.
- If you delete the root, the root may now be red.
- Every path from an ancestor of x to a descendant leaf of x changes the number of black nodes; black height property might be violated.

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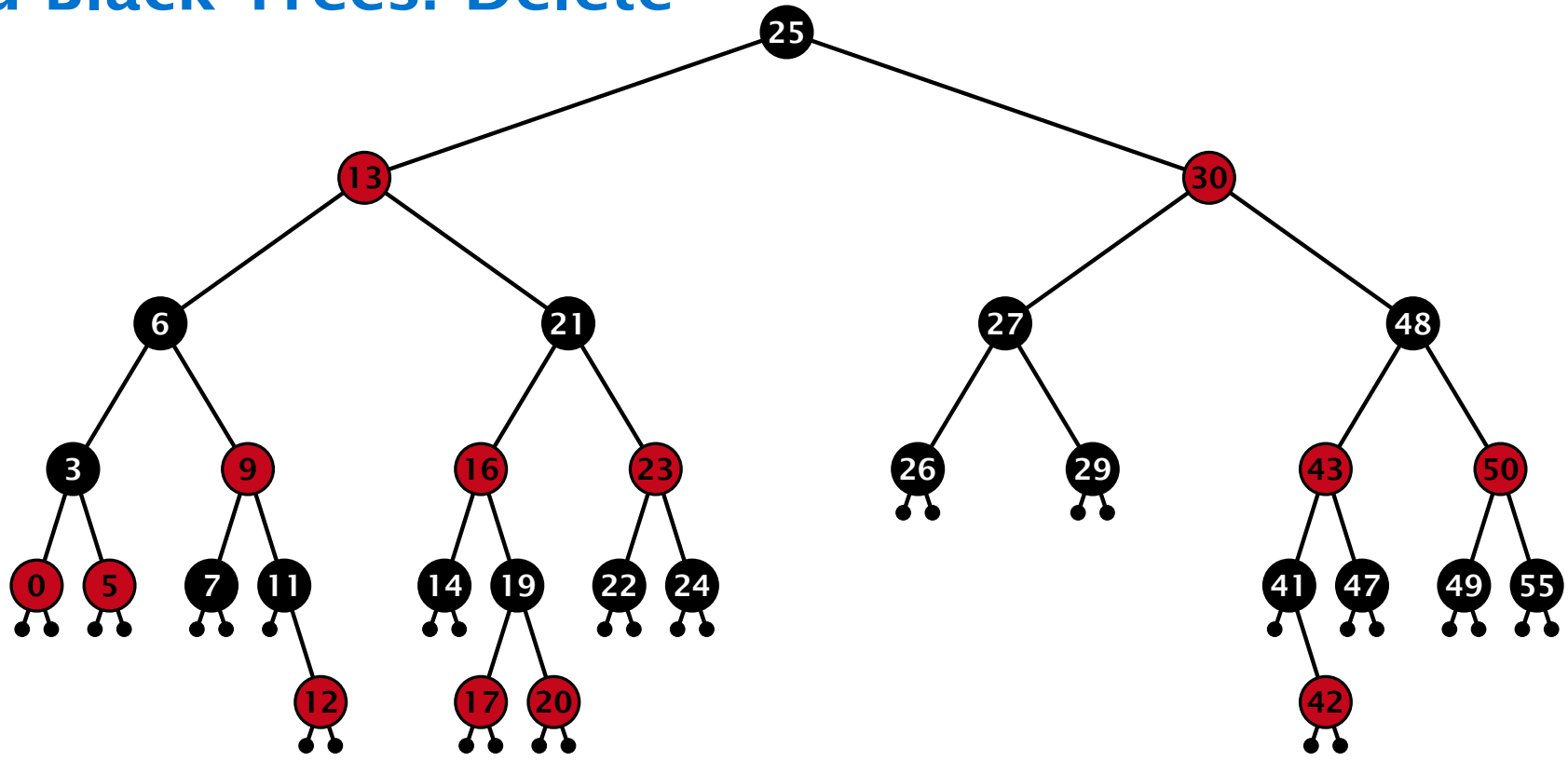

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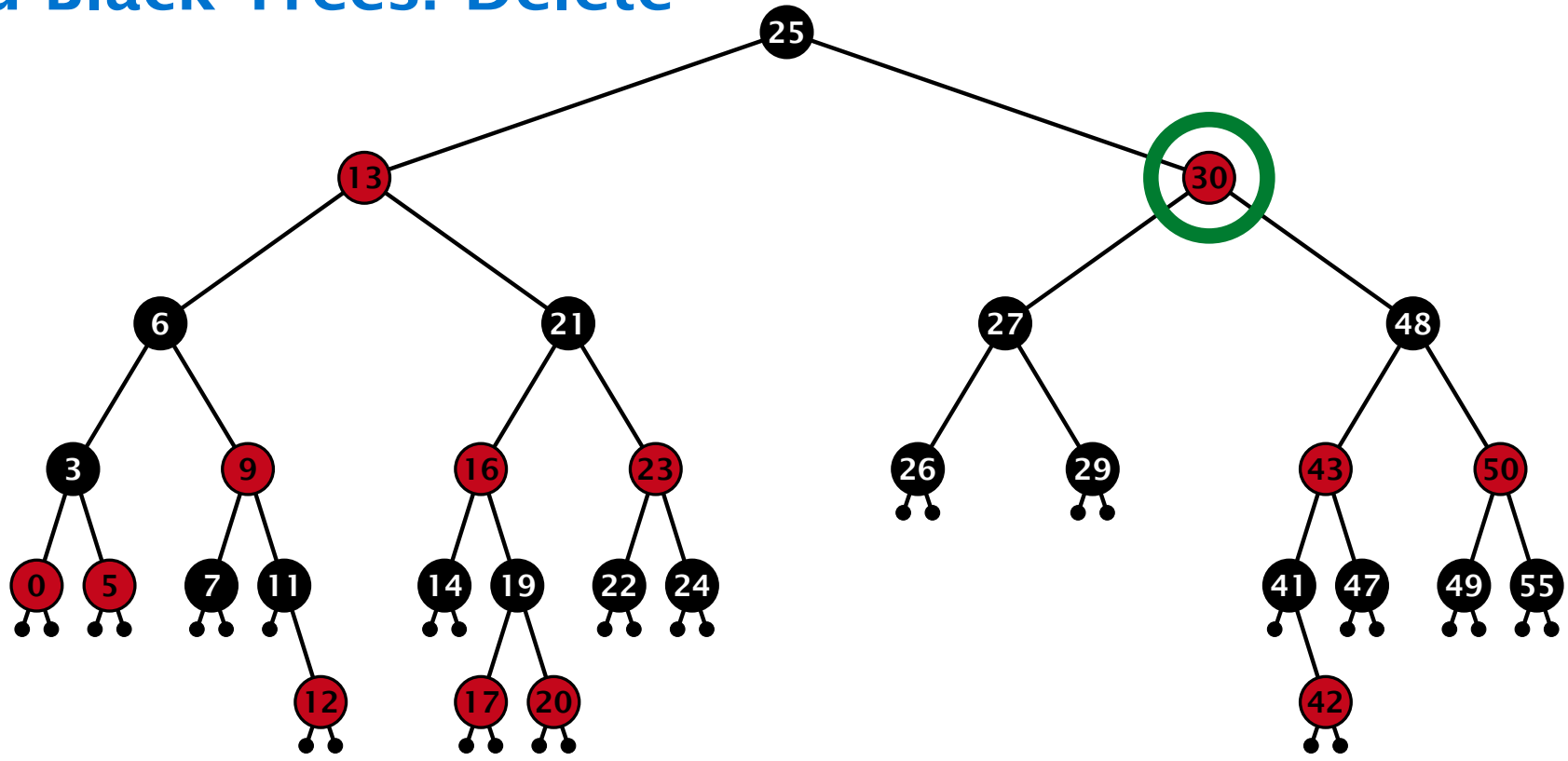

Case 3:

Element has two children

- do normal delete
- when replacing content by content of successor, don't change color of node

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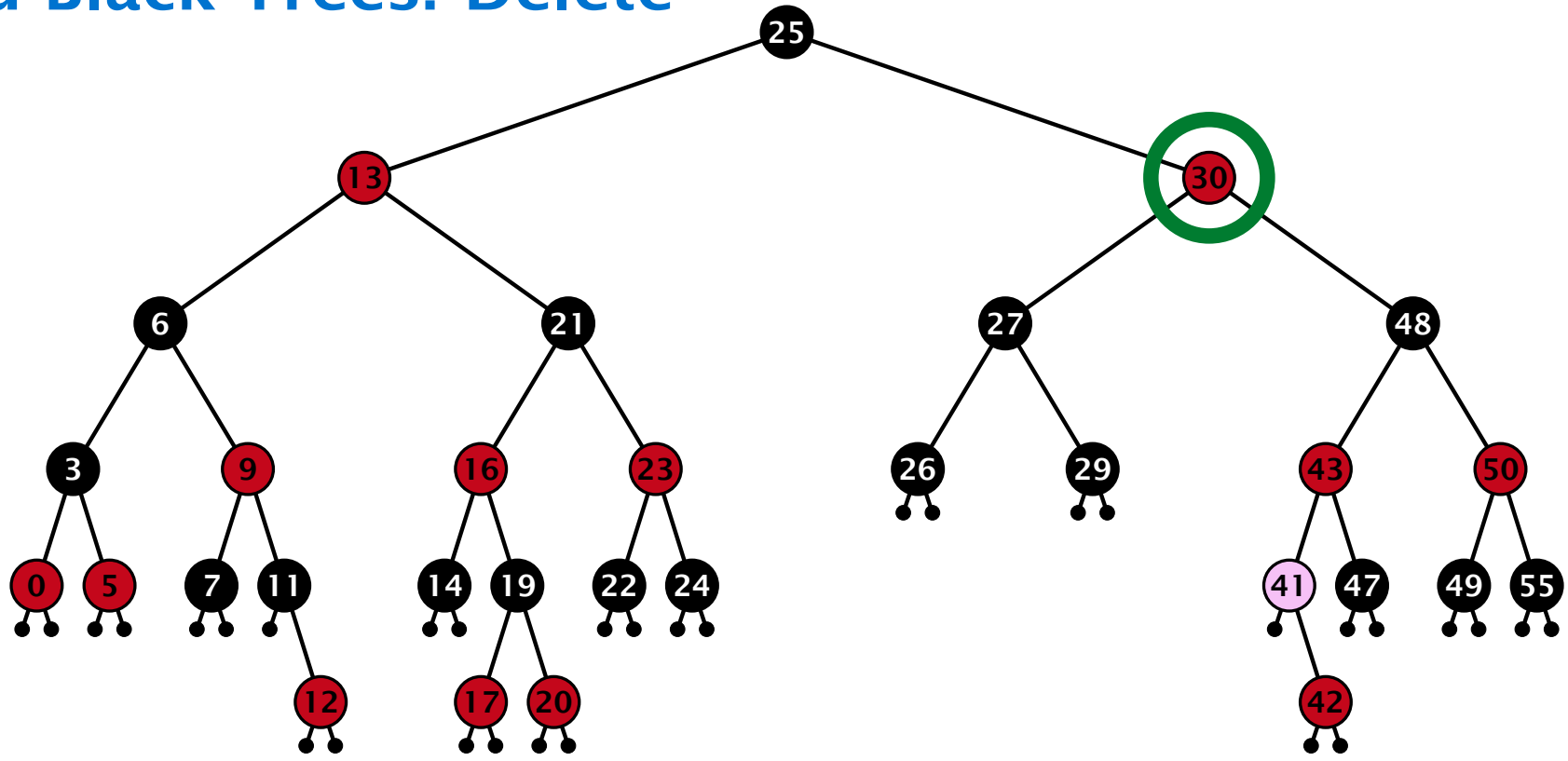

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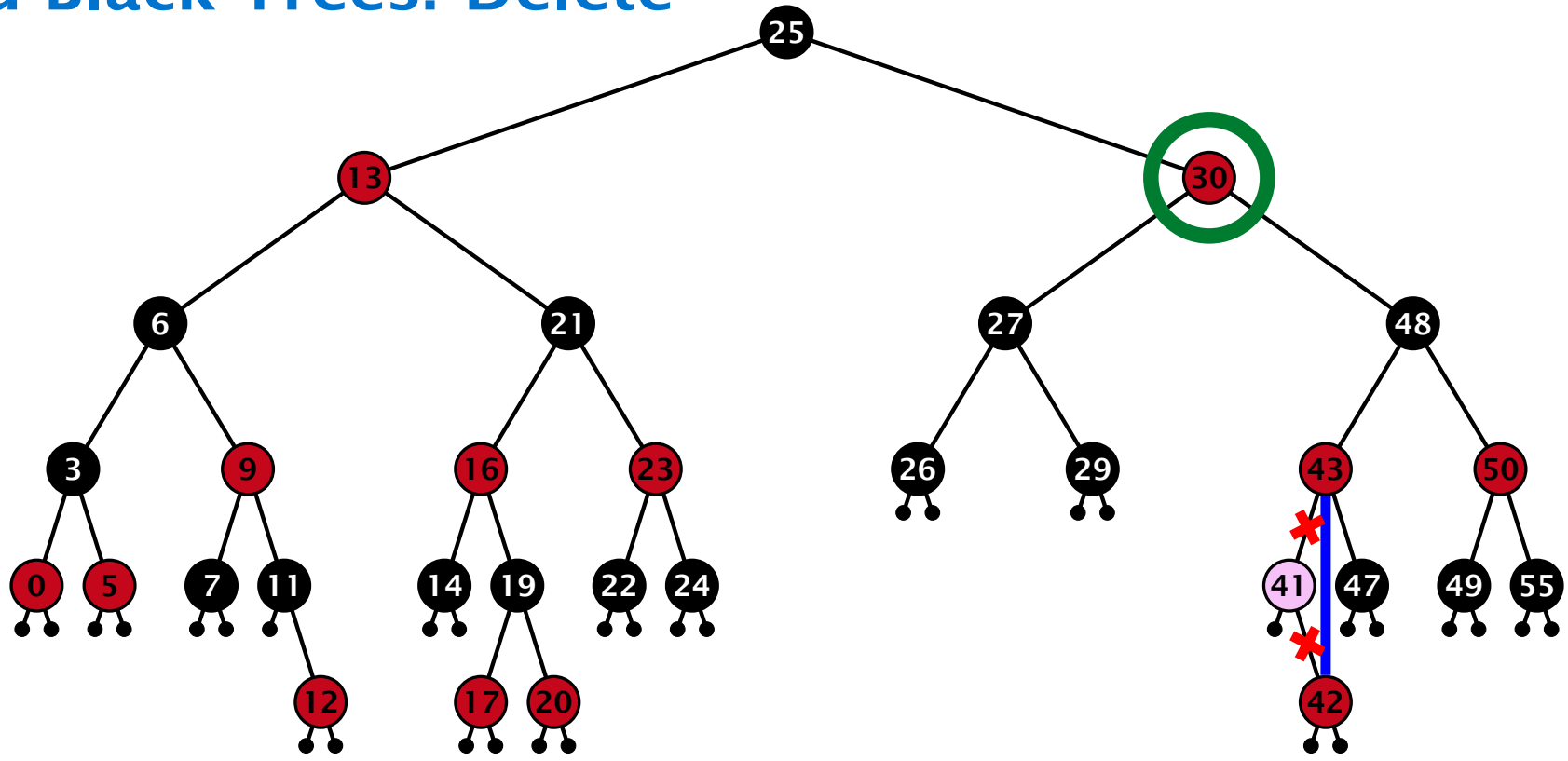

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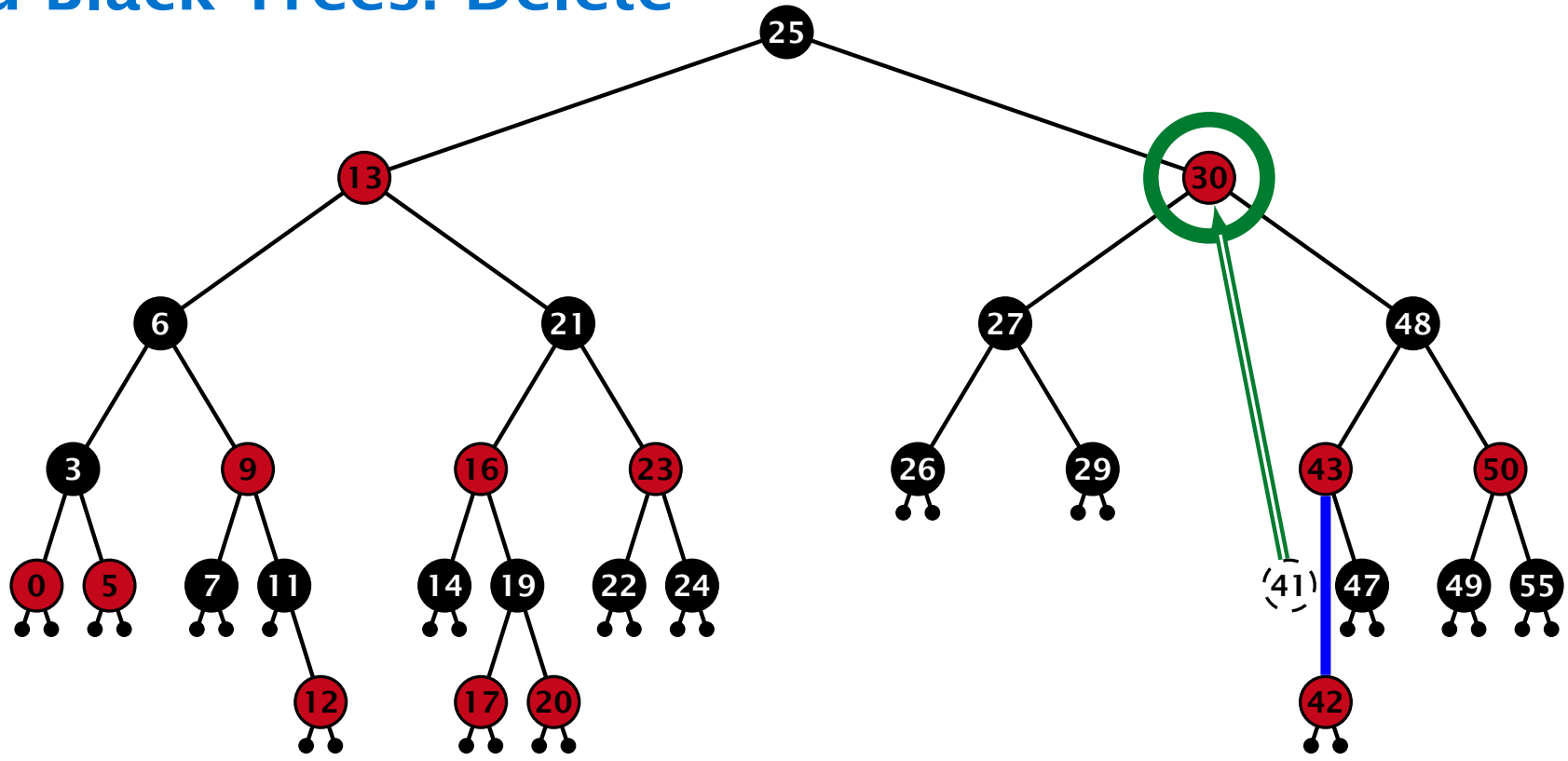

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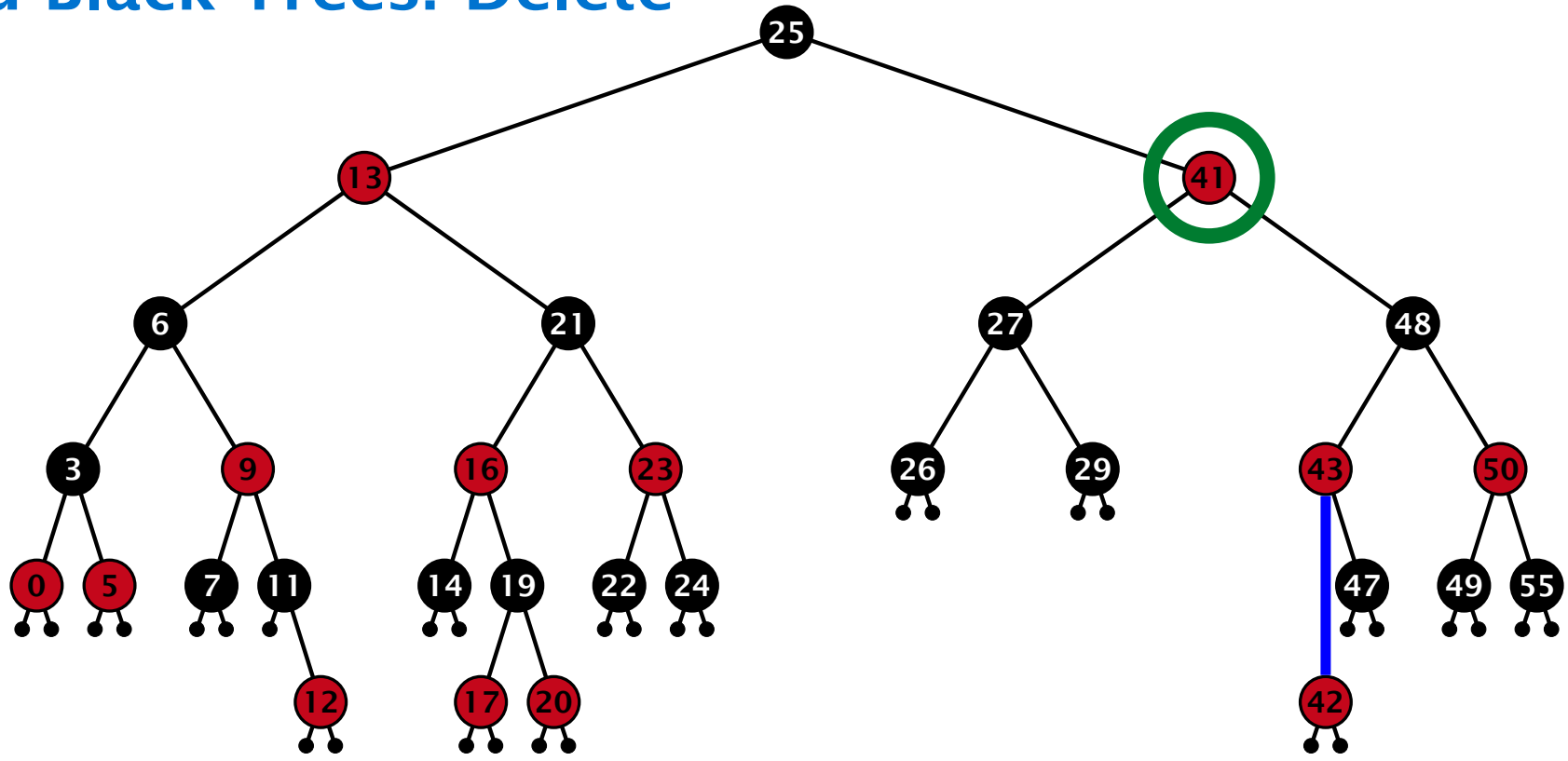

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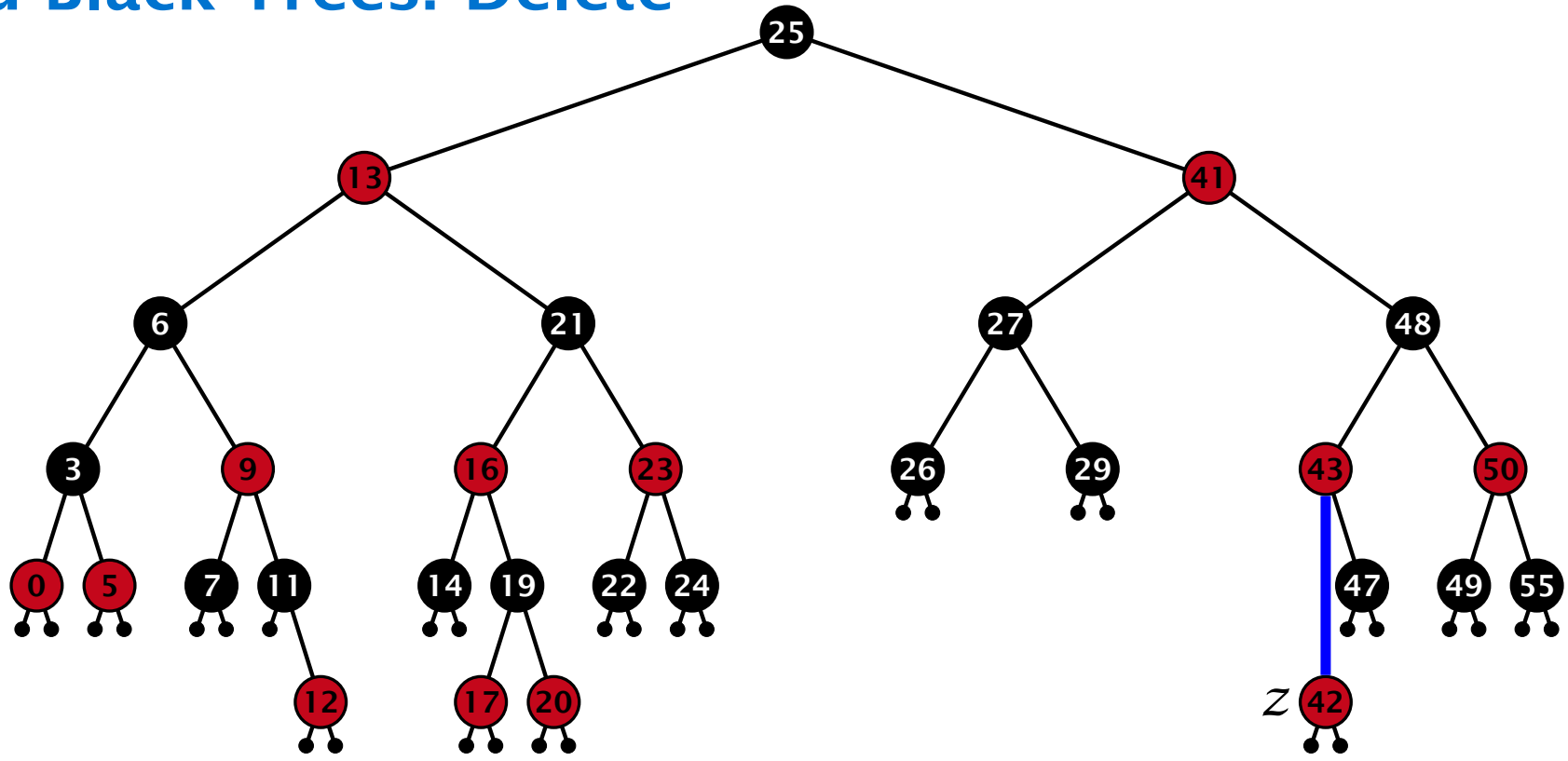

Delete:

- deleting black node messes up black-height property
- if z is red, we can simply color it black and everything is fine
- the problem is if z is black (e.g. a dummy-leaf); we call a fix-up procedure to fix the problem.

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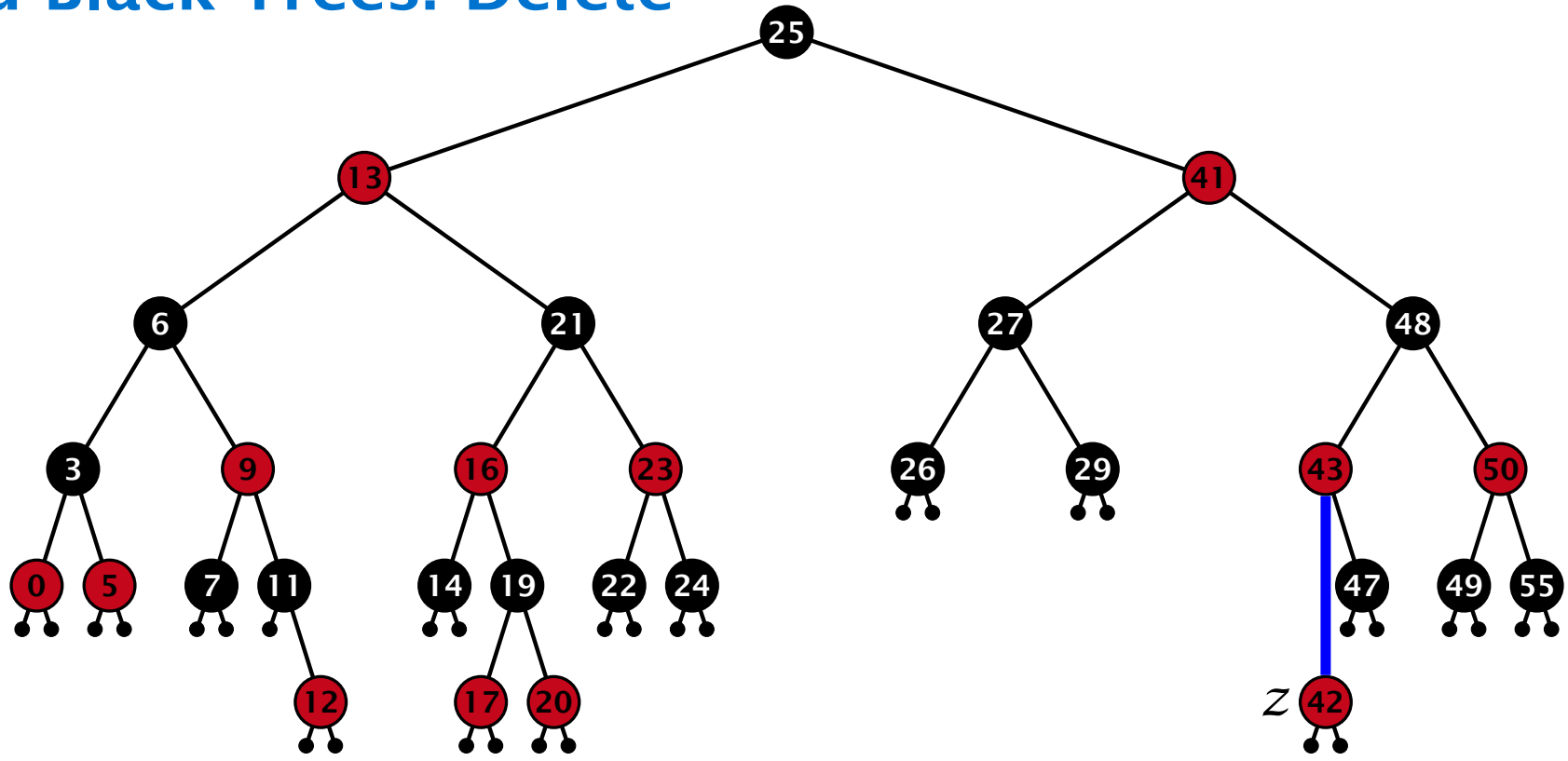

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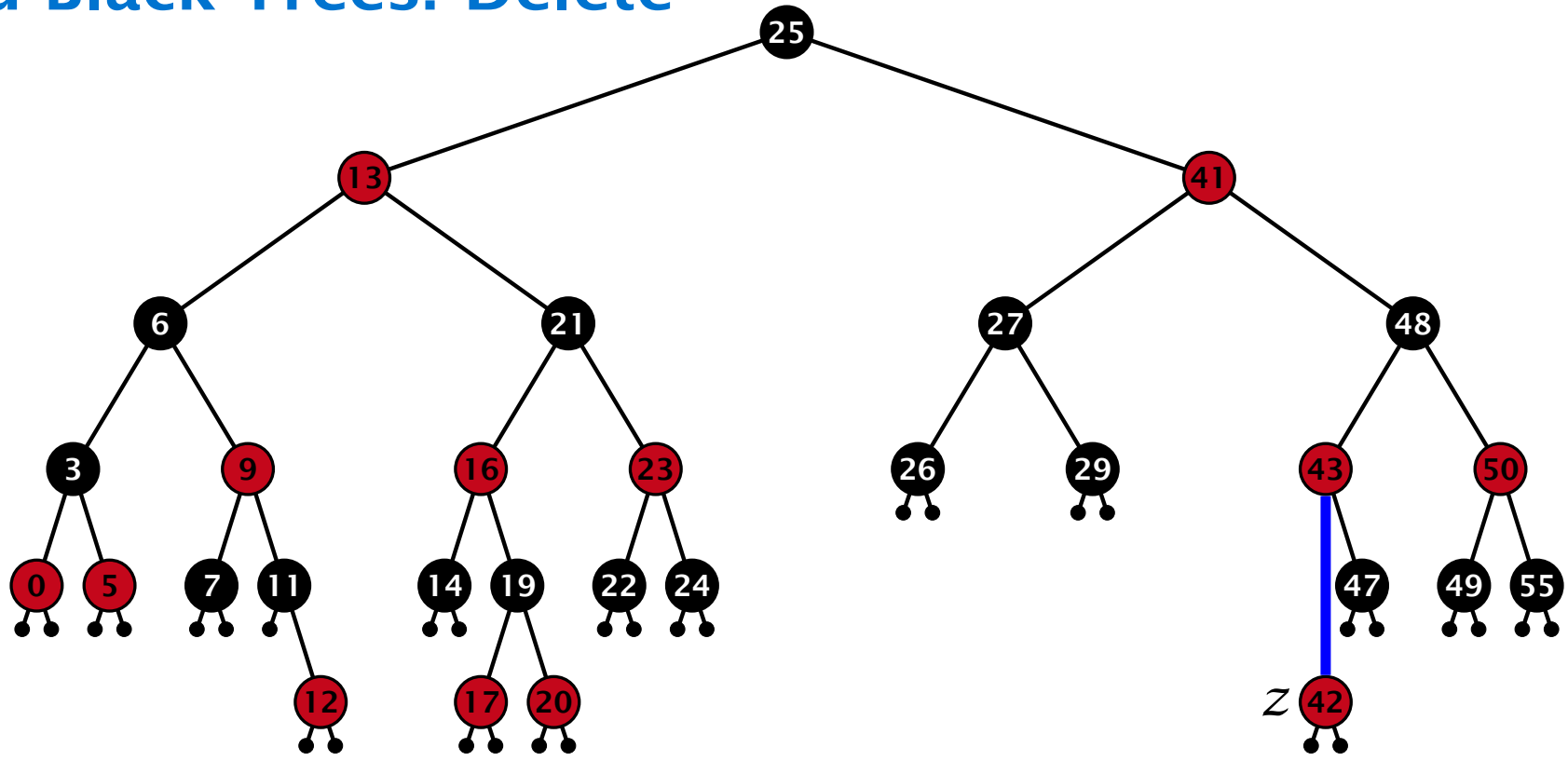

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Invariant of the fix-up algorithm

- the node z is black
- if we "assign" a fake black unit to the edge from z to its parent then the black-height property is fulfilled

Goal: make rotations in such a way that you at some point can remove the fake black unit from the edge.

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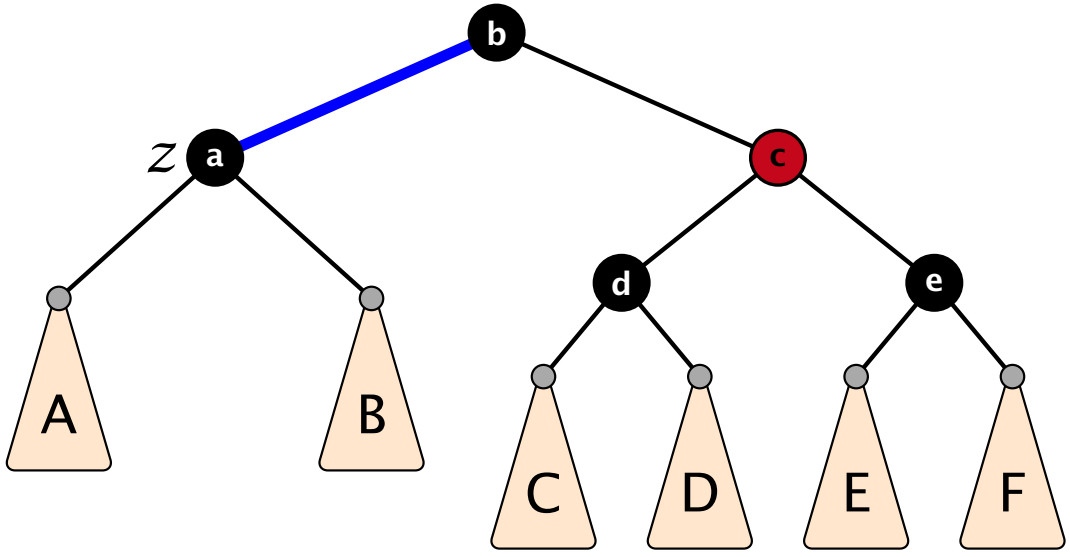
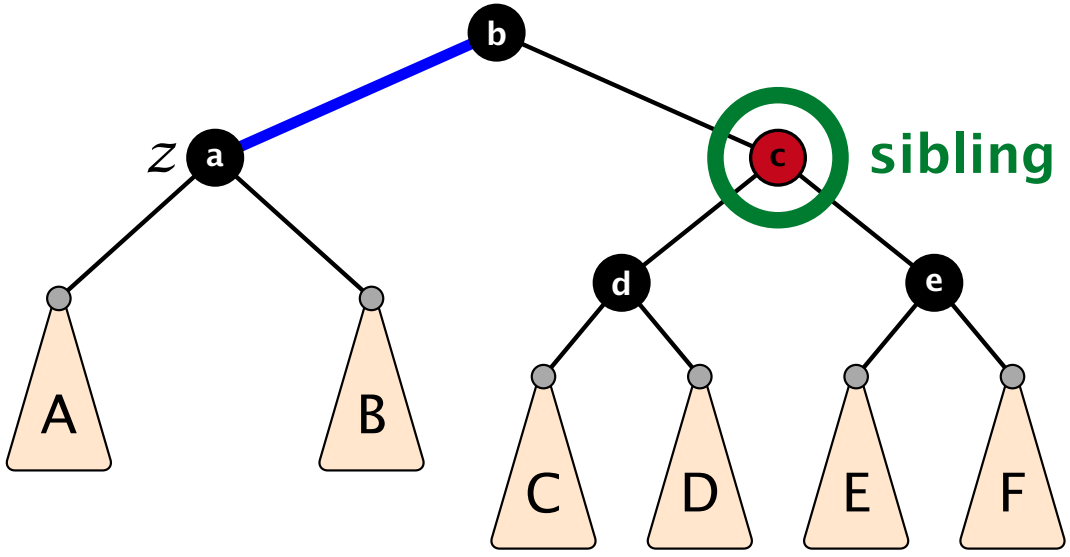


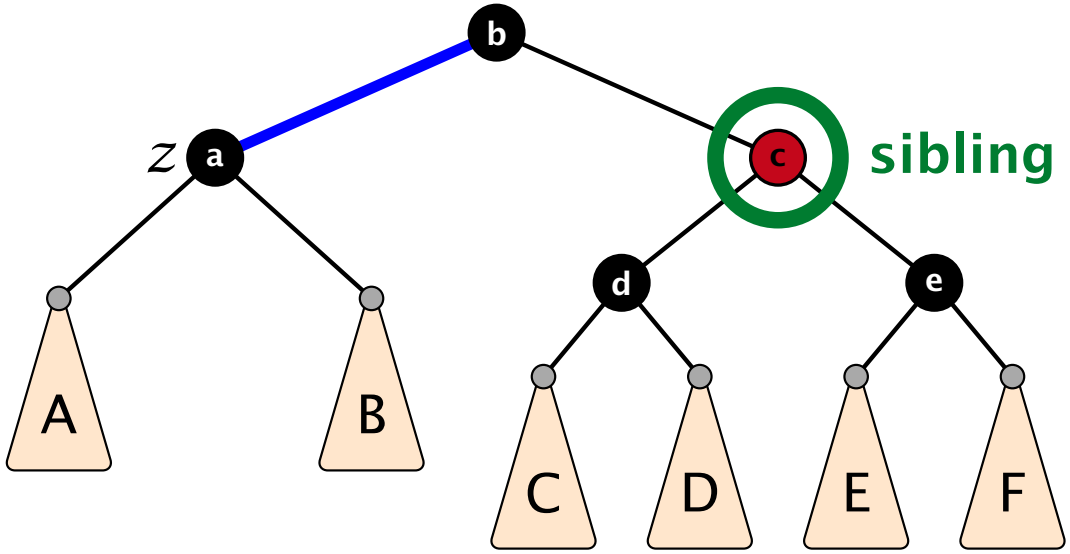
1. left-rotate around parent of z
2. recolor nodes b and c
3. the new sibling is black
   (and parent of z is red)
4. Case 2 (special),
   or Case 3, or Case 4

Case 1: Sibling of z is red



1. left-rotate around parent of z
2. recolor nodes b and c
3. the new sibling is black
   (and parent of z is red)
4. Case 2 (special),
   or Case 3, or Case 4

Case 1: Sibling of z is red



1. left-rotate around parent of z
2. recolor nodes b and c
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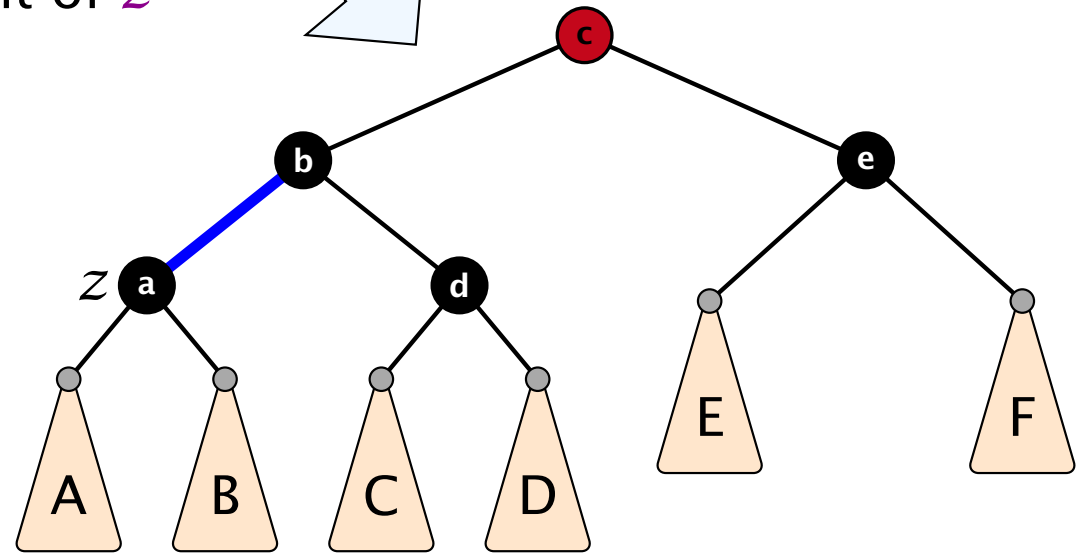
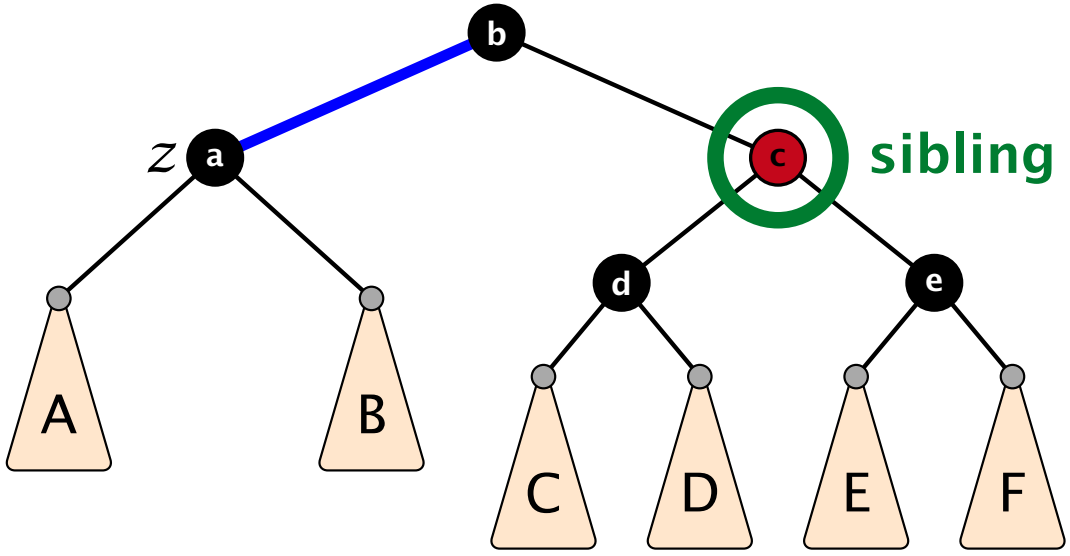
   (and parent of z is red)
4. Case 2 (special),
   or Case 3, or Case 4



Case 1: Sibling of z is red



1. left-rotate around parent of z
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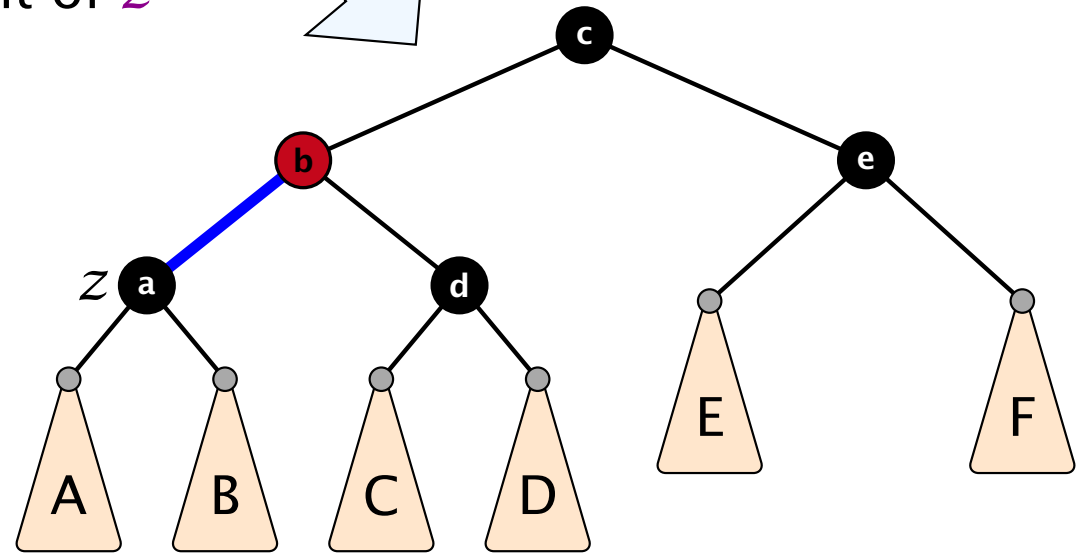
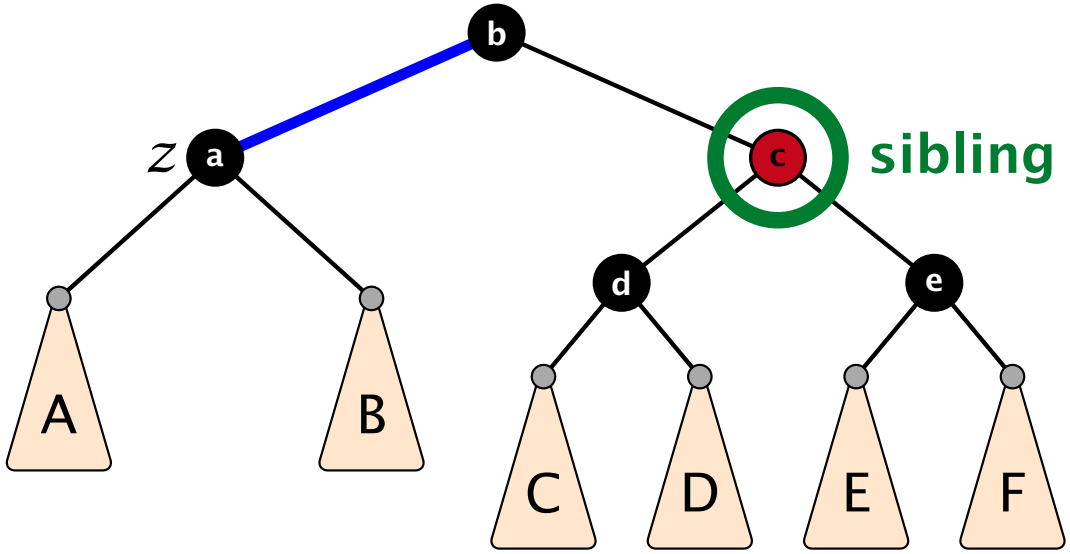
   or Case 3, or Case 4



Case 1: Sibling of z is red



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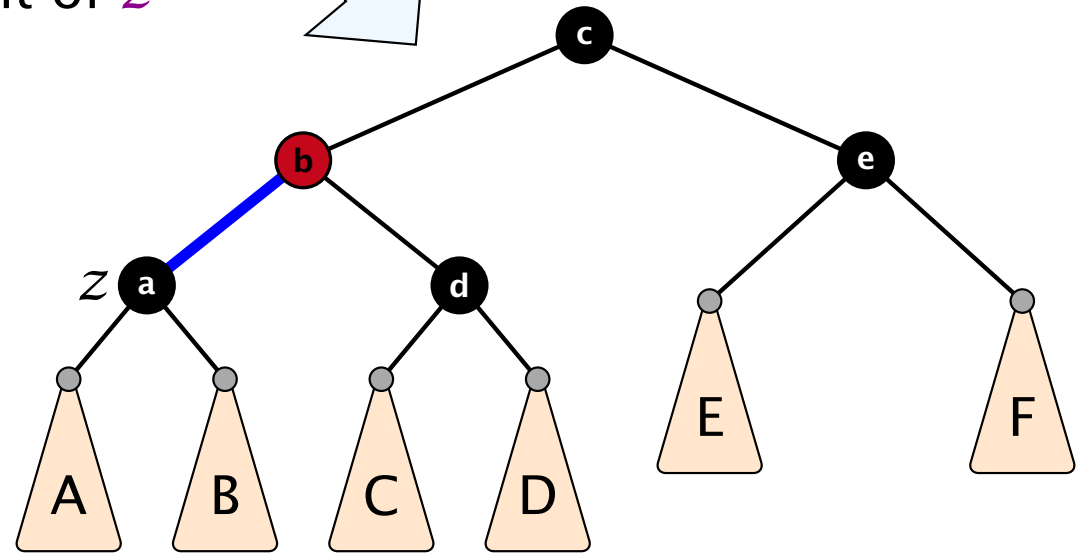
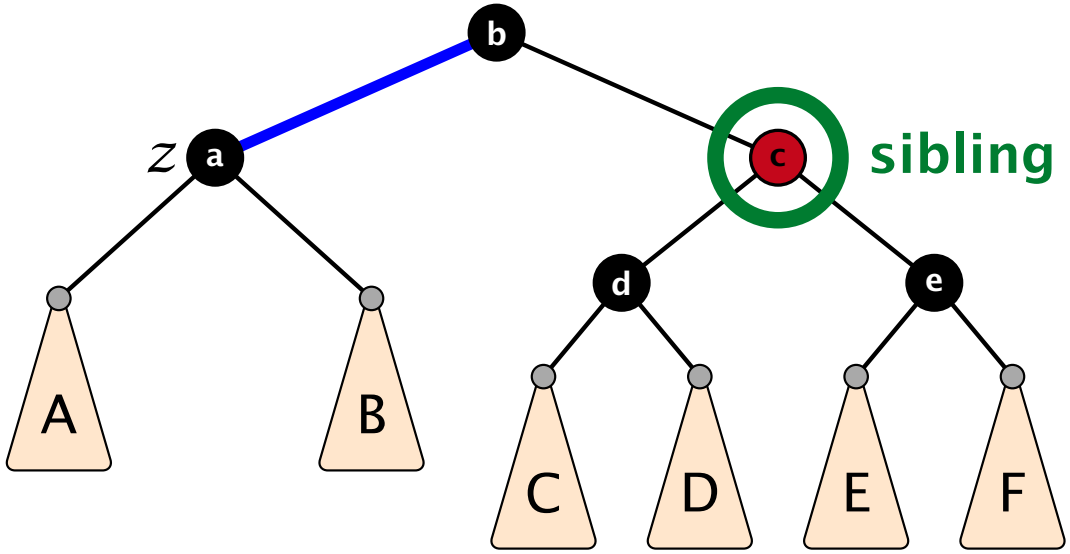


Case 1: Sibling of z is red



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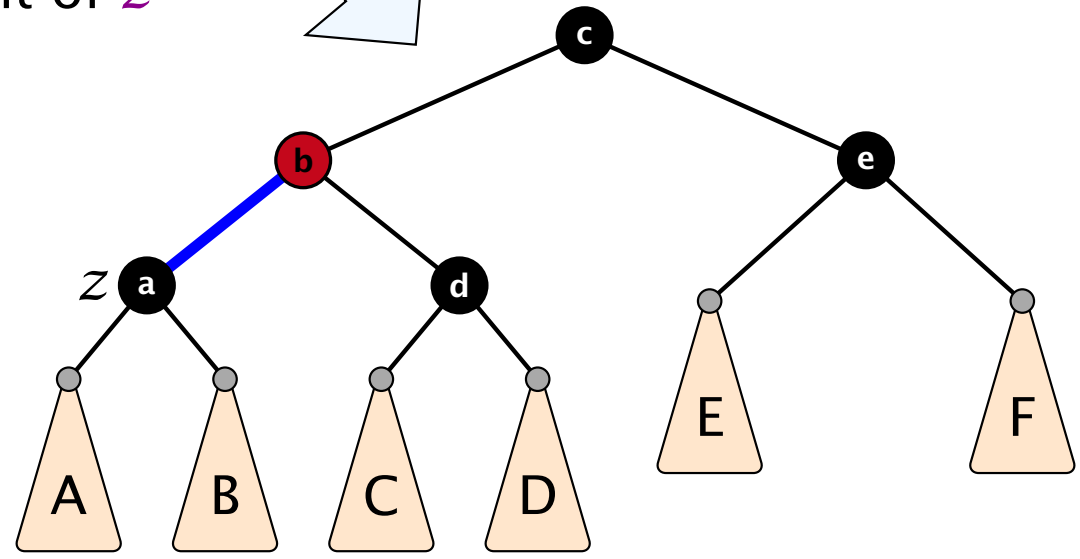
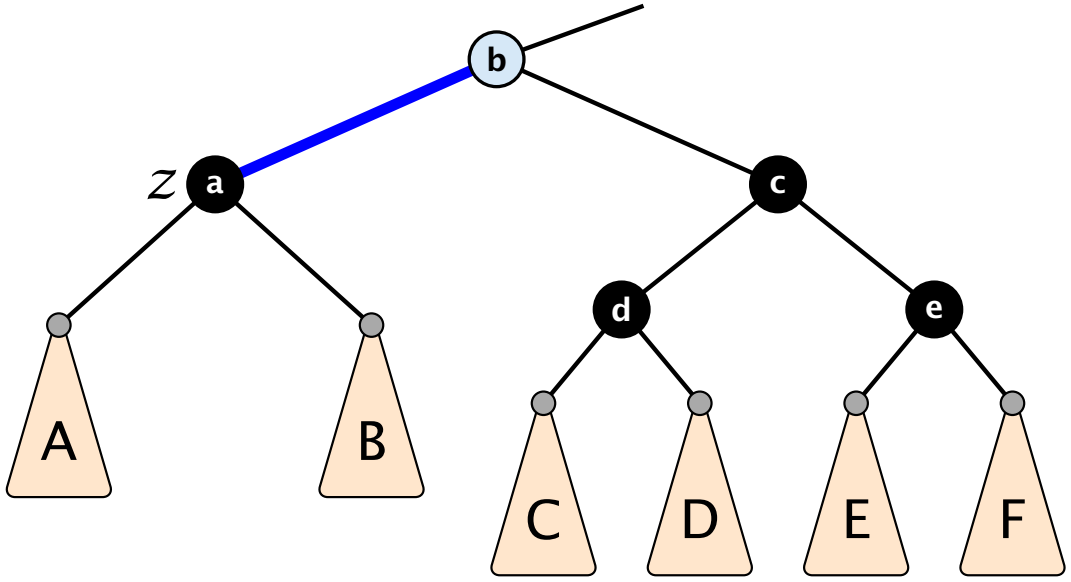
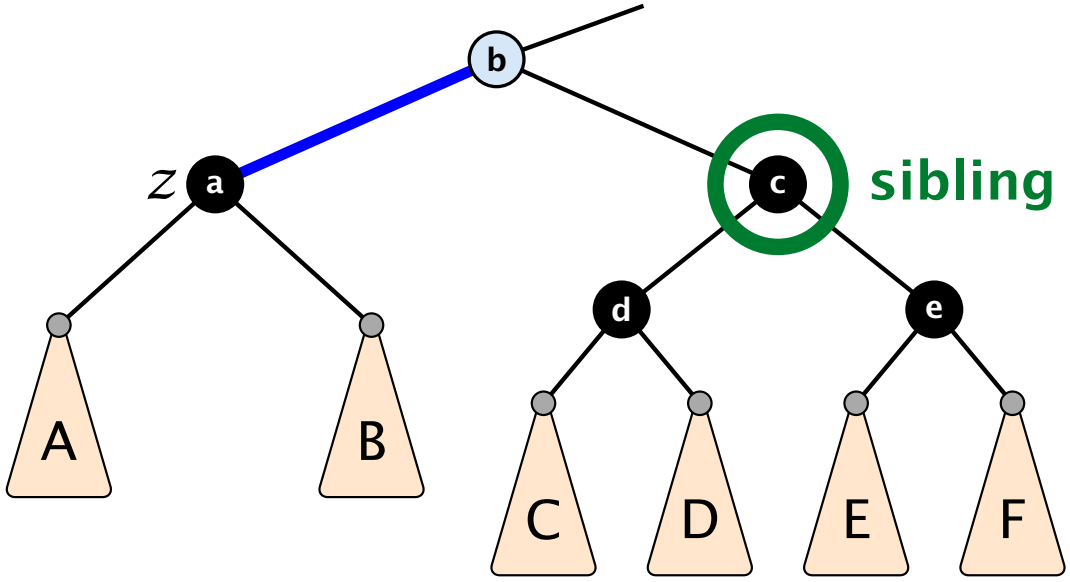


Case 2: Sibling is black with two black children



1. re-color node *c*
2. move fake black unit upwards
3. move z upwards
4. we made progress
5. if *b* is red we color it black and are done

Case 2: Sibling is black with two black children



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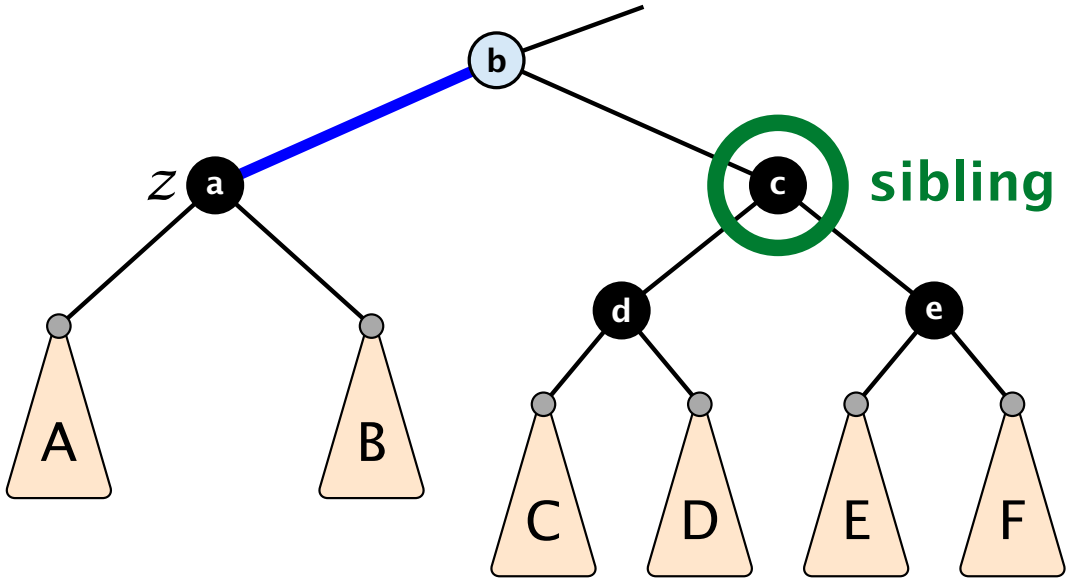

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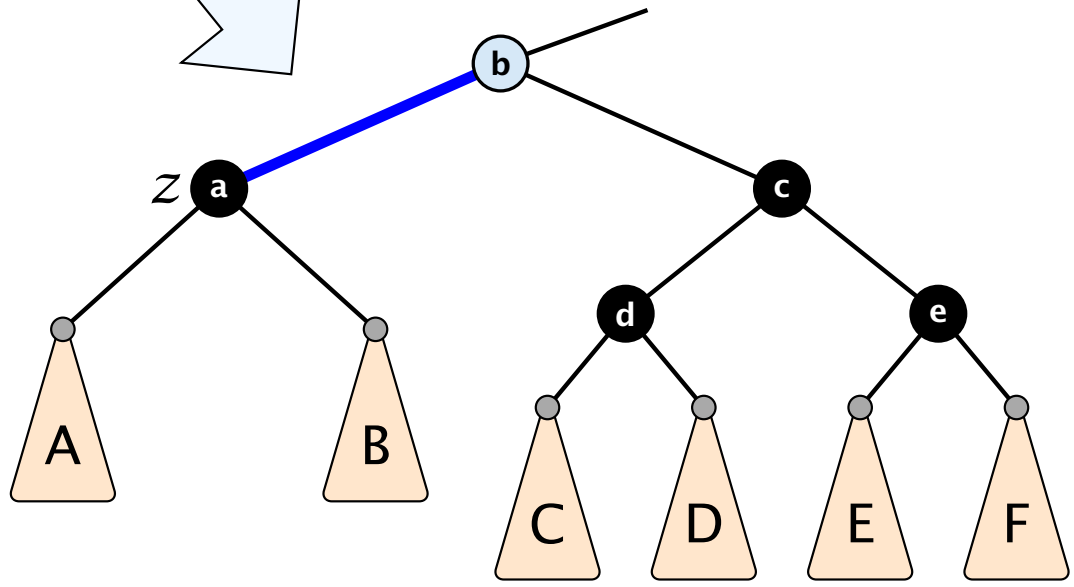

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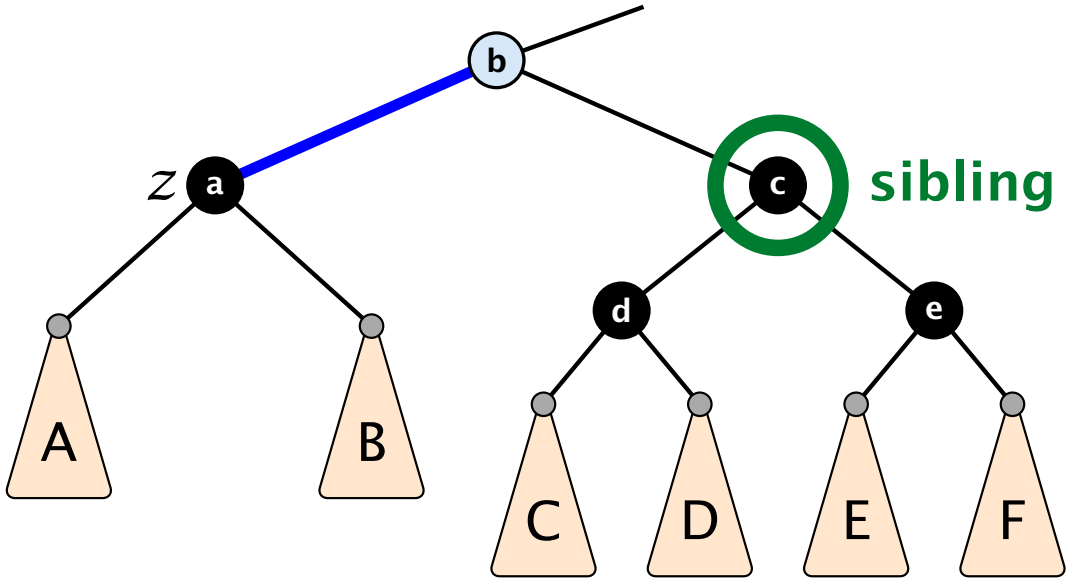

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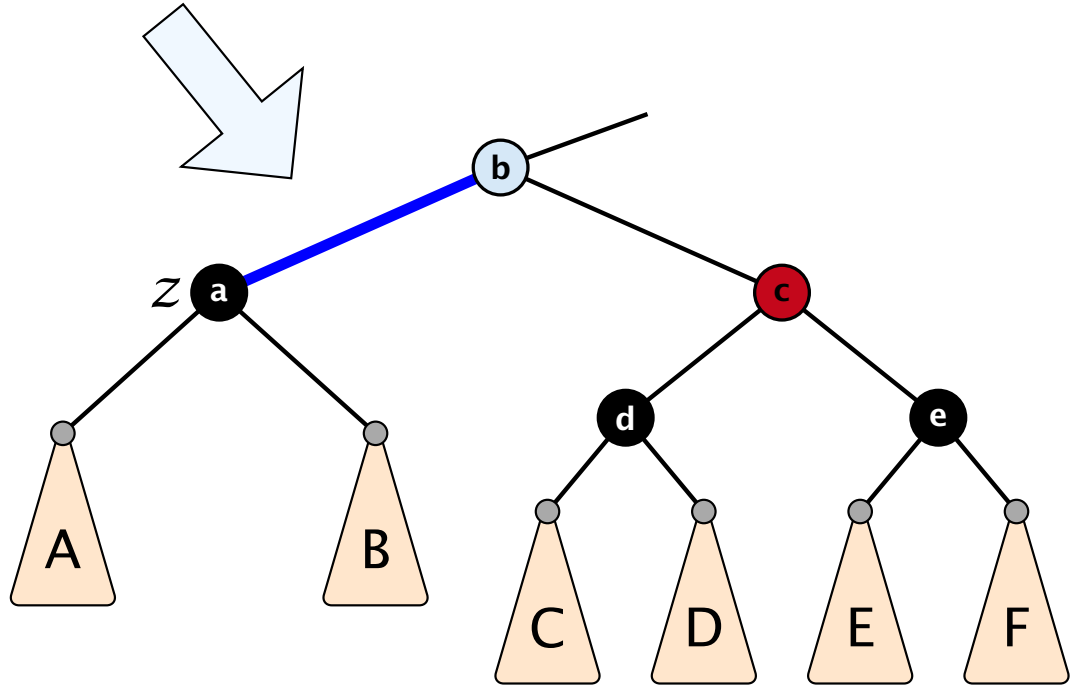

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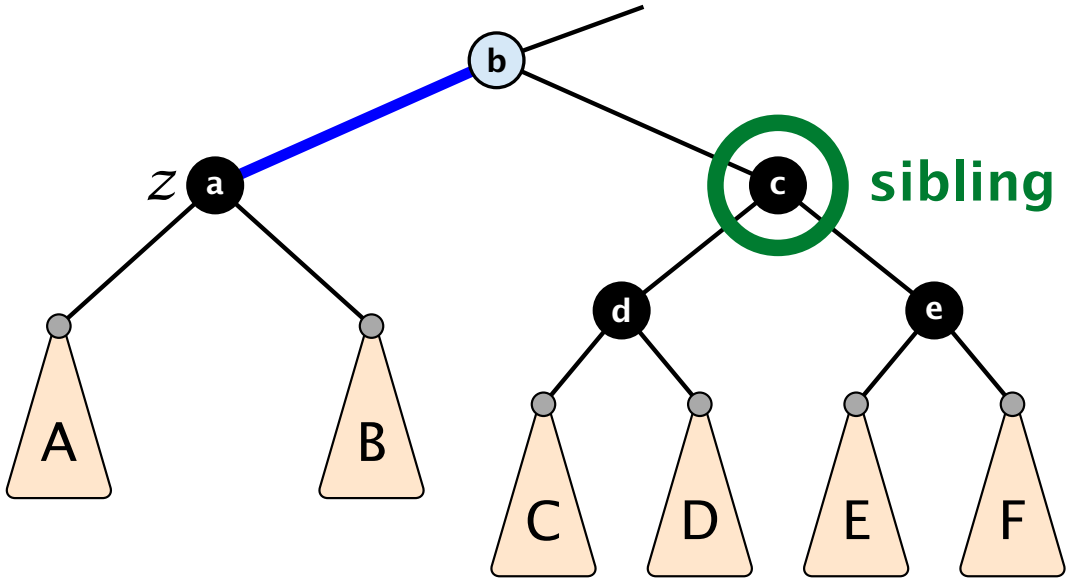

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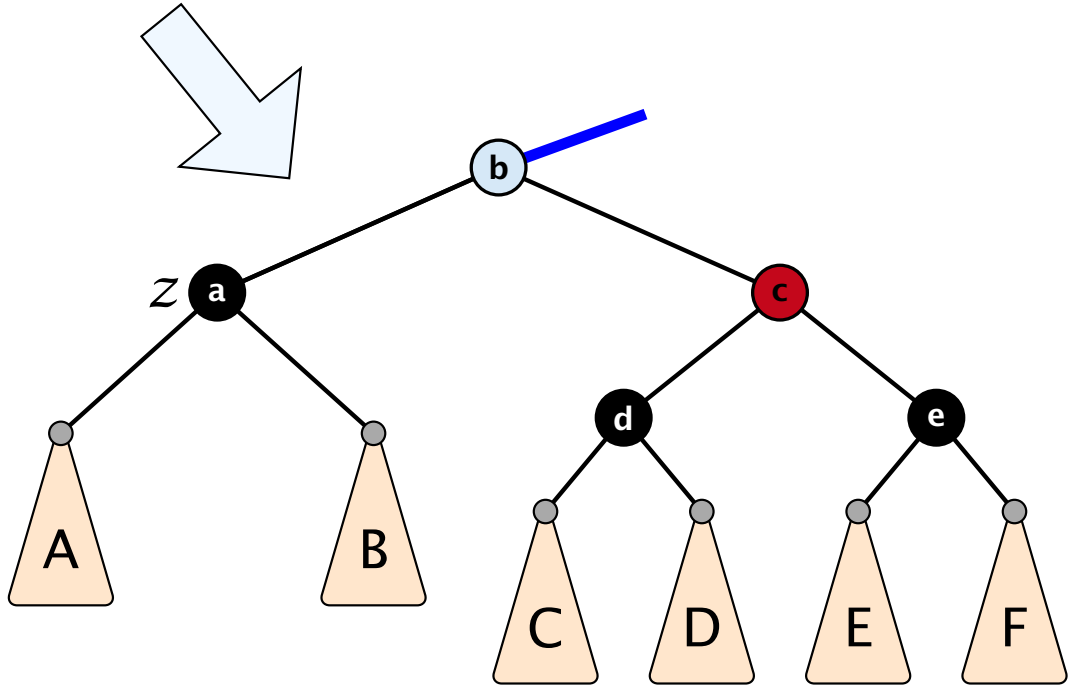

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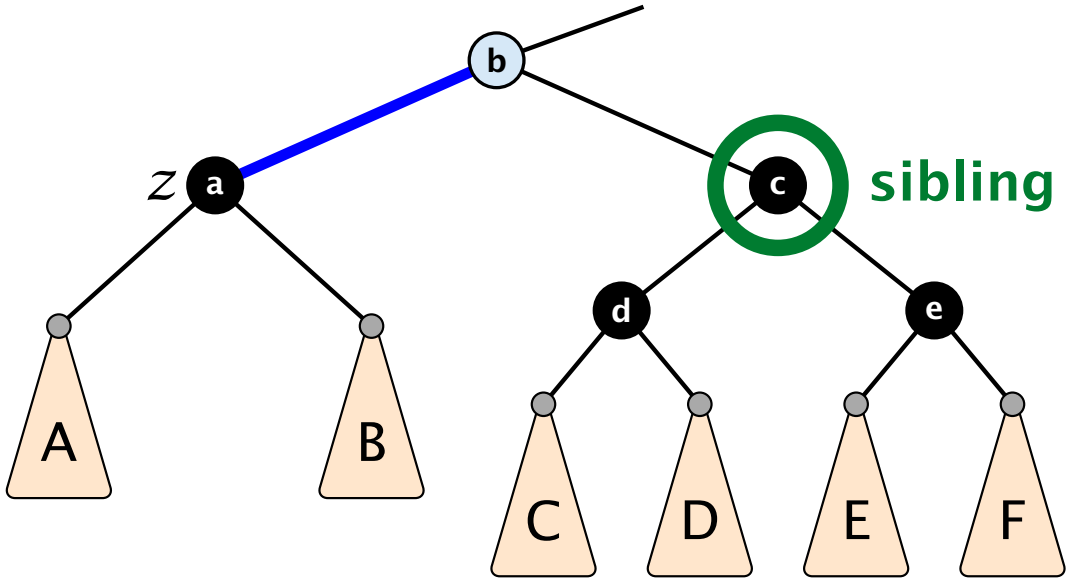

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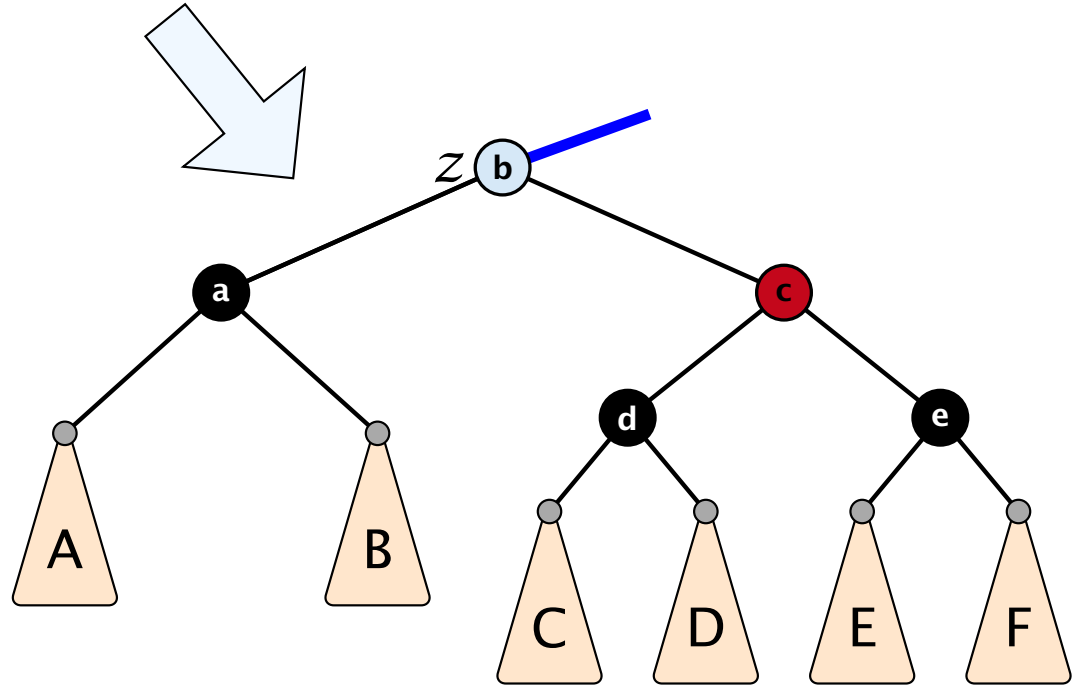

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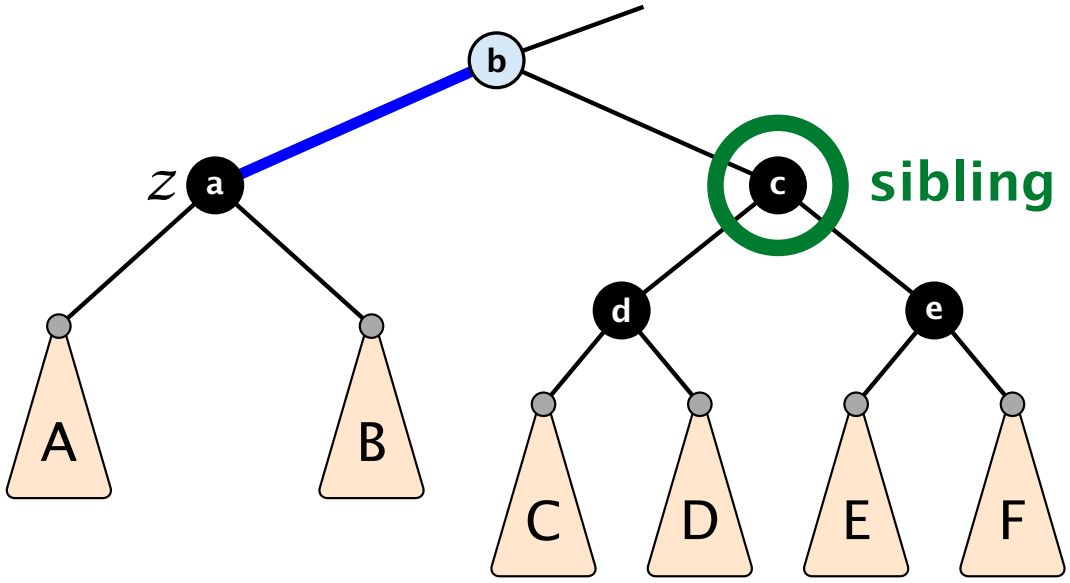

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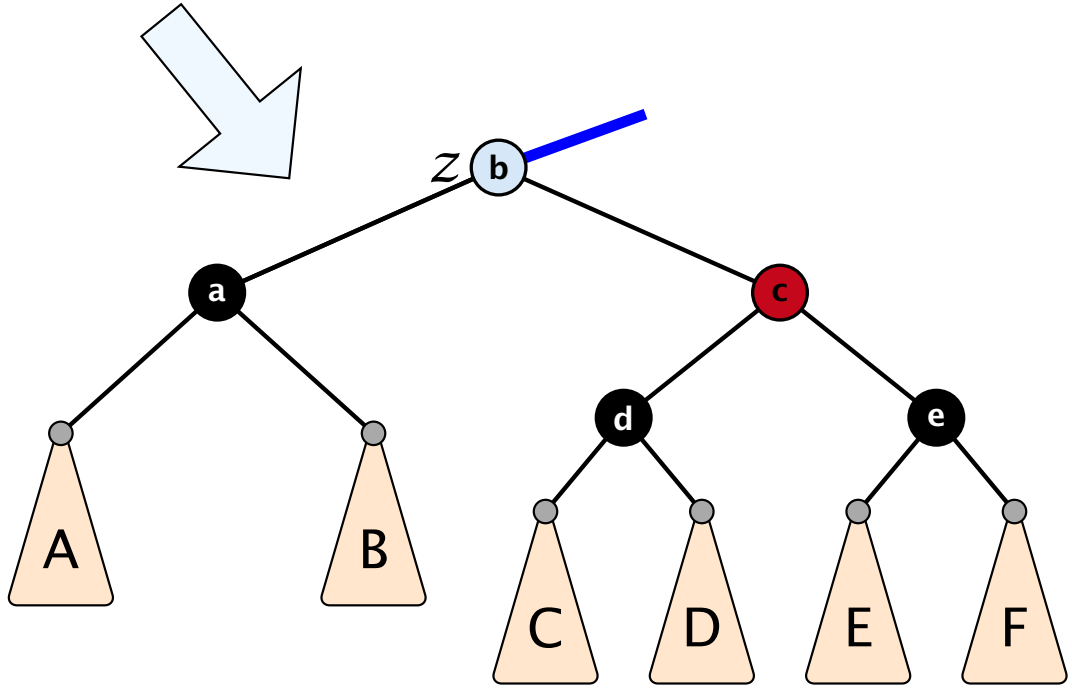

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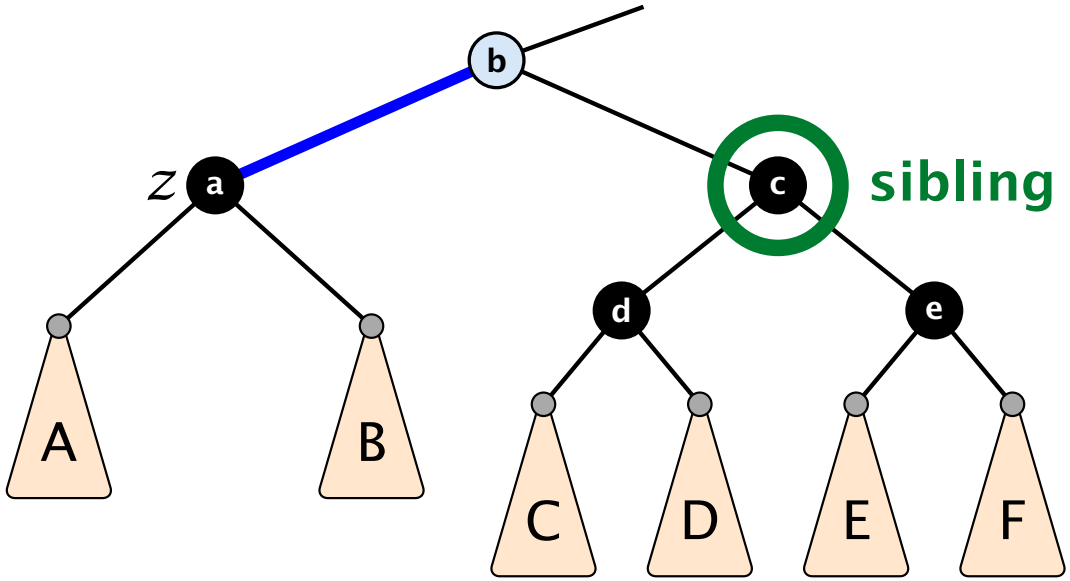

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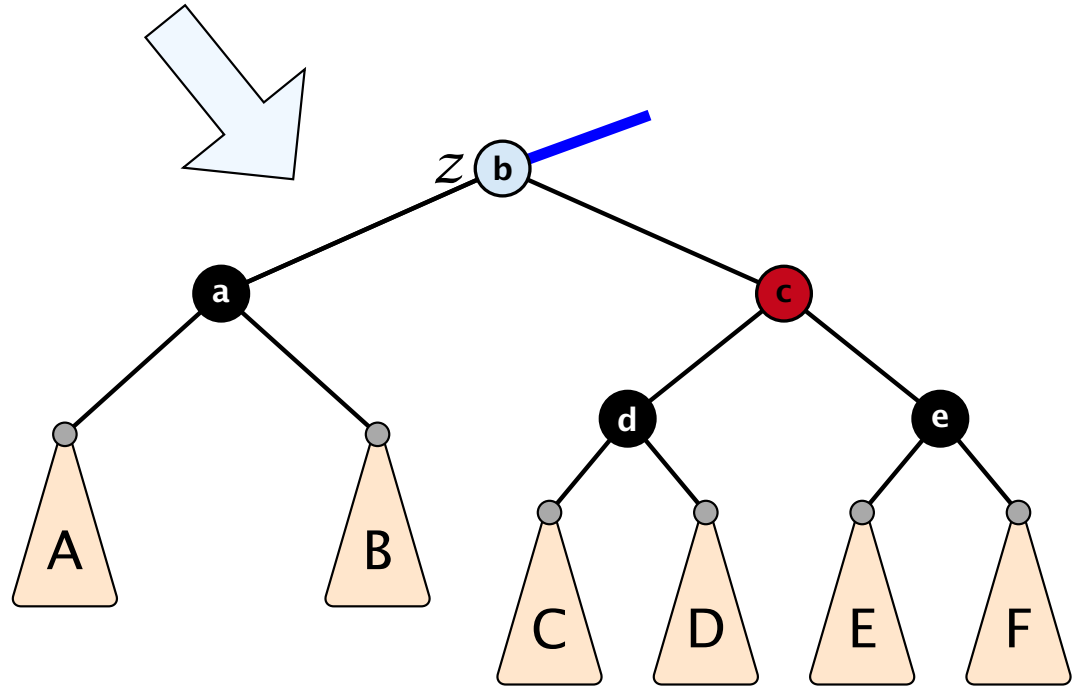
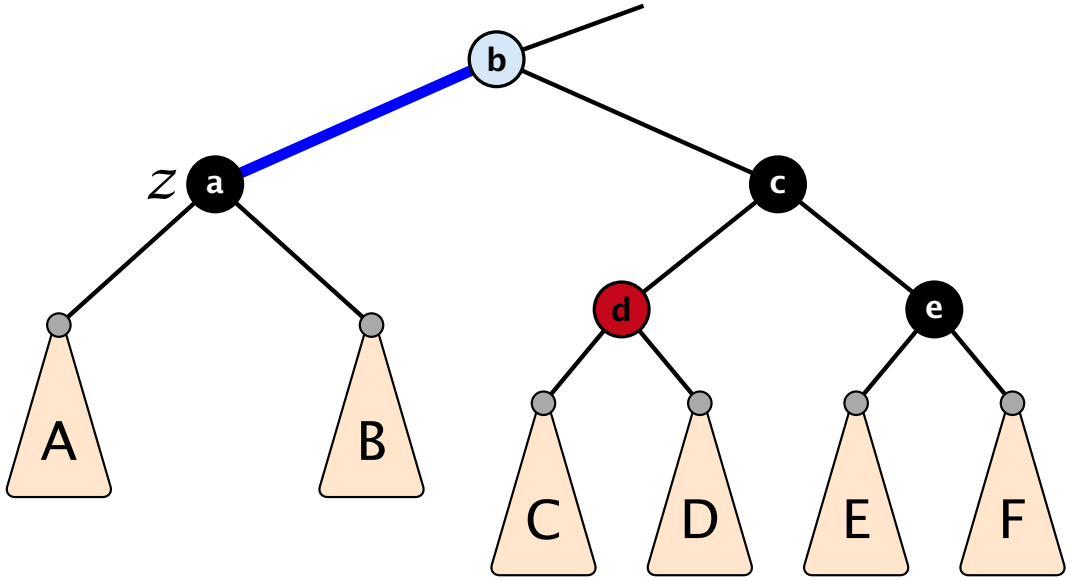


Case 3: Sibling black with one black child to the right

1. do a right-rotation at sibling
2. recolor c and d
3. new sibling is black with red right child (Case 4)



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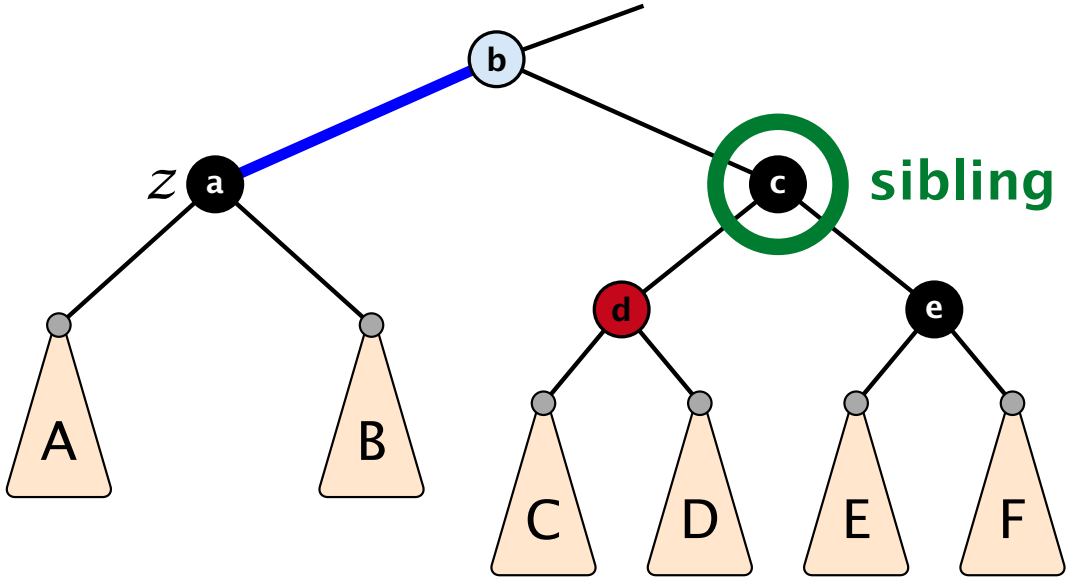

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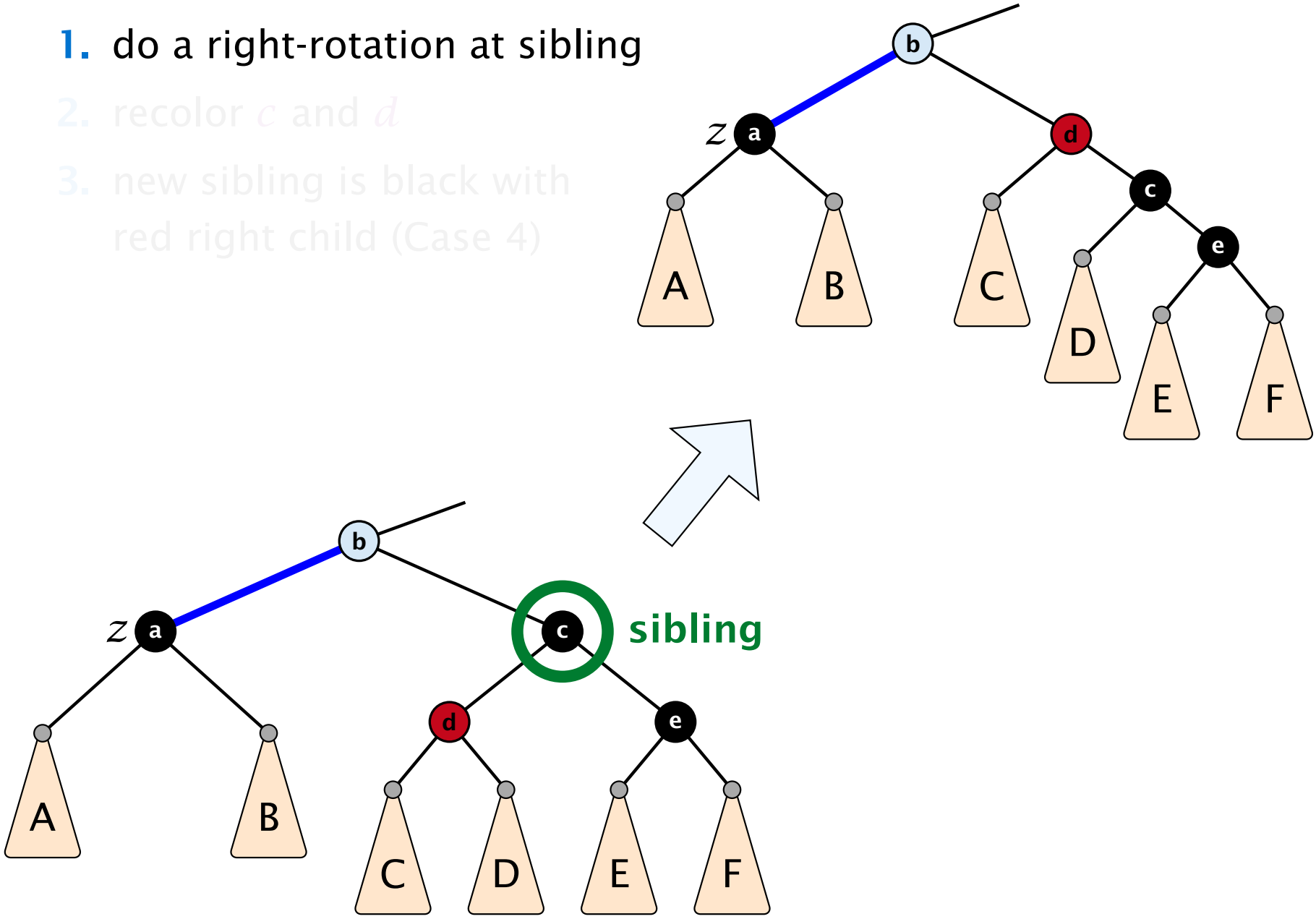

Case 3: Sibling black with one black child to the right

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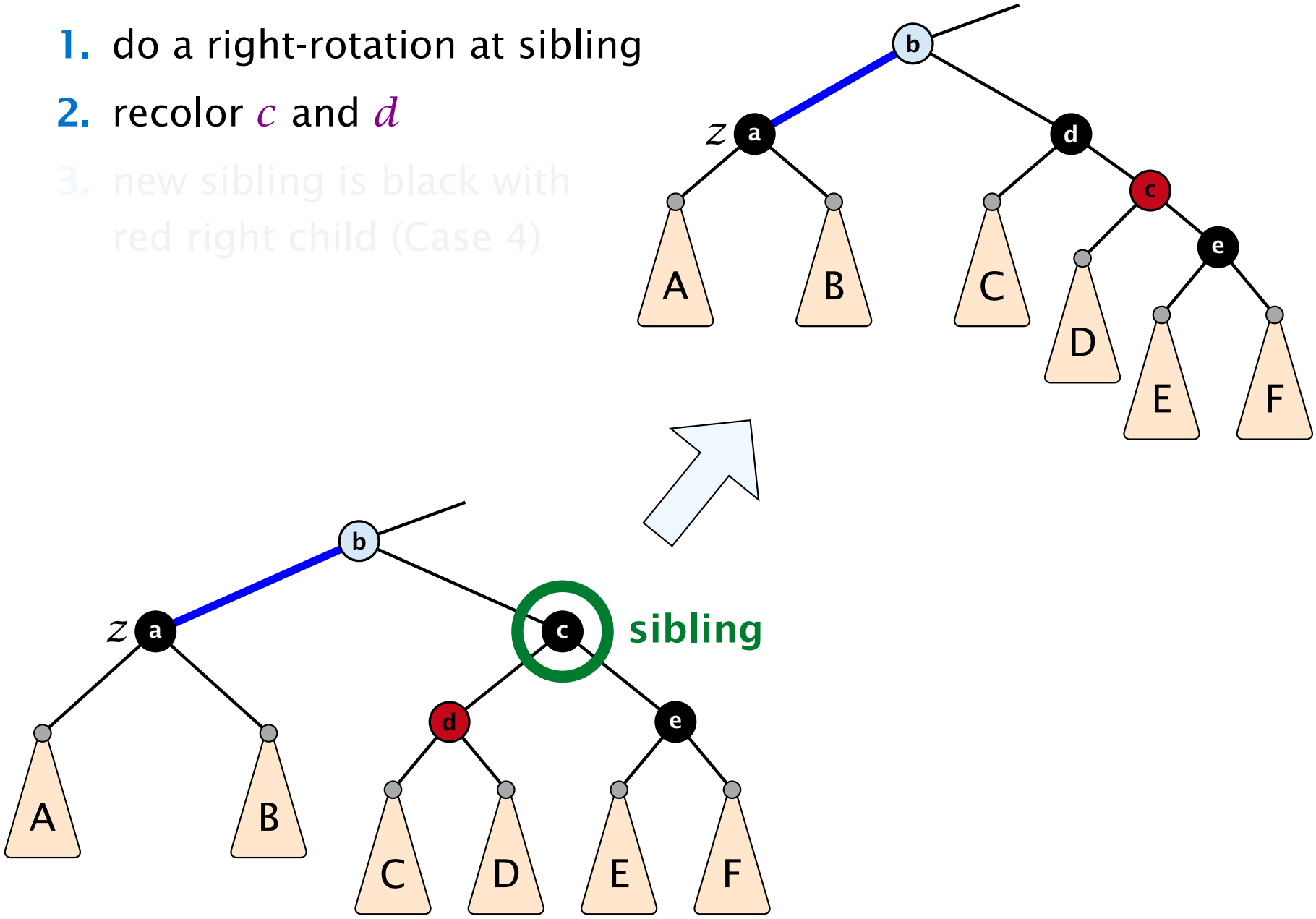

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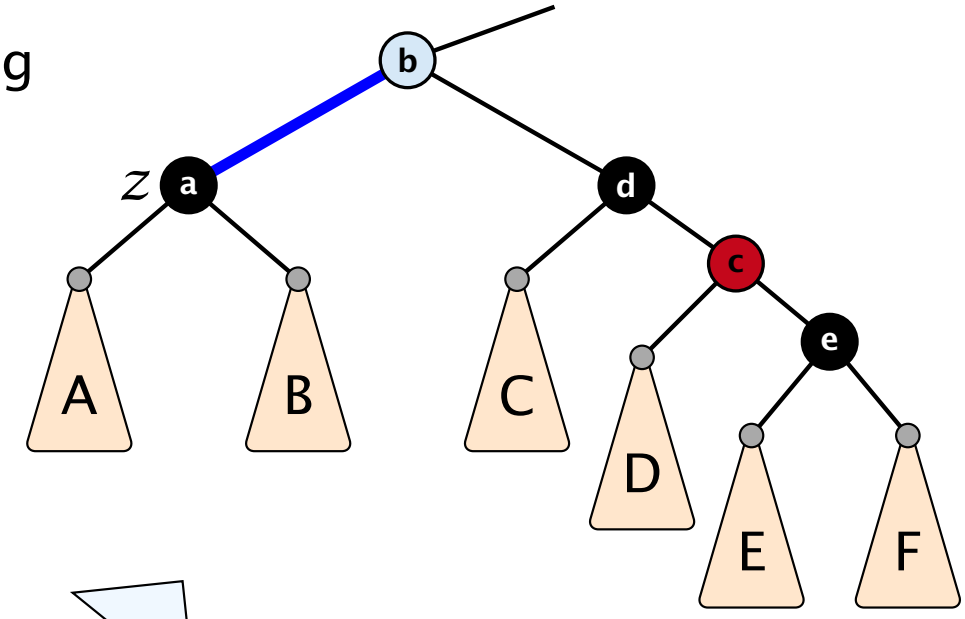
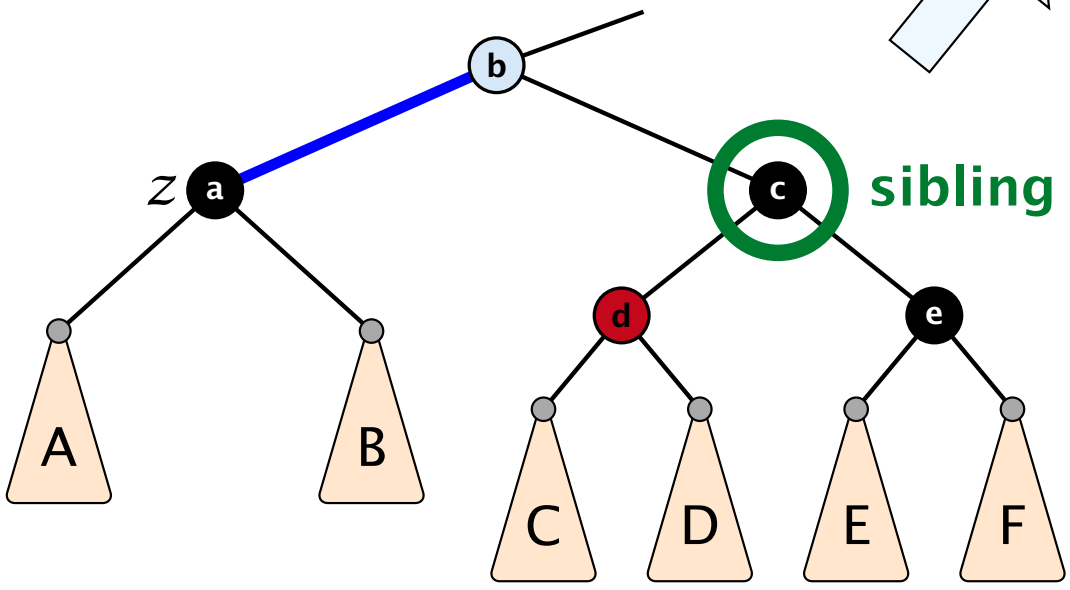
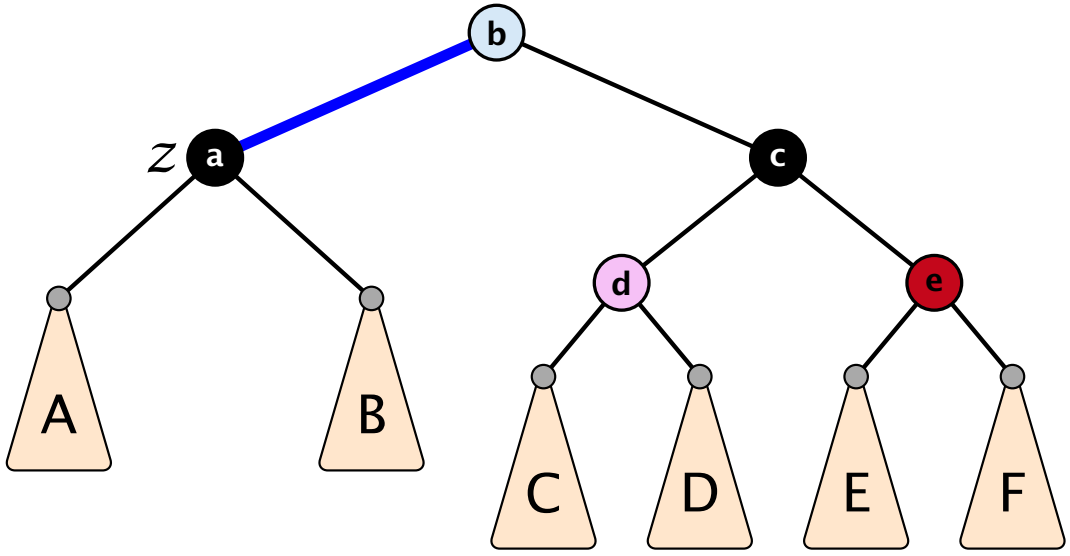


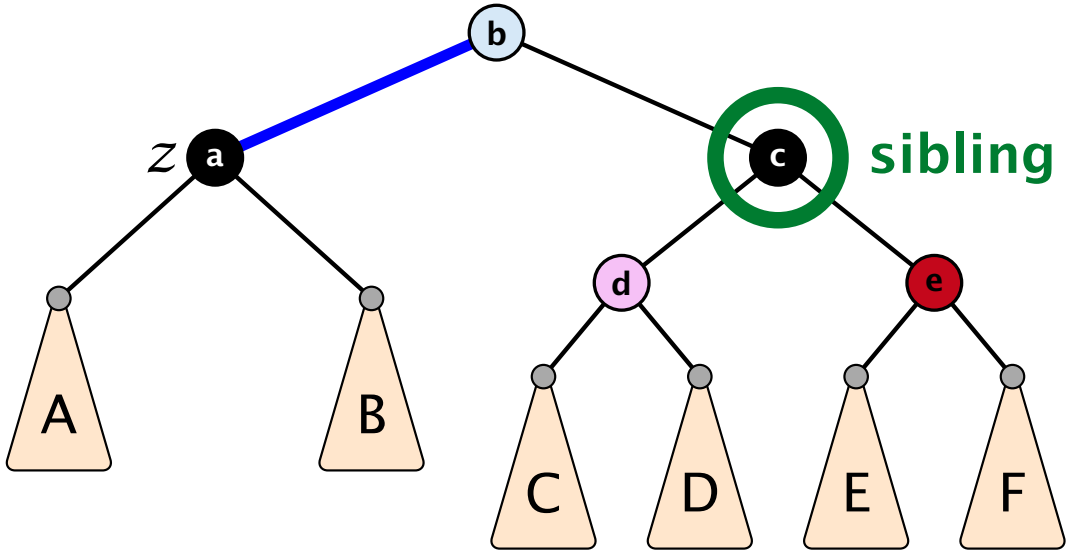


Case 4: Sibling is black with red right child



1. left-rotate around b
2. recolor nodes b, c, and e
3. remove the fake black unit
4. you have a valid red black tree

Case 4: Sibling is black with red right child



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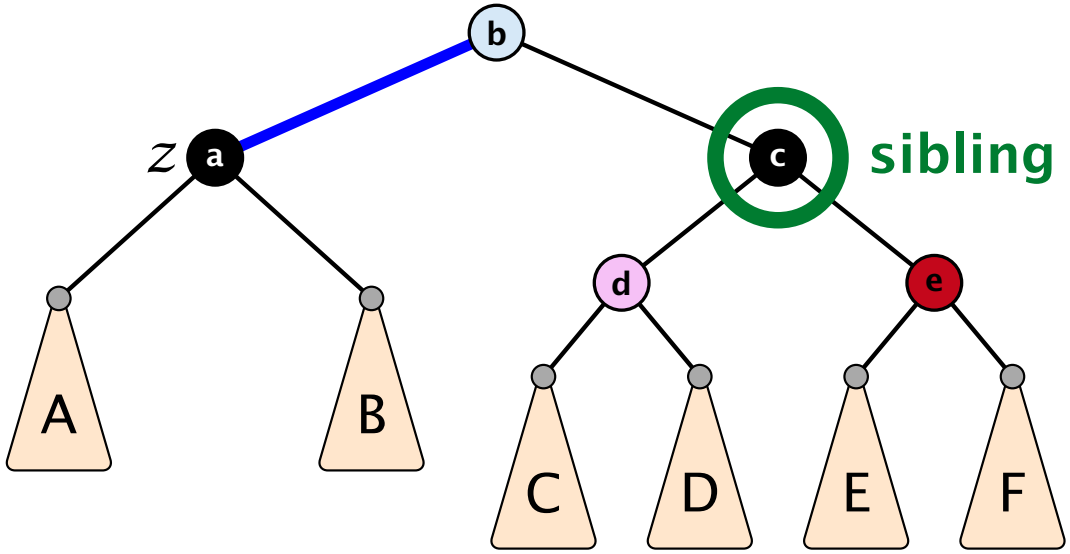
Case 4: Sibling is black with red right child



1. left-rotate around b
2. recolor nodes b, c, and e
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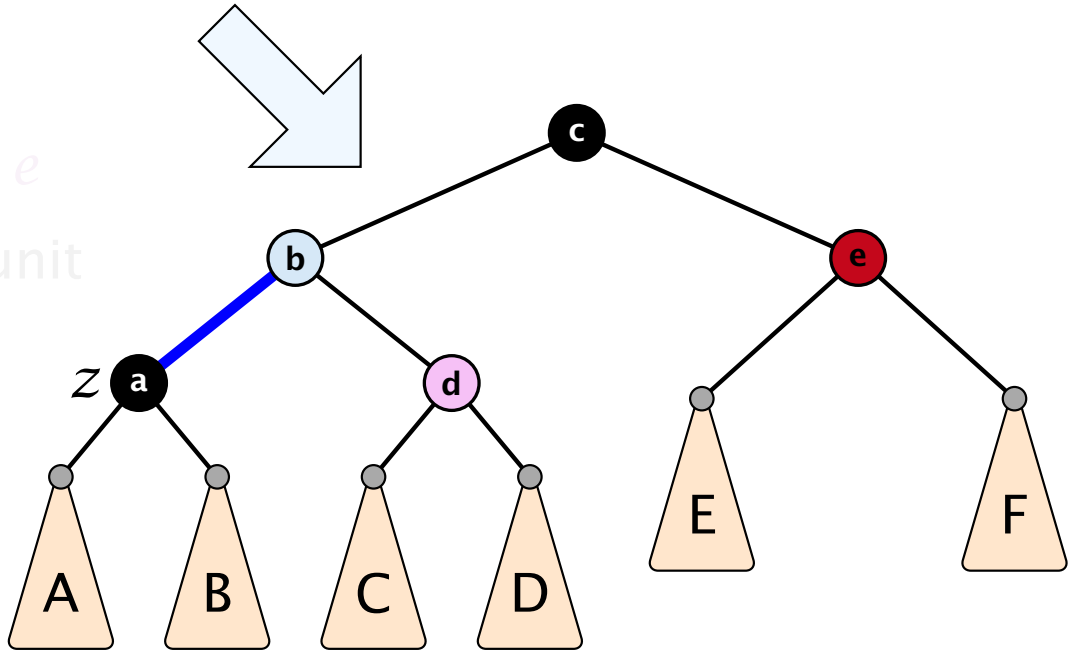
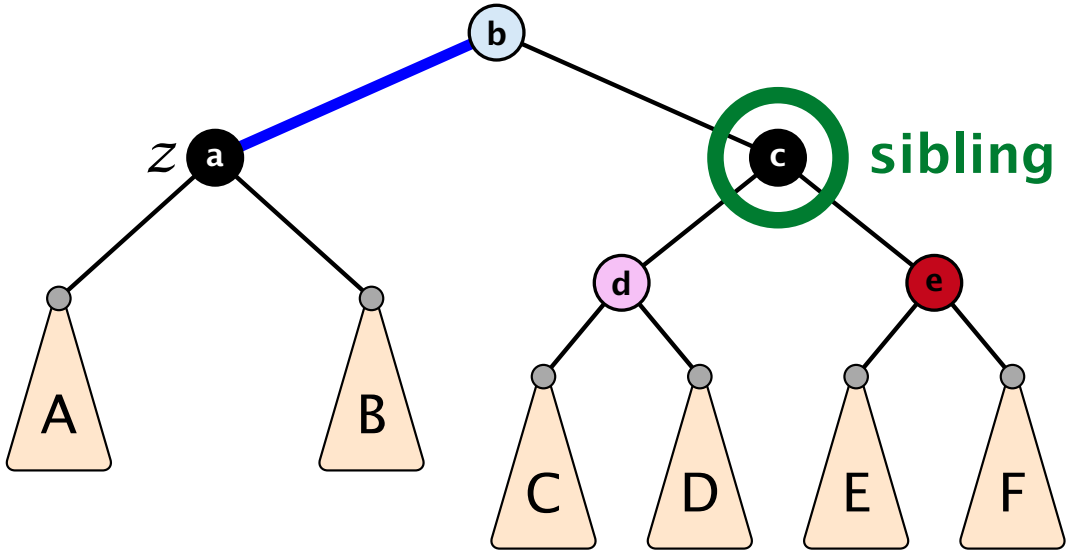
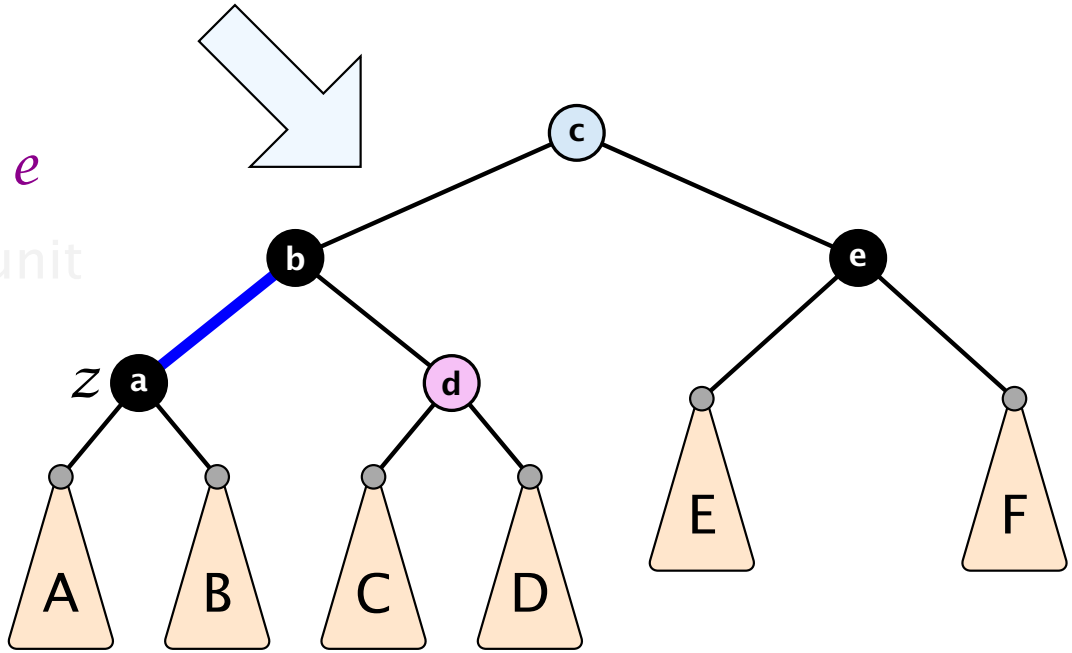


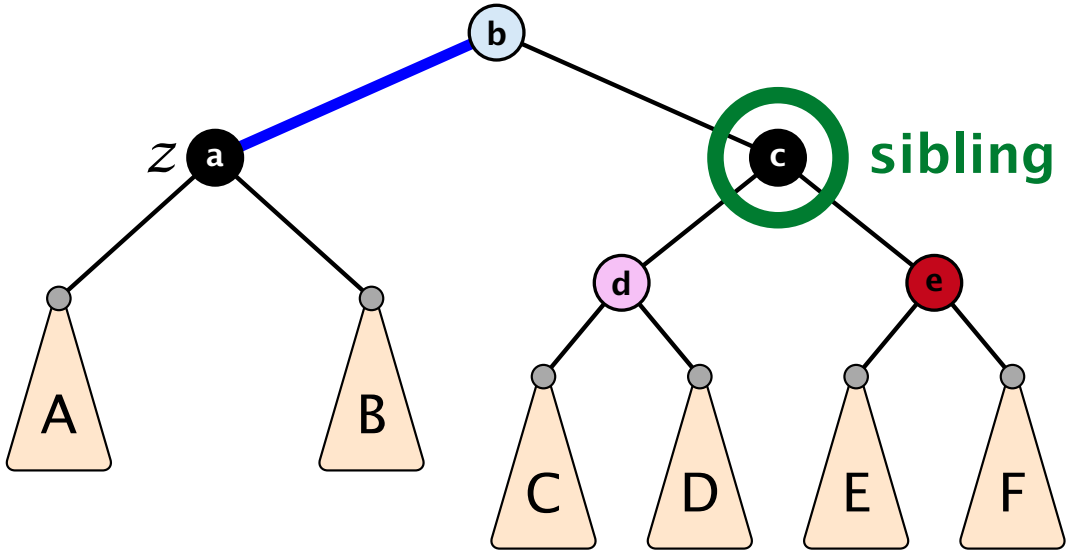
Case 4: Sibling is black with red right child



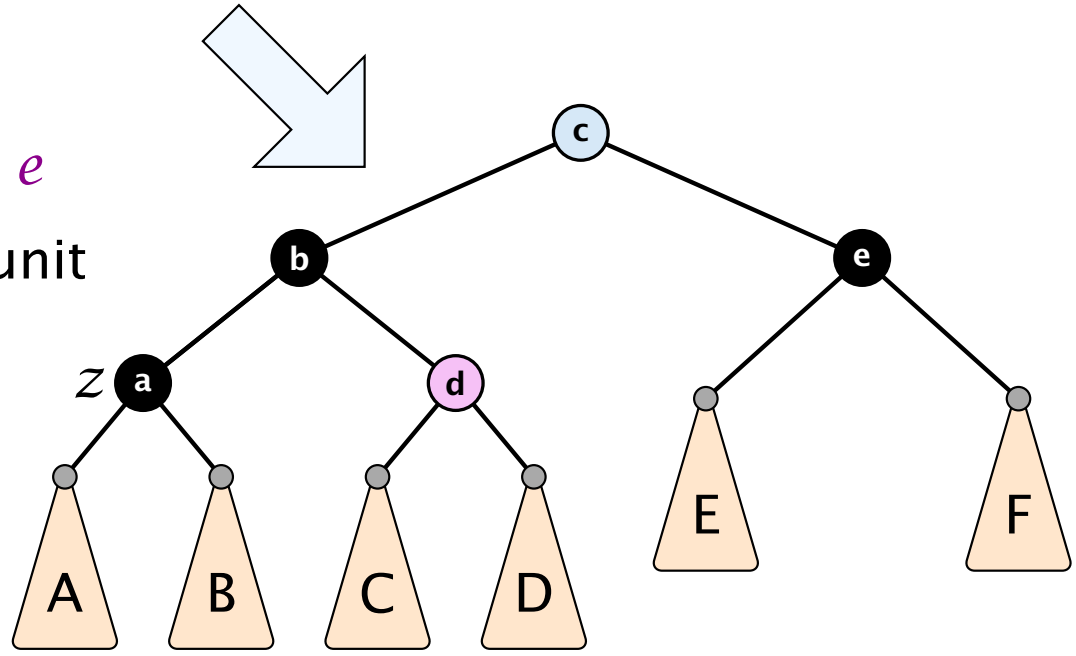
1. left-rotate around b
2. recolor nodes b, c, and e
3. remove the fake black unit
4. you have a valid red black tree



Case 4: Sibling is black with red right child



1. left-rotate around b
2. recolor nodes b, c, and e
3. remove the fake black unit
4. you have a valid red black tree



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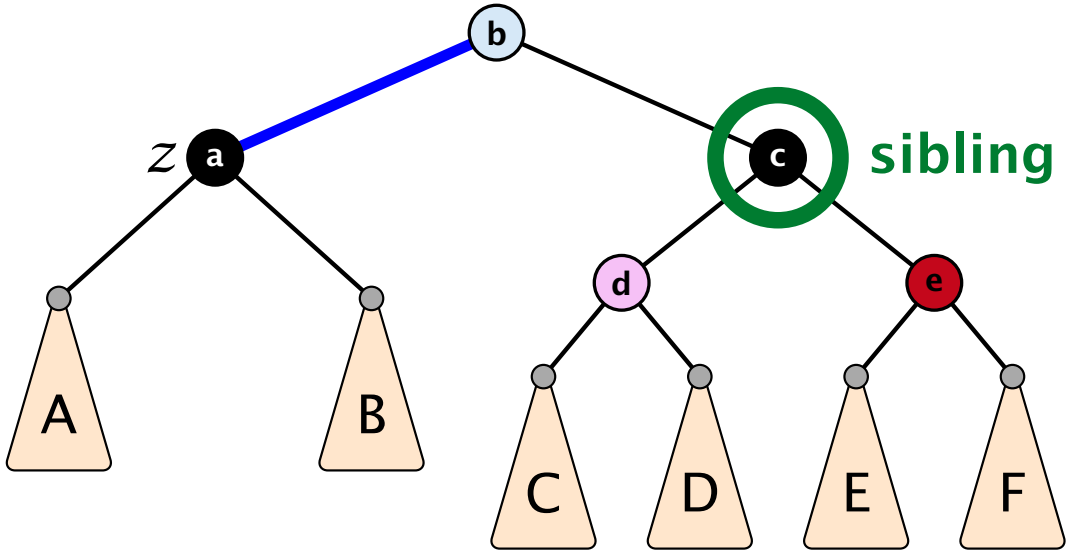

1. left-rotate around b
2. recolor nodes b, c, and e
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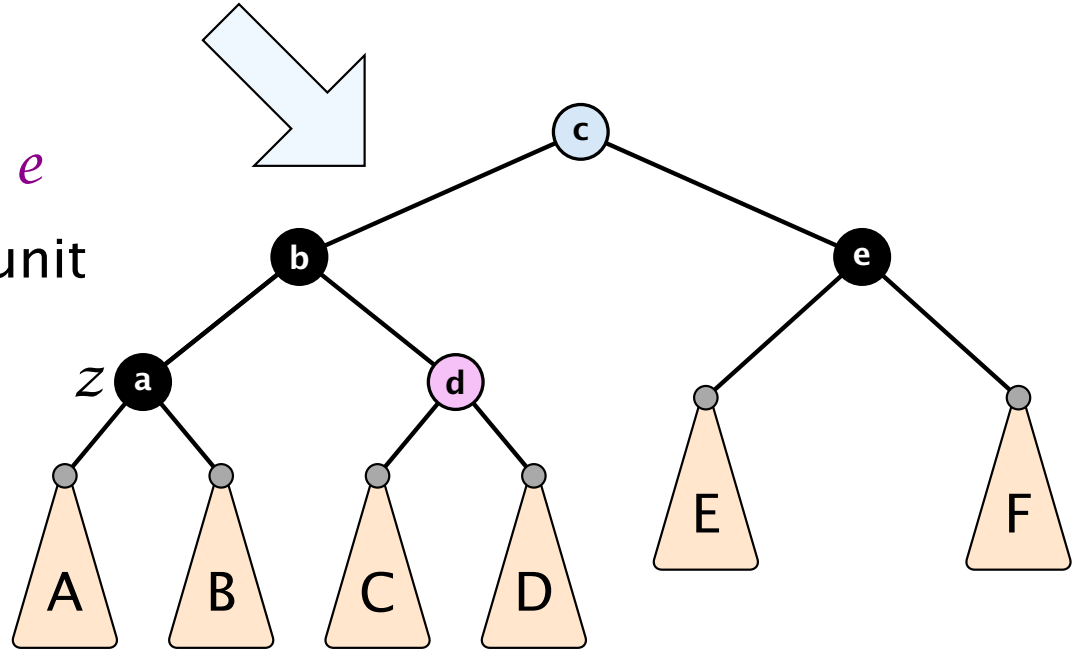

Running time:

- only Case 2 can repeat; but only h many steps, where h is the height of the tree
- Case 1 → Case 2 (special) → red black tree
  Case 1 → Case 3 → Case 4 → red black tree
  Case 1 → Case 4 → red black tree
- Case 3 → Case 4 → red black tree
- Case 4 → red black tree

Performing Case 2 at most $\mathcal{O}(\log n)$ times and every other step at most once, we get a red black tree. Hence, $\mathcal{O}(\log n)$ re-colorings and at most 3 rotations.

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- Case 4 → red black tree

Performing Case 2 at most $\mathcal{O}(\log n)$ times and every other step at most once, we get a red black tree. Hence, $\mathcal{O}(\log n)$ re-colorings and at most 3 rotations.

7.3 AVL-Trees

Definition 15

AVL-trees are binary search trees that fulfill the following balance condition. For every node v

$$|\text{height}(\text{left sub-tree}(v)) - \text{height}(\text{right sub-tree}(v))| \leq 1 \ .$$

Lemma 16

An AVL-tree of height h contains at least $F_{h+2} - 1$ and at most $2^h - 1$ internal nodes, where F_n is the n-th Fibonacci number ($F_0 = 0$, $F_1 = 1$), and the height is the maximal number of edges from the root to an (empty) dummy leaf.

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AVL trees

Proof.

The upper bound is clear, as a binary tree of height h can only contain

$$\sum_{j=0}^{h-1} 2^j = 2^h - 1$$

internal nodes.

AVL trees

Proof (cont.)

Induction (base cases):

1. an AVL-tree of height $h = 1$ contains at least one internal node, $1 \geq F_3 - 1 = 2 - 1 = 1$.
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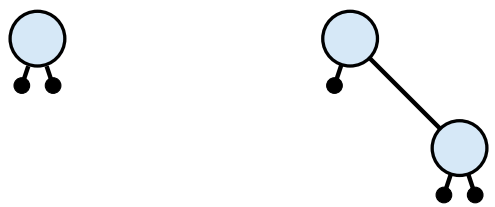

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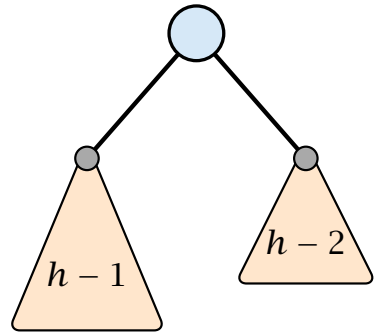

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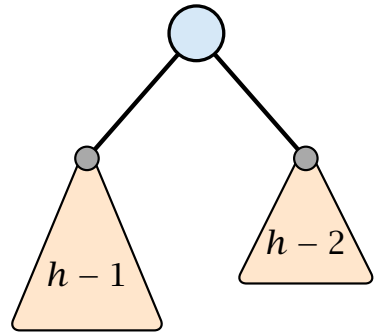

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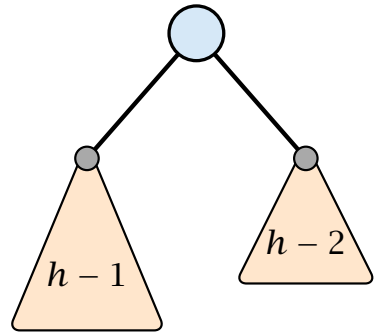

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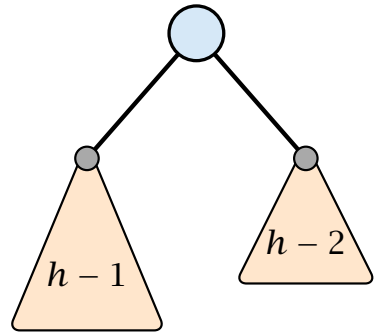

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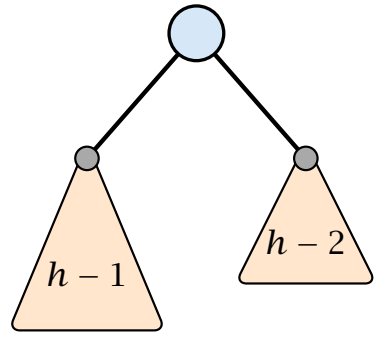

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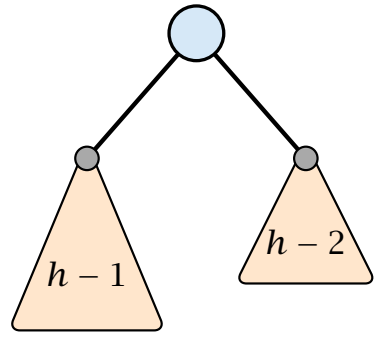

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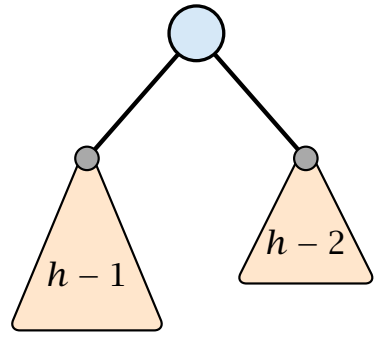

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$$g_h := 1 + \text{minimal size of AVL-tree of height } h \ .$$

Then

$$\begin{aligned} g_1 &= 2 && = F_3 \\ g_2 &= 3 && = F_4 \\ g_h - 1 &= 1 + g_{h-1} - 1 + g_{h-2} - 1\,, && \text{hence} \\ g_h &= g_{h-1} + g_{h-2} && = F_{h+2} \end{aligned}$$

7.3 AVL-Trees

An AVL-tree of height h contains at least $F_{h+2} - 1$ internal nodes. Since

$$n + 1 \geq F_{h+2} = \Omega\left(\left(\frac{1+\sqrt{5}}{2}\right)^h\right) \ ,$$

we get

$$n \geq \Omega\left(\left(\frac{1+\sqrt{5}}{2}\right)^h\right) \ ,$$

and, hence, $h = \mathcal{O}(\log n)$.

7.3 AVL-Trees

We need to maintain the balance condition through rotations.

For this we store in every internal tree-node v the balance of the node. Let v denote a tree node with left child c_ℓ and right child c_r.

$$\text{balance}[v] := \text{height}(T_{c_\ell}) - \text{height}(T_{c_r}) \ ,$$

where T_{c_ℓ} and T_{c_r}, are the sub-trees rooted at c_ℓ and c_r, respectively.

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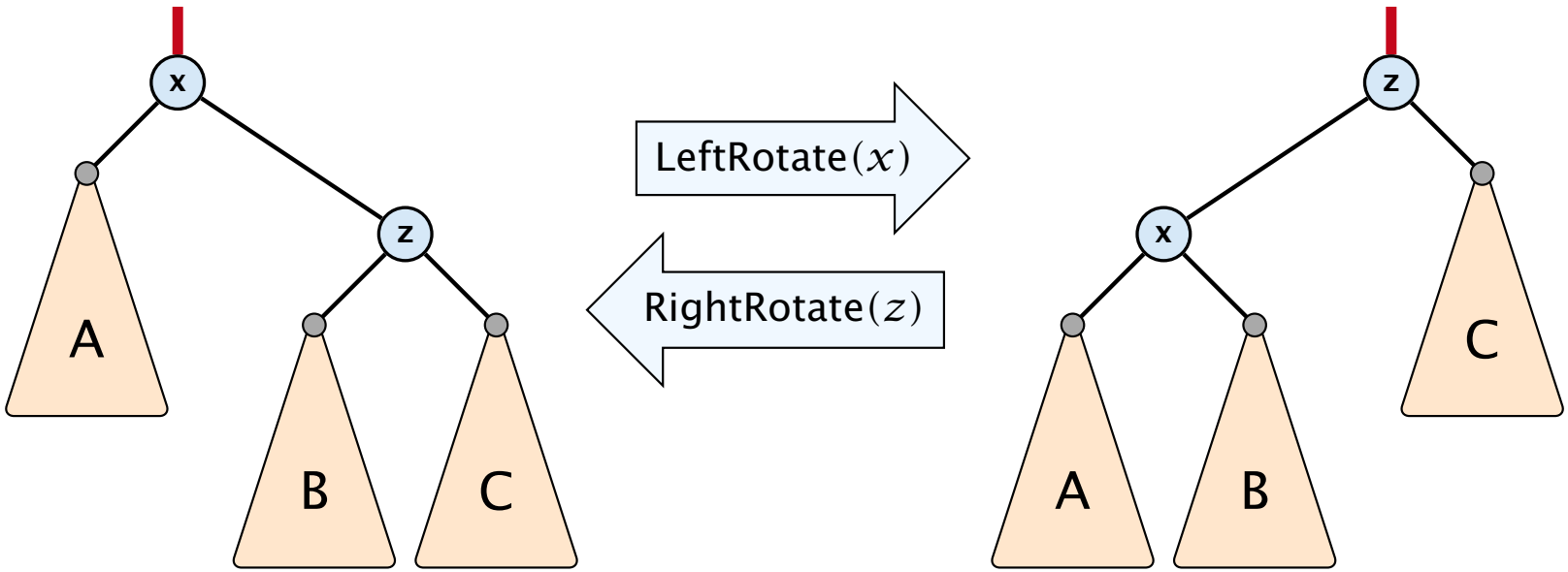
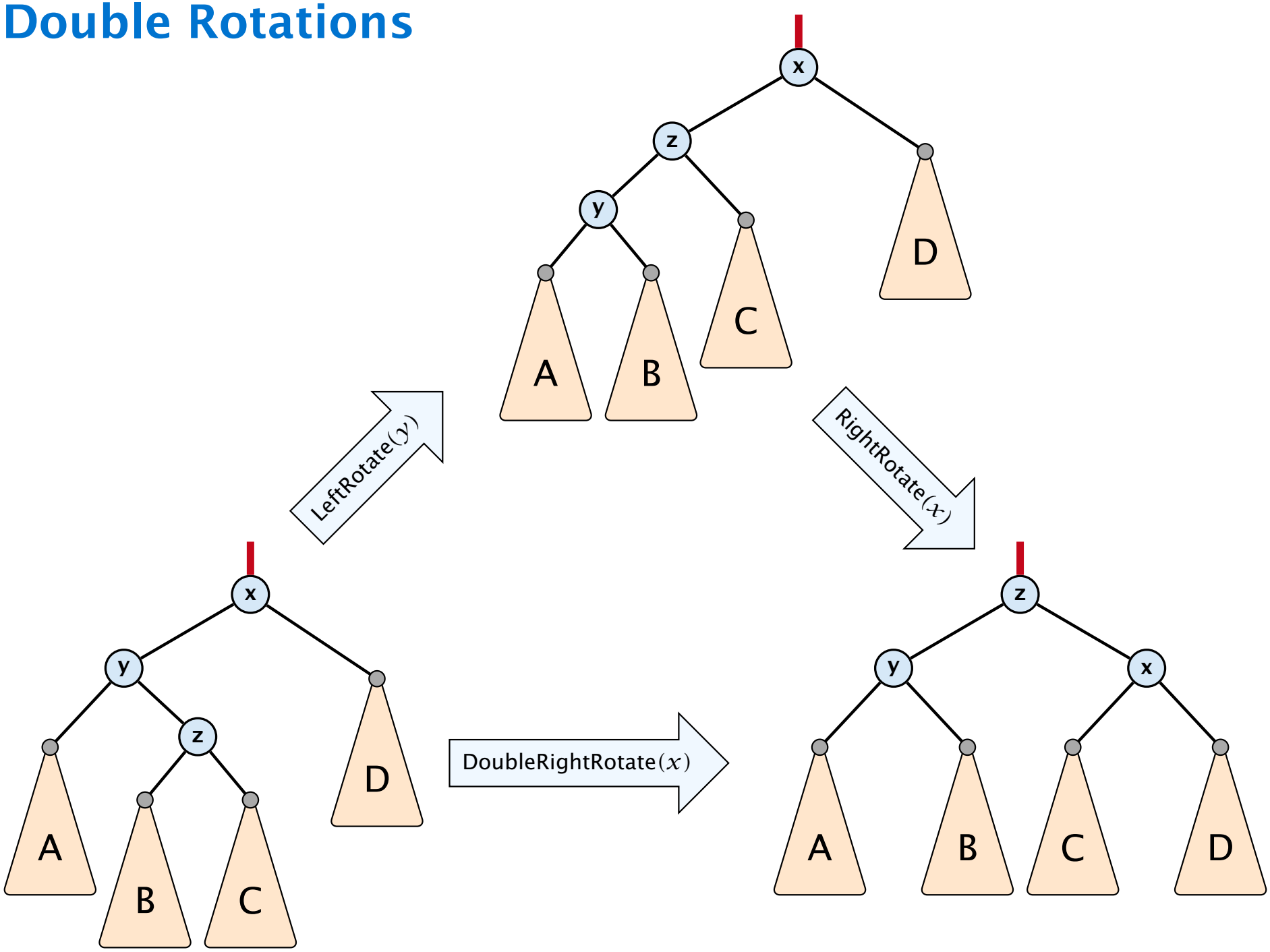
Rotations

The properties will be maintained through rotations:



Double Rotations



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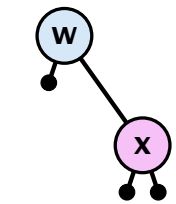
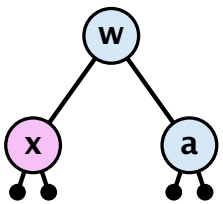


$\mathrm{bal}(w) = -1$



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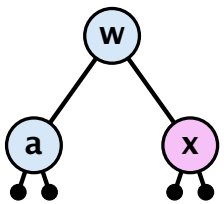

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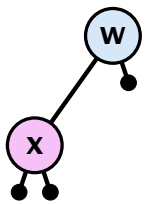

$\mathrm{bal}(w) = 1$

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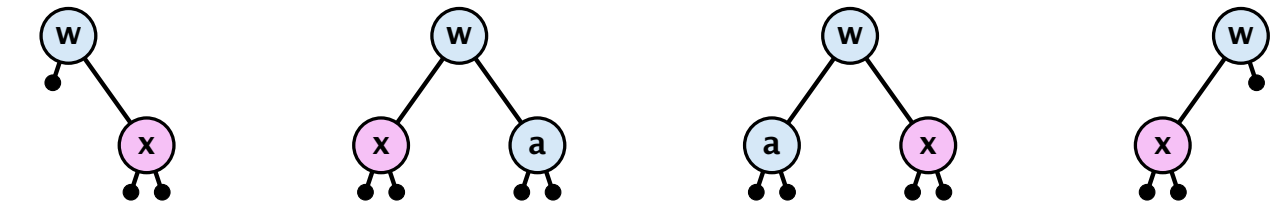

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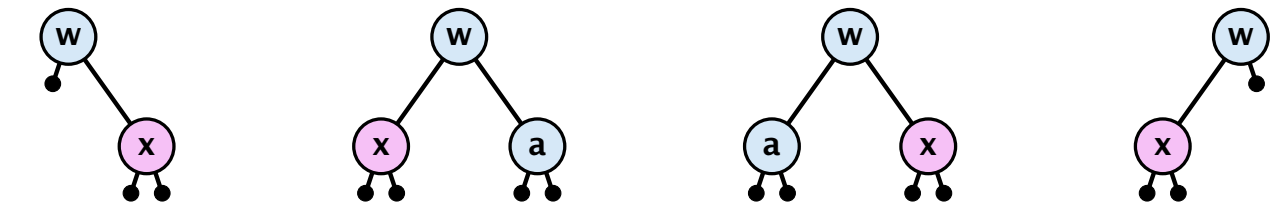

- If $\mathrm{bal}[w] \neq 0$, T_w has changed height; the balance-constraint may be violated at ancestors of w.
- Call AVL-fix-up-insert($\mathrm{parent}[w]$) to restore the balance-condition.

AVL-trees: Insert

Invariant at the beginning of AVL-fix-up-insert(v):

1. The balance constraints hold at all descendants of v.
2. A node has been inserted into T_c, where c is either the right or left child of v.
3. T_c has increased its height by one (otw. we would already have aborted the fix-up procedure).
4. The balance at node c fulfills $\text{balance}[c] \in \{-1, 1\}$. This holds because if the balance of c is 0, then T_c did not change its height, and the whole procedure would have been aborted in the previous step.

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AVL-trees: Insert

Algorithm 11 AVL-fix-up-insert(v)

```
1: if balance[v] ∈ {−2, 2} then DoRotationInsert(v);
2: if balance[v] ∈ {0} return;
3: AVL-fix-up-insert(parent[v]);
```

We will show that the above procedure is correct, and that it will do at most one rotation.

AVL-trees: Insert

Algorithm 12 DoRotationInsert(v)

```
 1: if balance[v] = −2 then // insert in right sub-tree
 2:         if balance[right[v]] = −1 then
 3:                 LeftRotate(v);
 4:         else
 5:                 DoubleLeftRotate(v);
 6: else // insert in left sub-tree
 7:         if balance[left[v]] = 1 then
 8:                 RightRotate(v);
 9:         else
10:                 DoubleRightRotate(v);
```

AVL-trees: Insert

It is clear that the invariant for the fix-up routine holds as long as no rotations have been done.

We have to show that after doing one rotation **all** balance constraints are fulfilled.

We show that after doing a rotation at v:

- v fulfills balance condition.
- All children of v still fulfill the balance condition.
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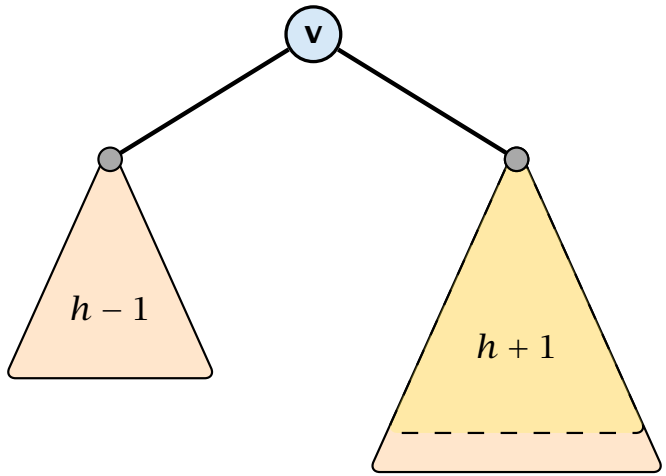

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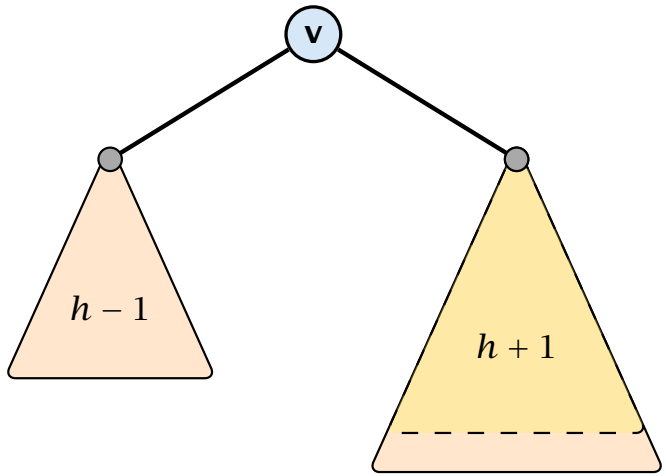

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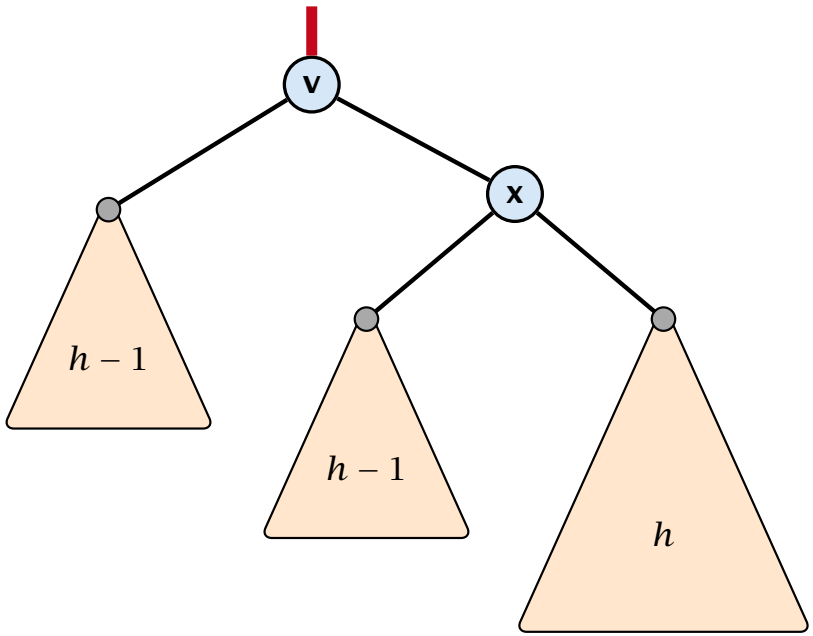

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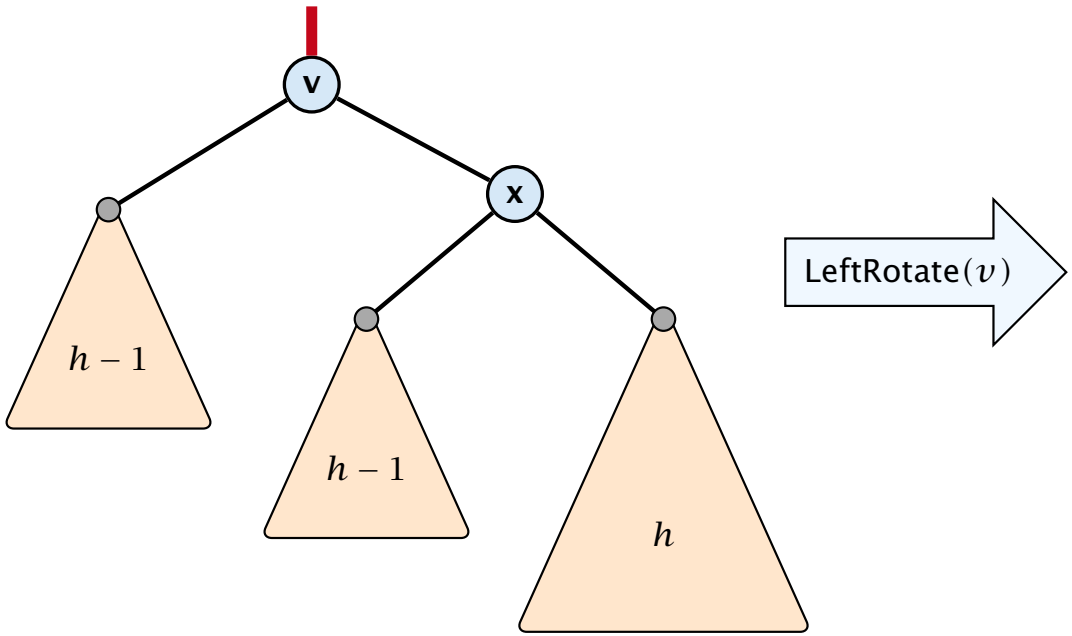

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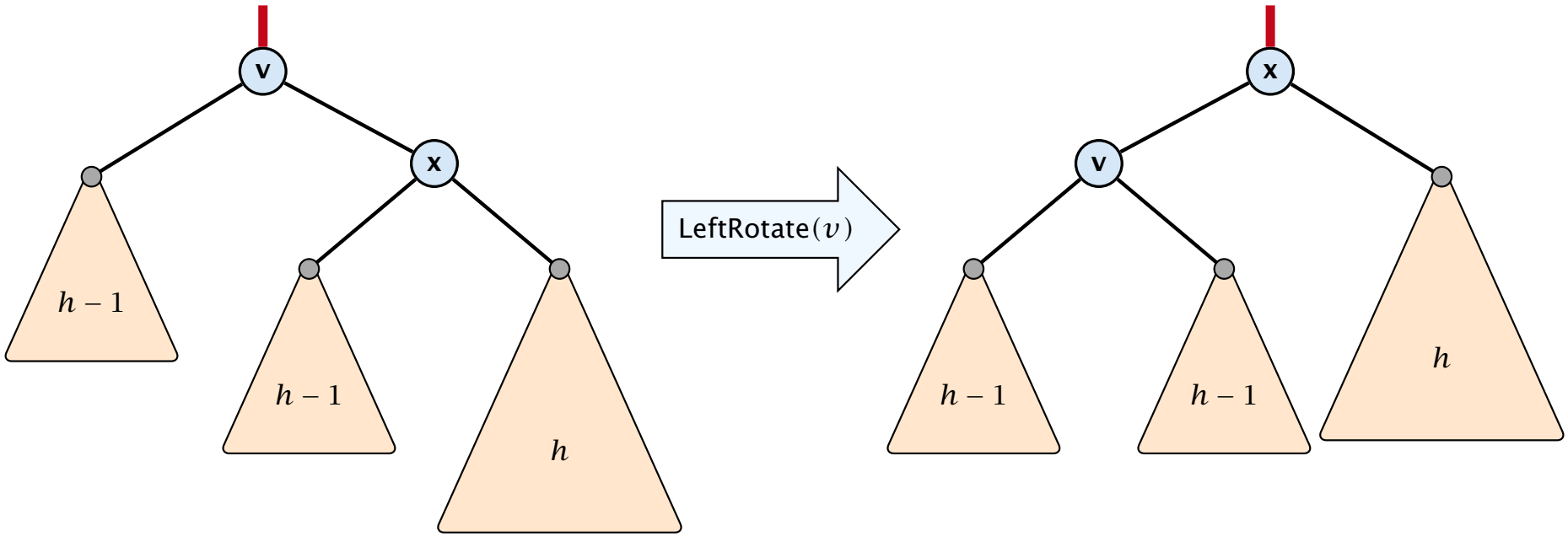

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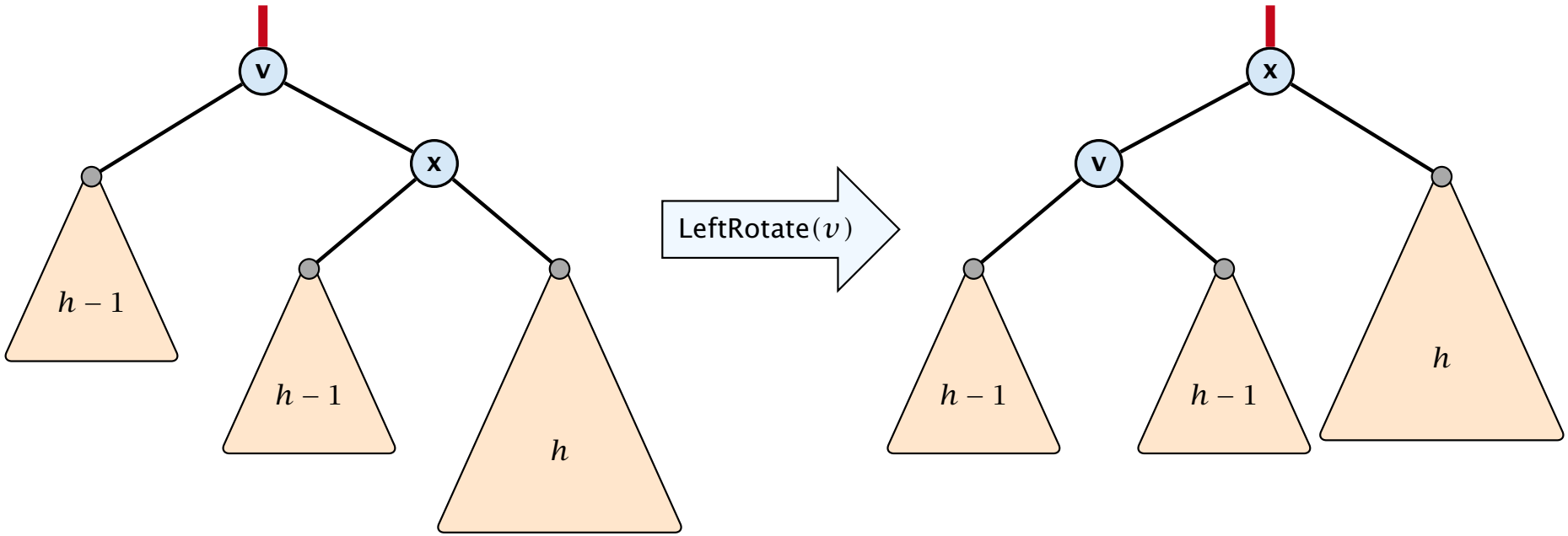

Now, the subtree has height $h + 1$ as before the insertion.

Hence, we do not need to continue.

Case 2: balance[right[v]] = 1

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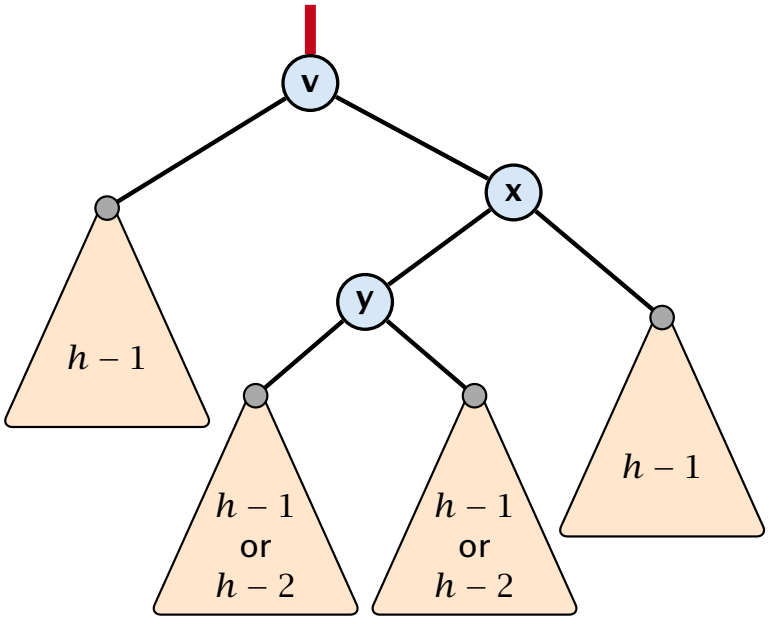

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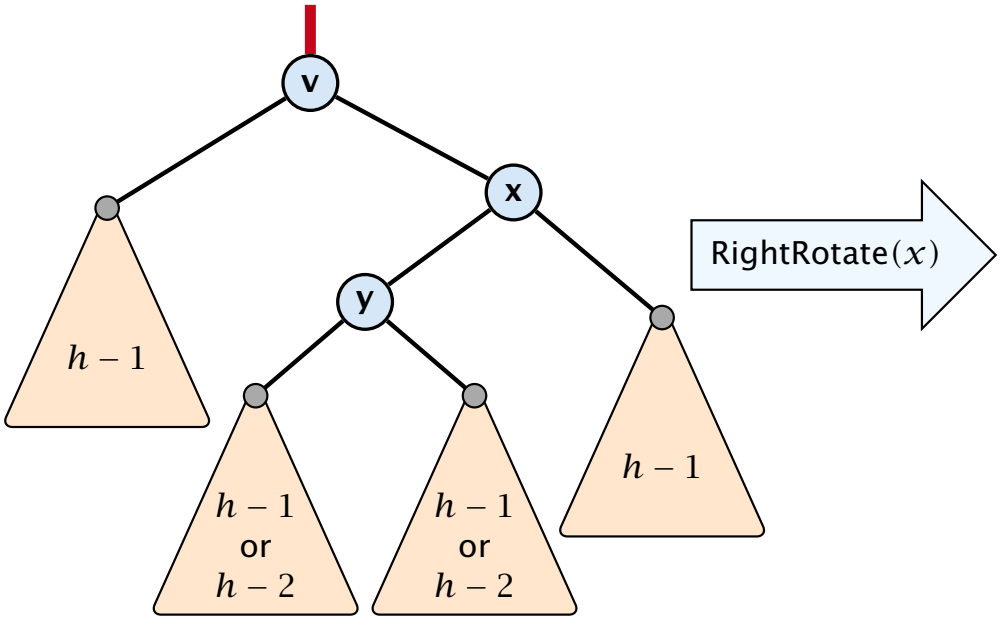

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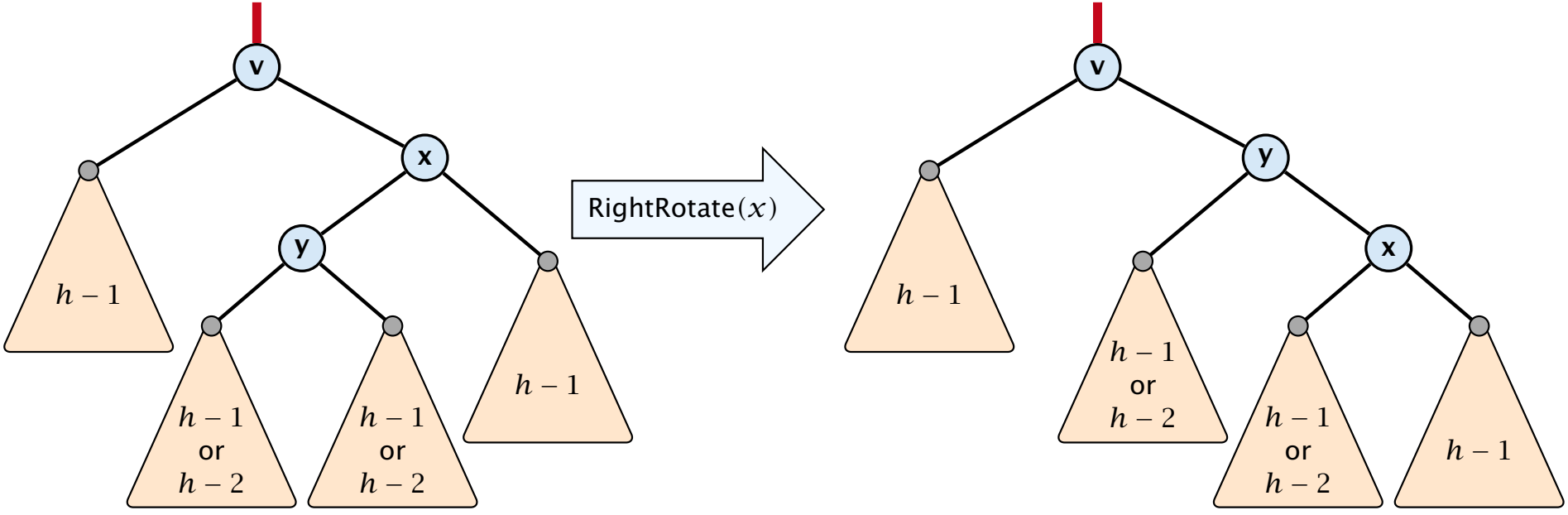

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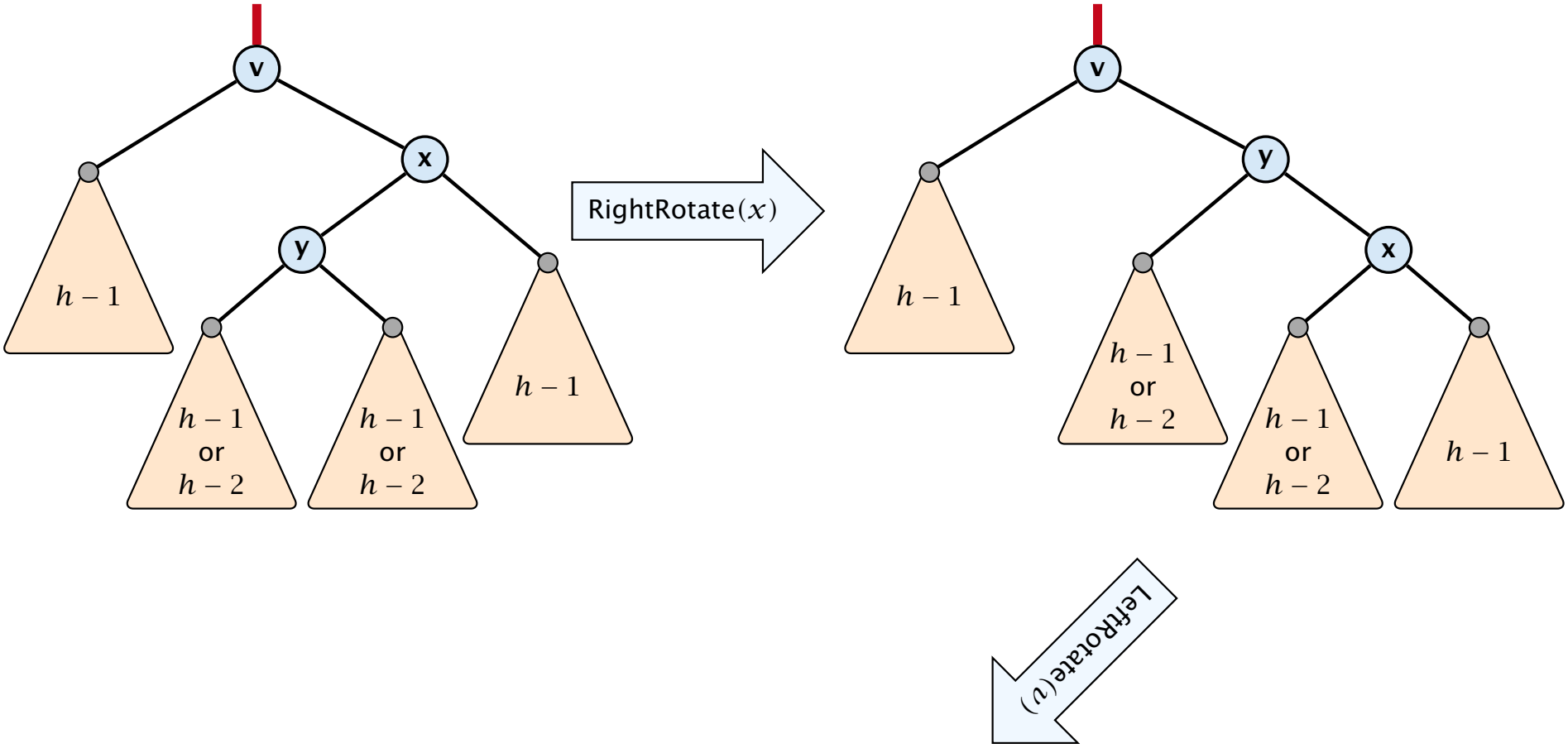

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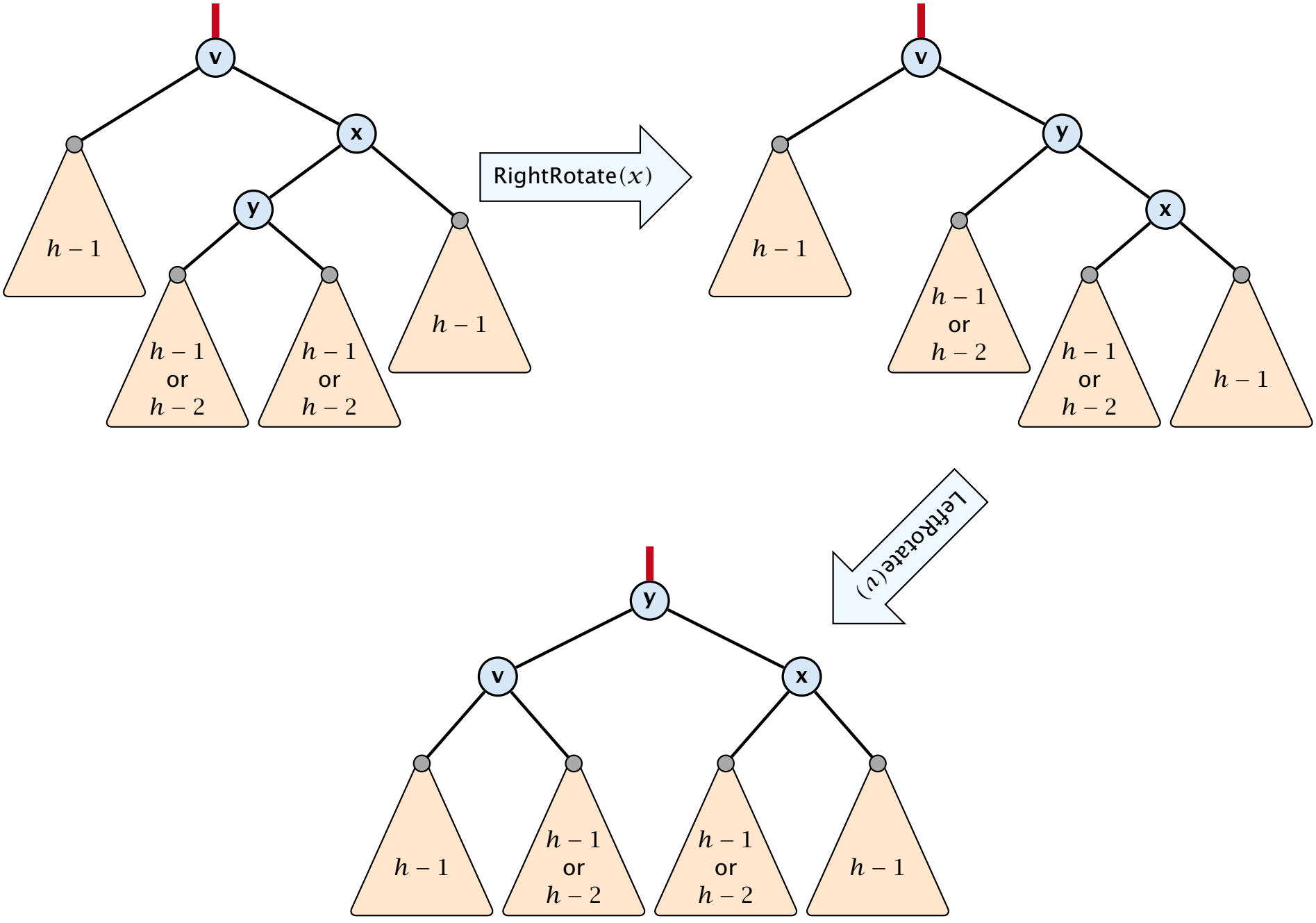

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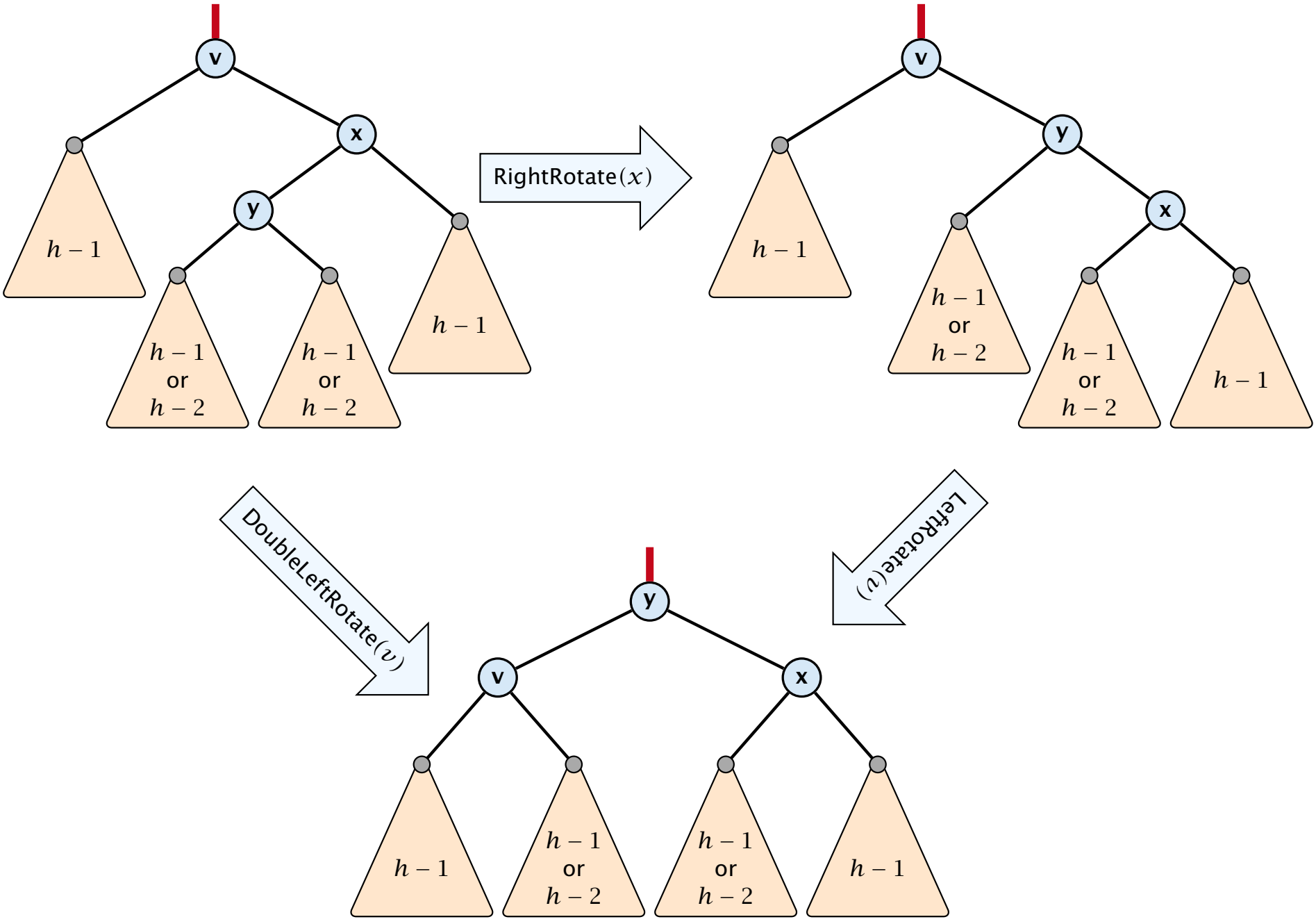

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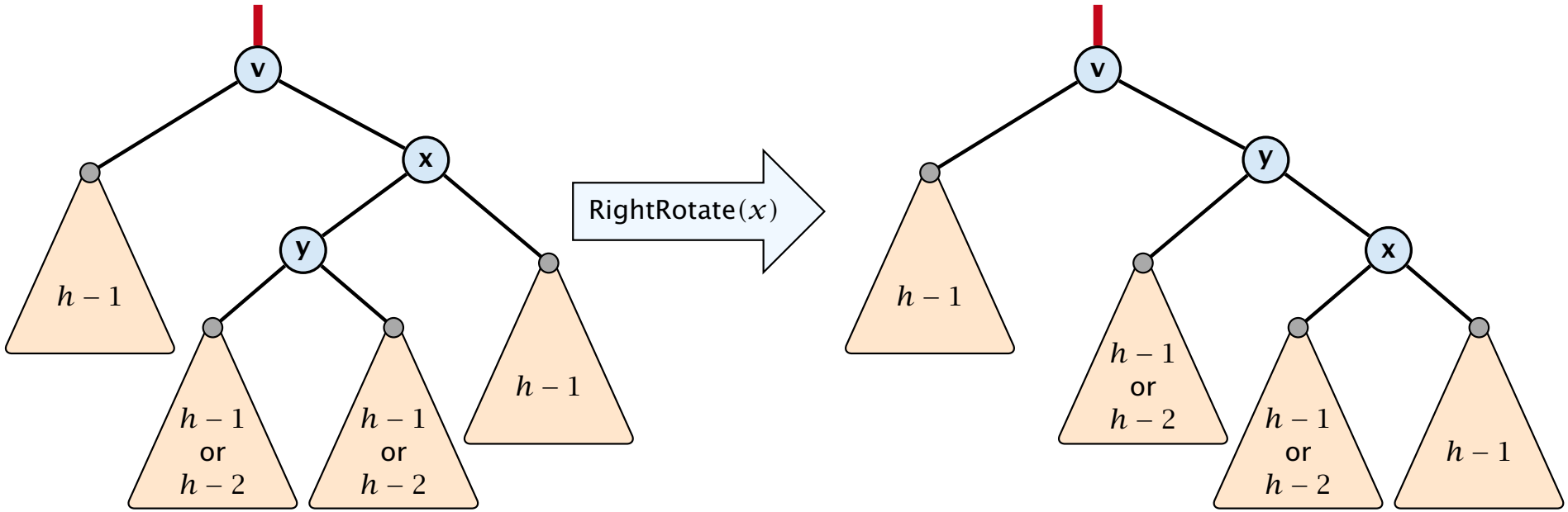

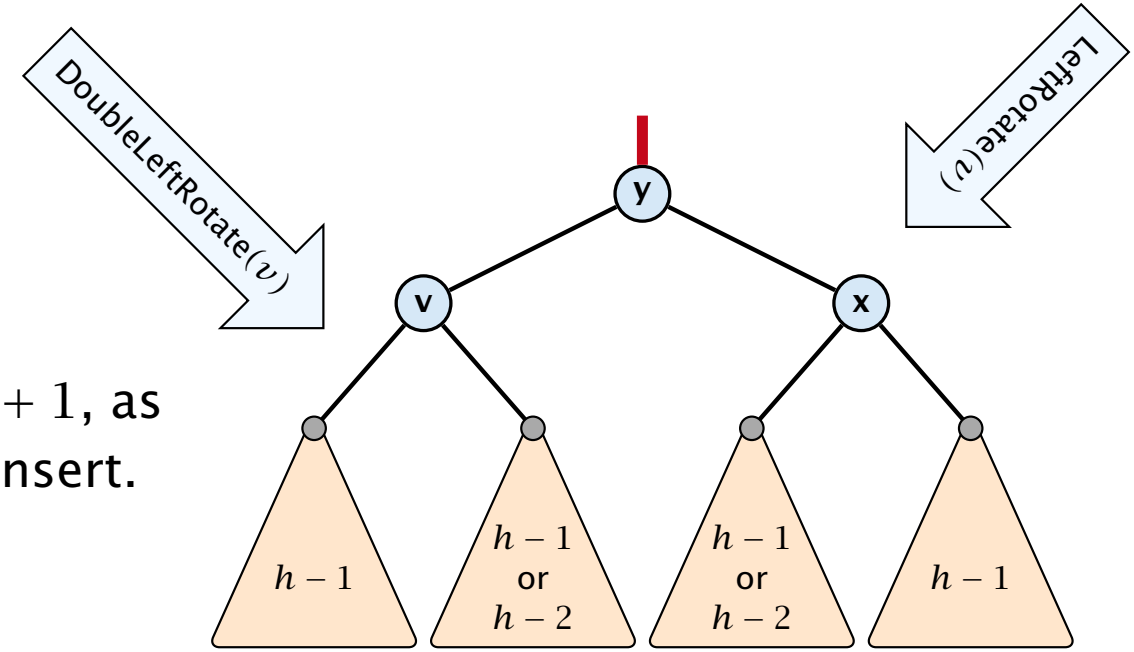

Height is $h + 1$, as before the insert.

AVL-trees: Delete

- Delete like in a binary search tree.
- Let v denote the parent of the node that has been spliced out.
- The balance-constraint may be violated at v, or at ancestors of v, as a sub-tree of a child of v has reduced its height.
- Initially, the node c—the new root in the sub-tree that has changed—is either a dummy leaf or a node with two dummy leafs as children.

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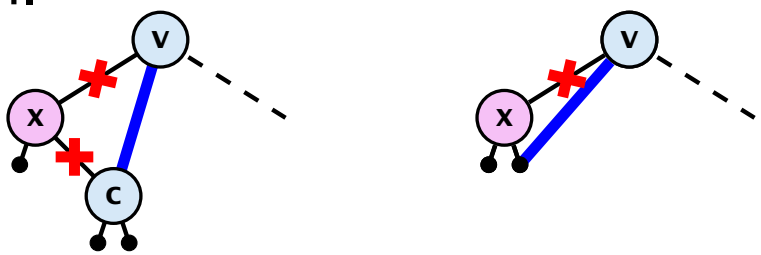

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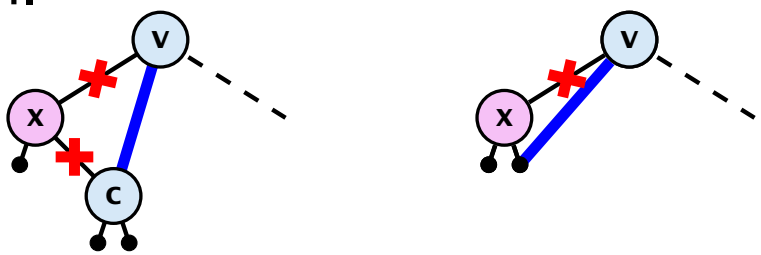

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Invariant at the beginning AVL-fix-up-delete(v):

1. The balance constraints holds at all descendants of v.
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Algorithm 13 AVL-fix-up-delete(v)

1: **if** balance$[v] \in \{-2, 2\}$ **then** DoRotationDelete(v);
2: **if** balance$[v] \in \{-1, 1\}$ **return**;
3: AVL-fix-up-delete(parent$[v]$);

We will show that the above procedure is correct. However, for the case of a delete there may be a logarithmic number of rotations.

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Algorithm 14 DoRotationDelete(v)

```
 1: if balance[v] = −2 then // deletion in left sub-tree
 2:         if balance[right[v]] ∈ {0, −1} then
 3:                 LeftRotate(v);
 4:         else
 5:                 DoubleLeftRotate(v);
 6: else // deletion in right sub-tree
 7:         if balance[left[v]] = {0, 1} then
 8:                 RightRotate(v);
 9:         else
10:                 DoubleRightRotate(v);
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AVL-trees: Delete

It is clear that the invariant for the fix-up routine hold as long as no rotations have been done.

We show that after doing a rotation at v:

- v fulfills the balance condition.
- All children of v still fulfill the balance condition.
- If now $\text{balance}[v] \in \{-1, 1\}$ we can stop as the height of T_v is the same as before the deletion.

We only look at the case where the deleted node was in the right sub-tree of v. The other case is symmetric.

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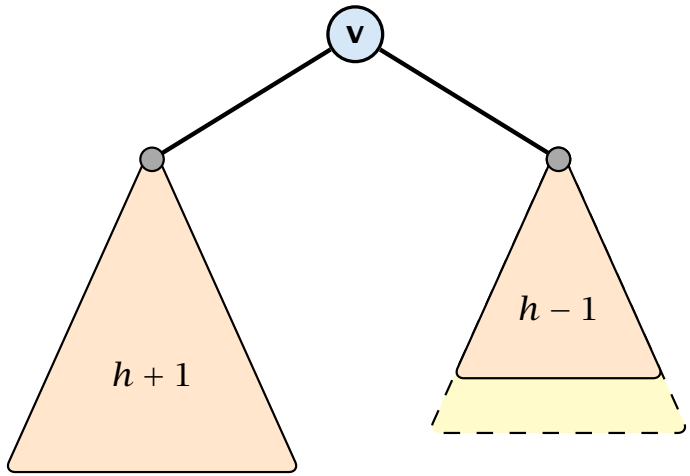

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Before the deletion the height of T_v was $h+2$.

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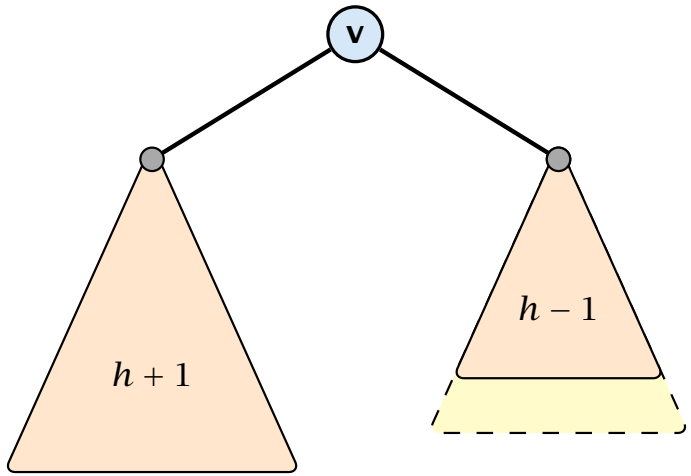

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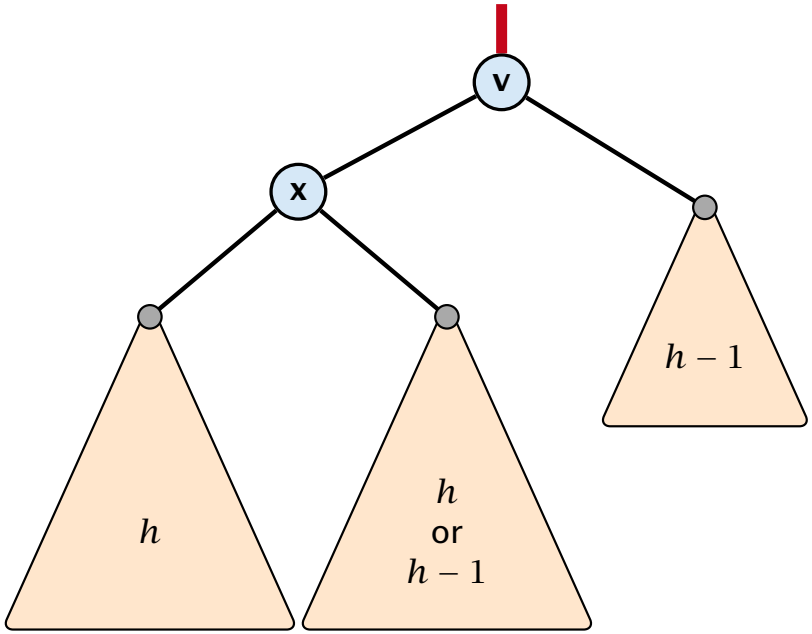

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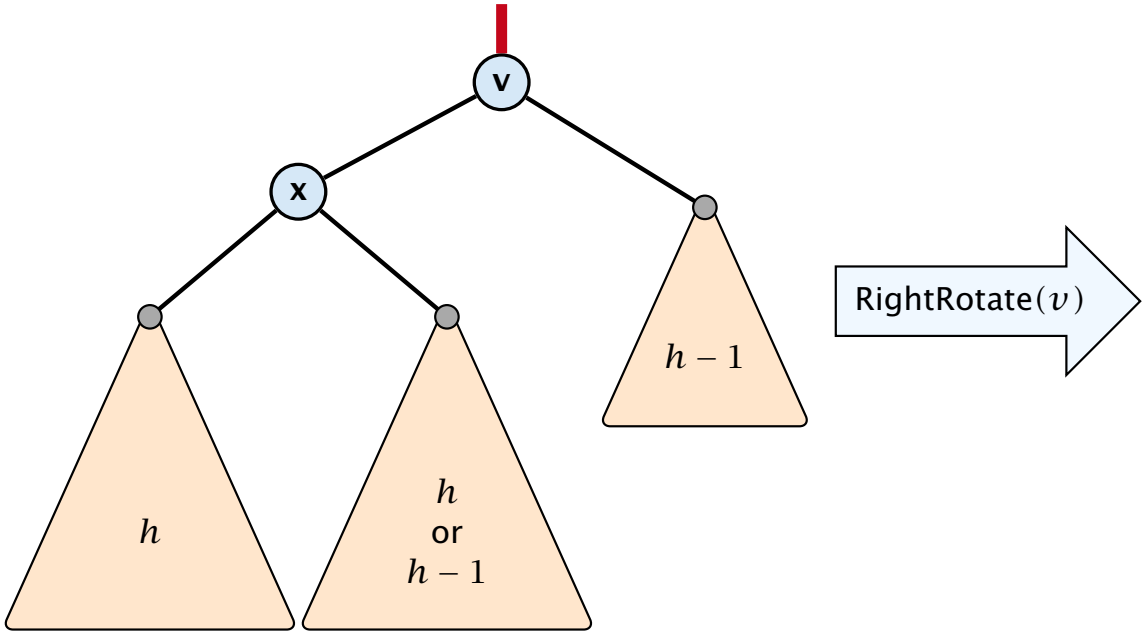

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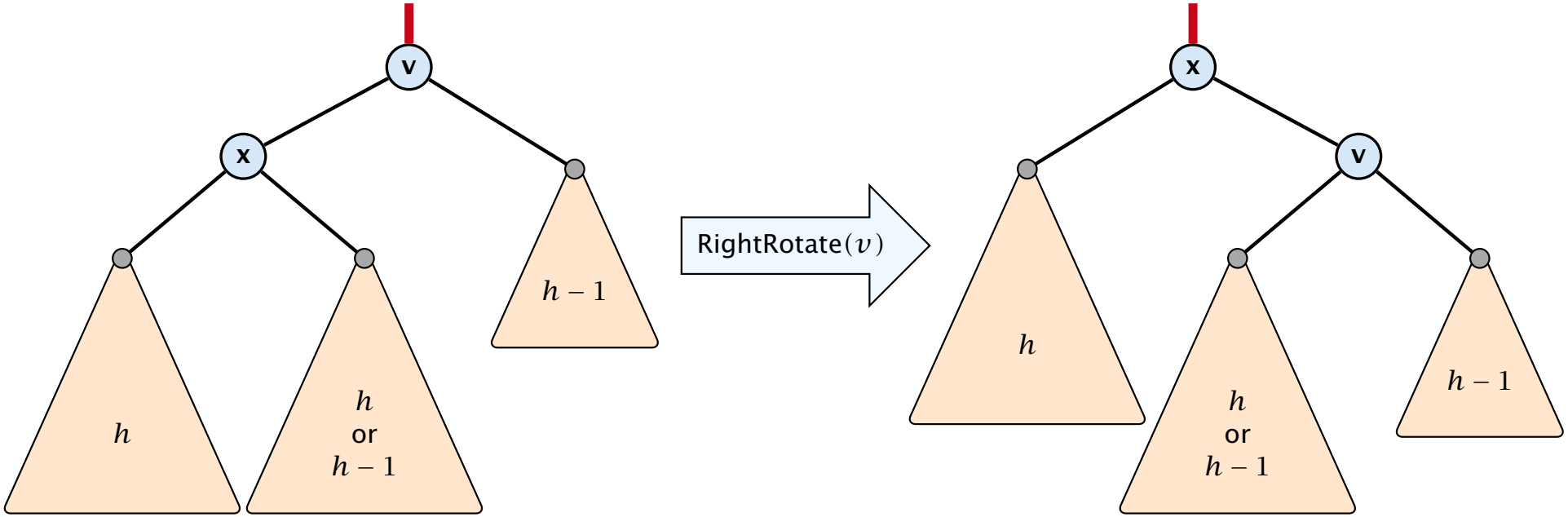

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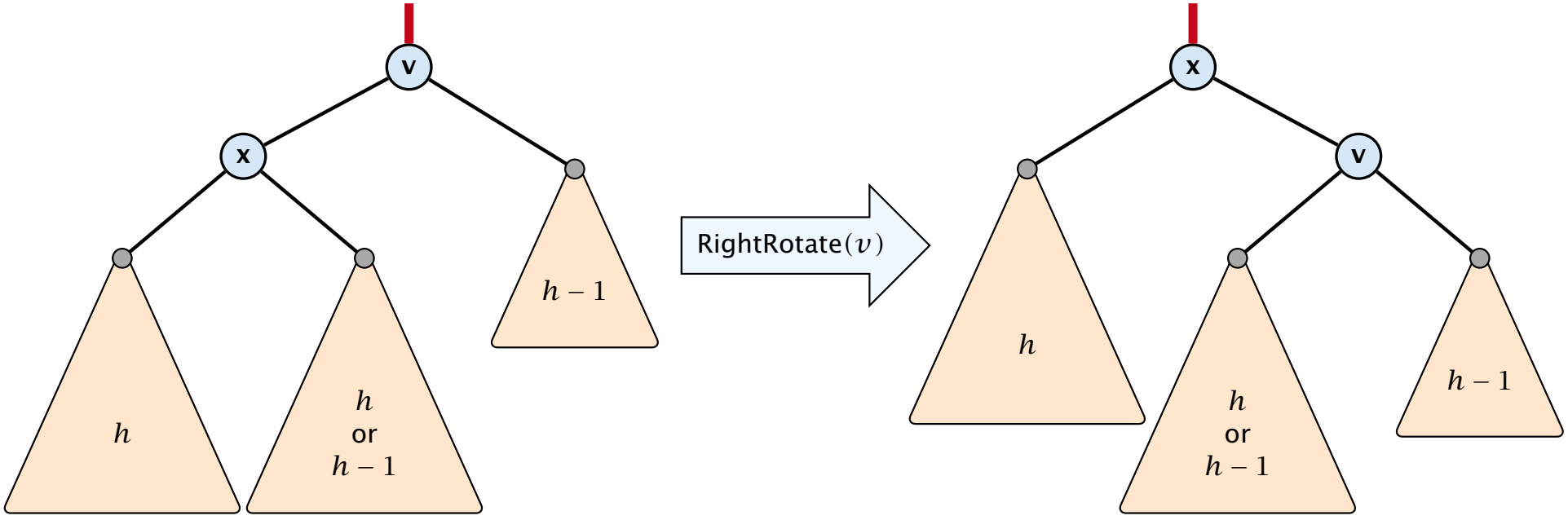

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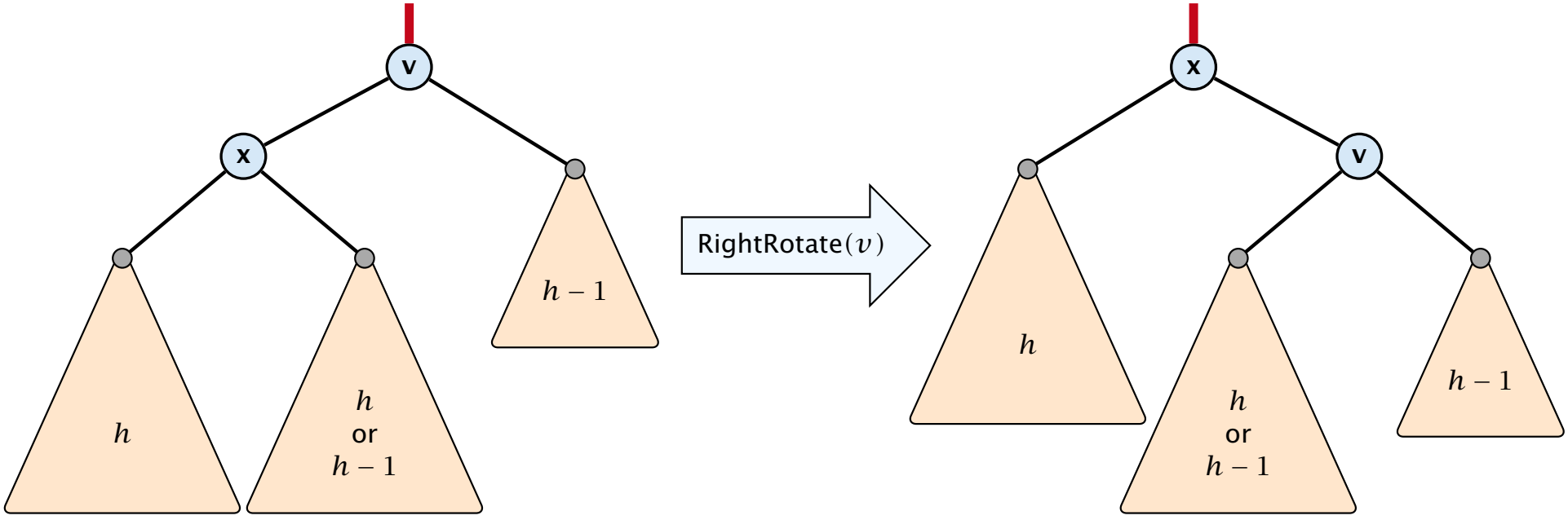

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If the middle subtree has height $h - 1$ the whole tree has decreased its height from $h + 2$ to $h + 1$. We do continue the fix-up procedure as the balance at the root is zero.

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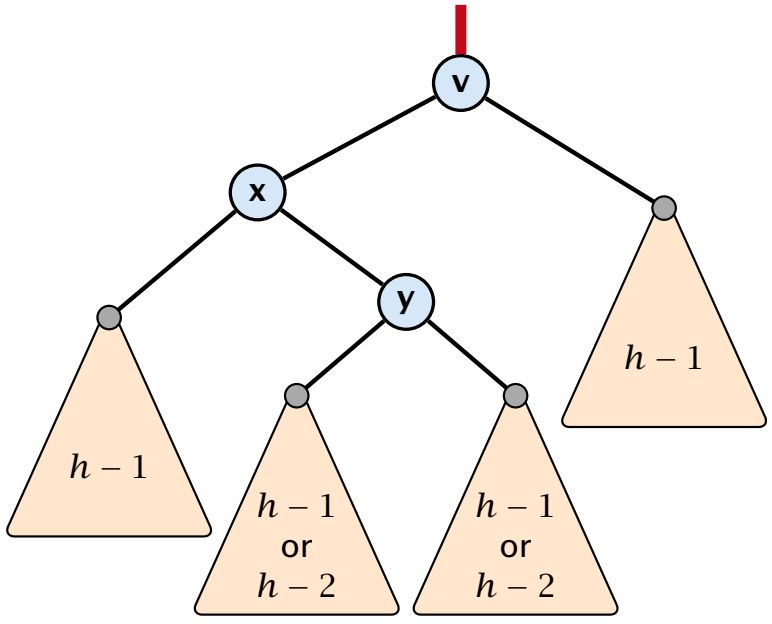

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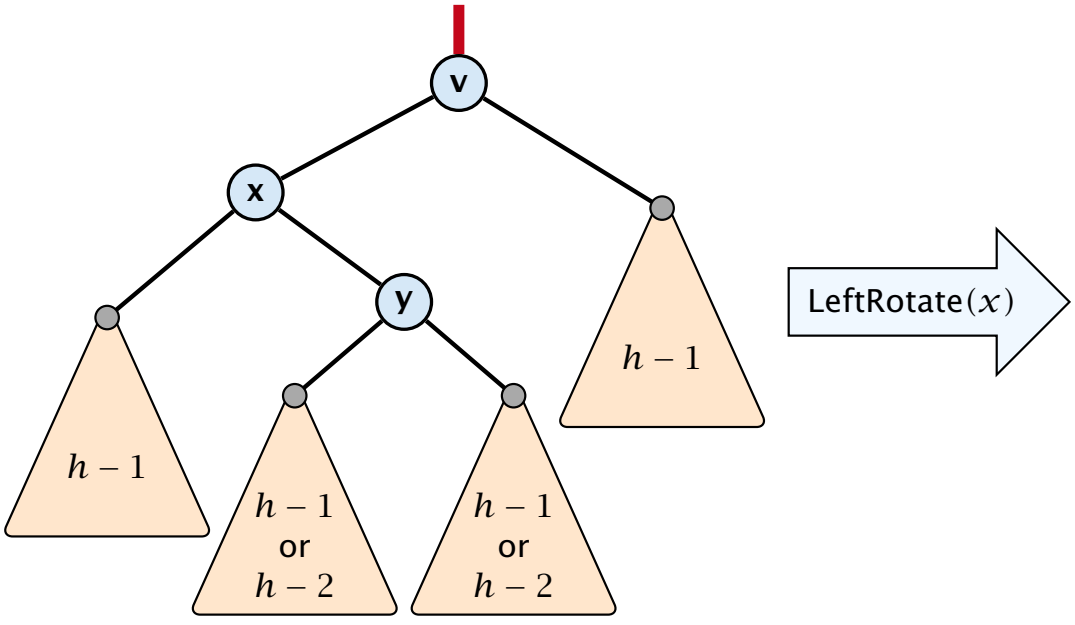

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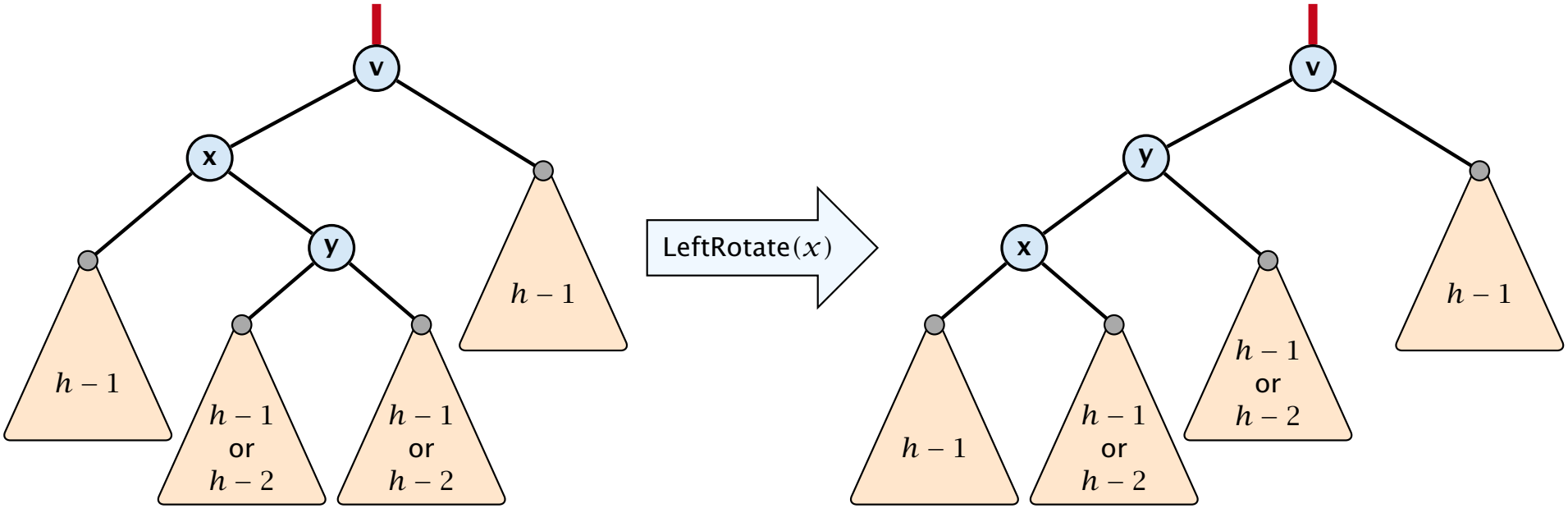

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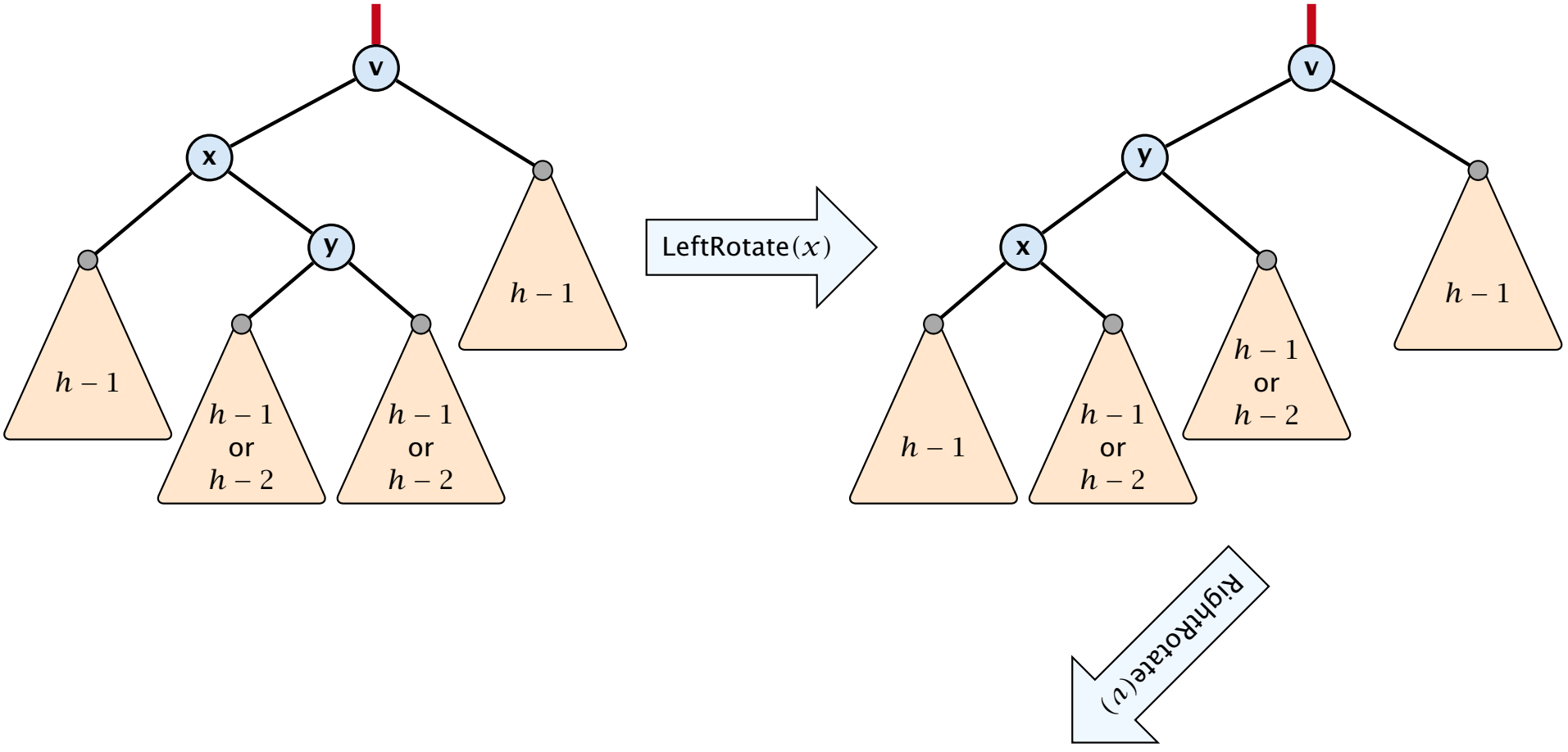

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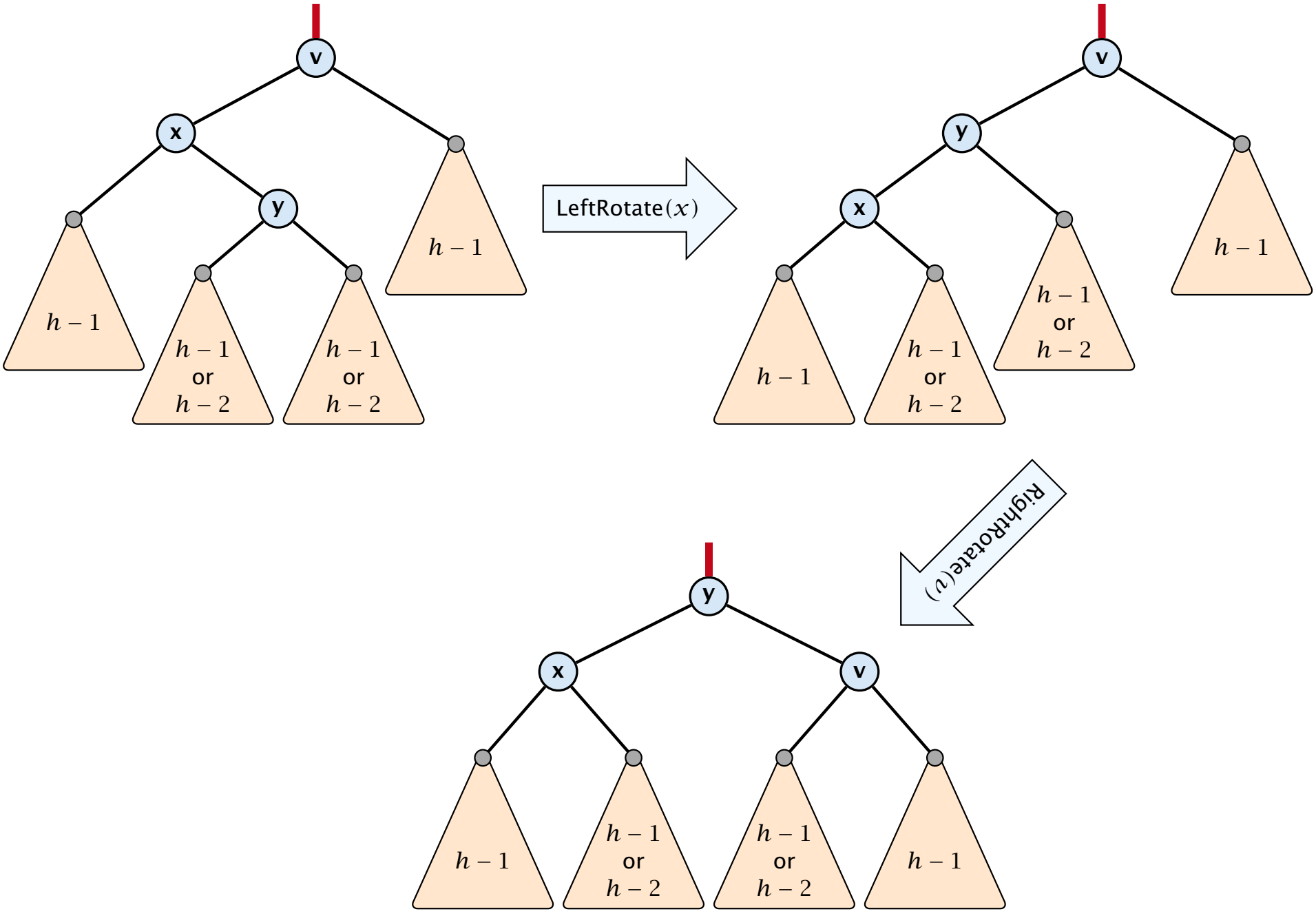

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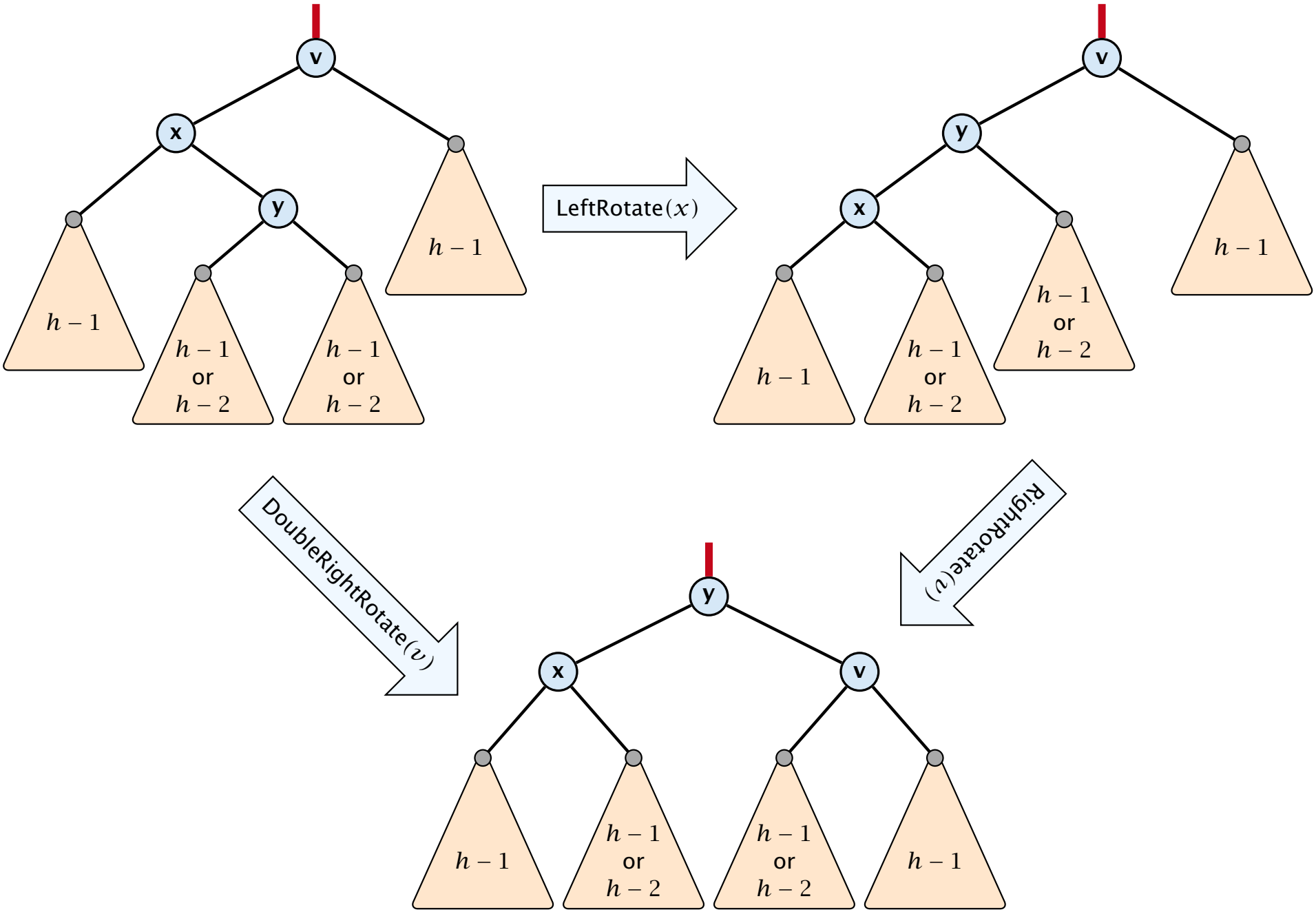

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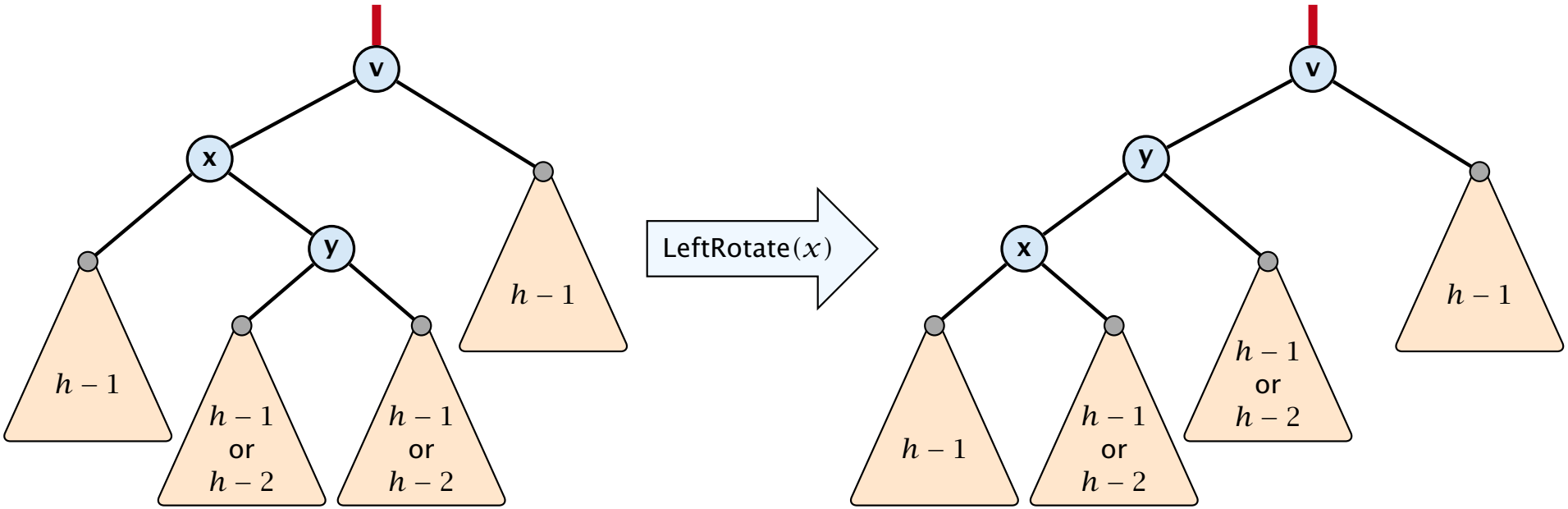

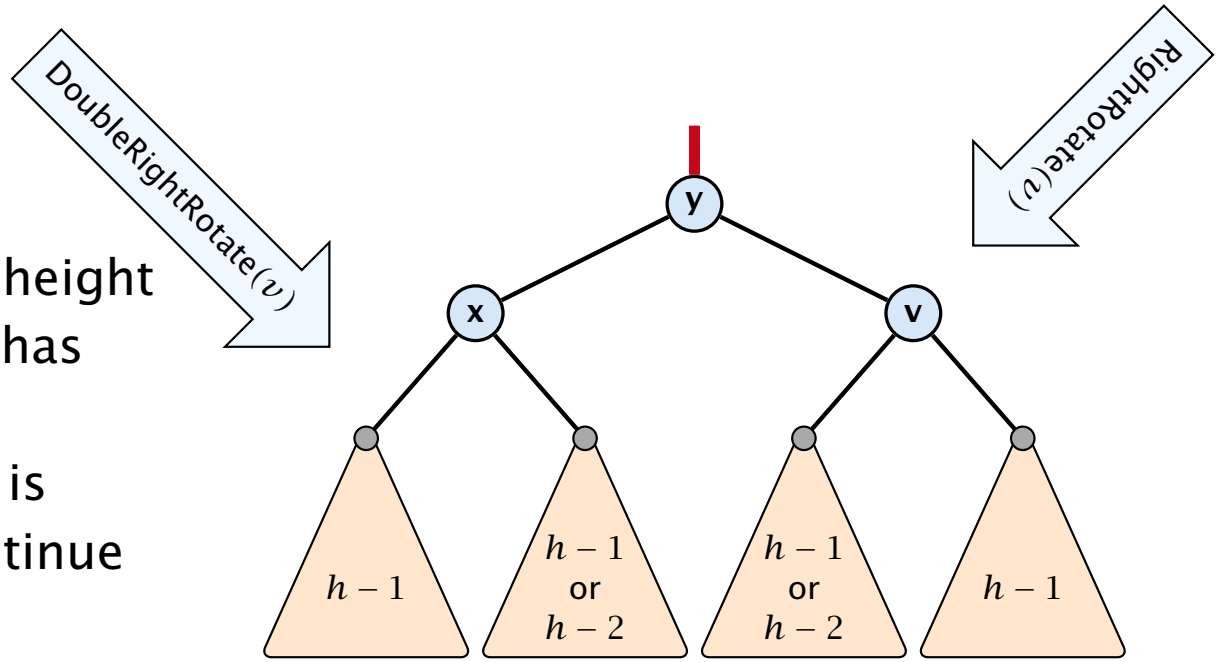

Sub-tree has height $h + 1$, i.e., it has shrunk. The balance at y is zero. We continue the iteration.

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Suppose you want to develop a data structure with:

- **Insert(x)**: insert element x.
- **Search(k)**: search for element with key k.
- **Delete(x)**: delete element referenced by pointer x.
- **find-by-rank(ℓ)**: return the ℓ-th element; return "error" if the data-structure contains less than ℓ elements.

Augment an existing data-structure instead of developing a new one.

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How to augment a data-structure

1. choose an underlying data-structure
2. determine additional information to be stored in the underlying structure
3. verify/show how the additional information can be maintained for the basic modifying operations on the underlying structure.
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Goal: Design a data-structure that supports insert, delete, search, and find-by-rank in time $\mathcal{O}(\log n)$.

1. We choose a red-black tree as the underlying data-structure.
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3. We need to be able to update the size-field in each node without asymptotically affecting the running time of insert, delete, and search. We come back to this step later...

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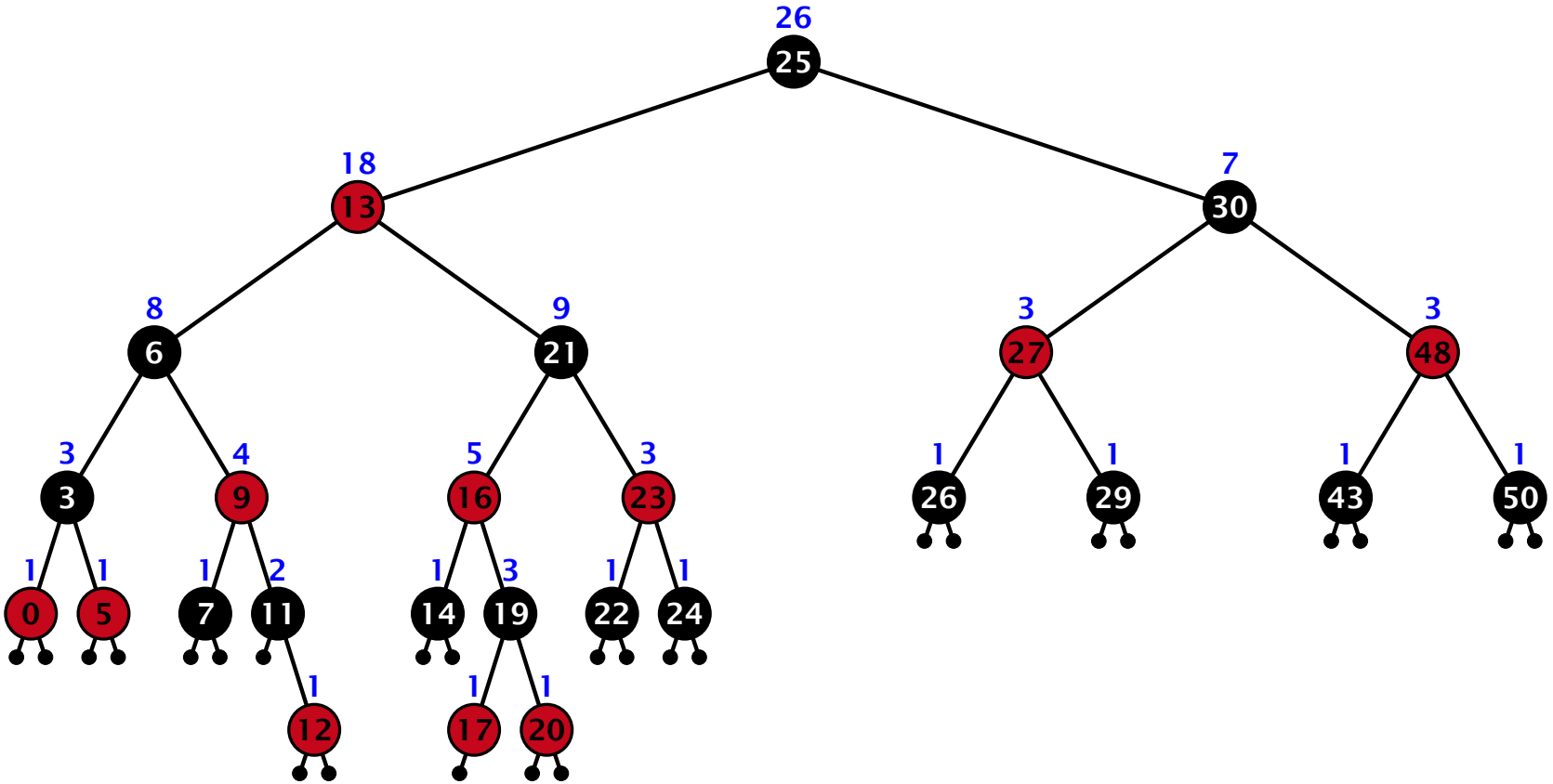
4. How does find-by-rank work?
   Find-by-rank(k) := Select(root,k) with

Algorithm 15 Select(x, i)

```
1: if x = null then return error
2: if left[x] ≠ null then r ← left[x].size + 1 else r ← 1
3: if i = r then return x
4: if i < r then
5:         return Select(left[x], i)
6: else
7:         return Select(right[x], i − r)
```

Select(x, i)

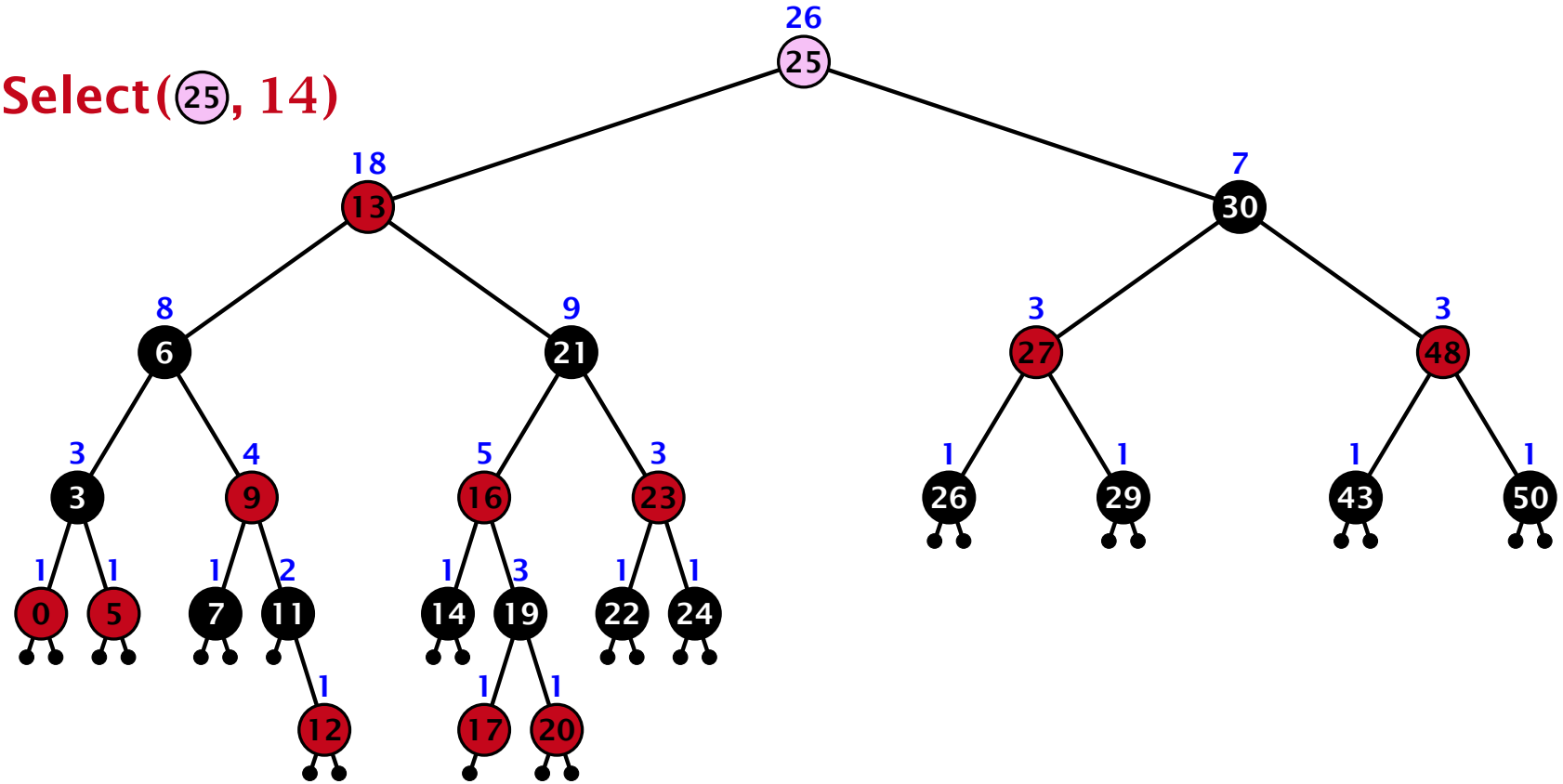


Find-by-rank:

- decide whether you have to proceed into the left or right sub-tree
- adjust the rank that you are searching for if you go right

Select(x, i)

Select(25, 14)



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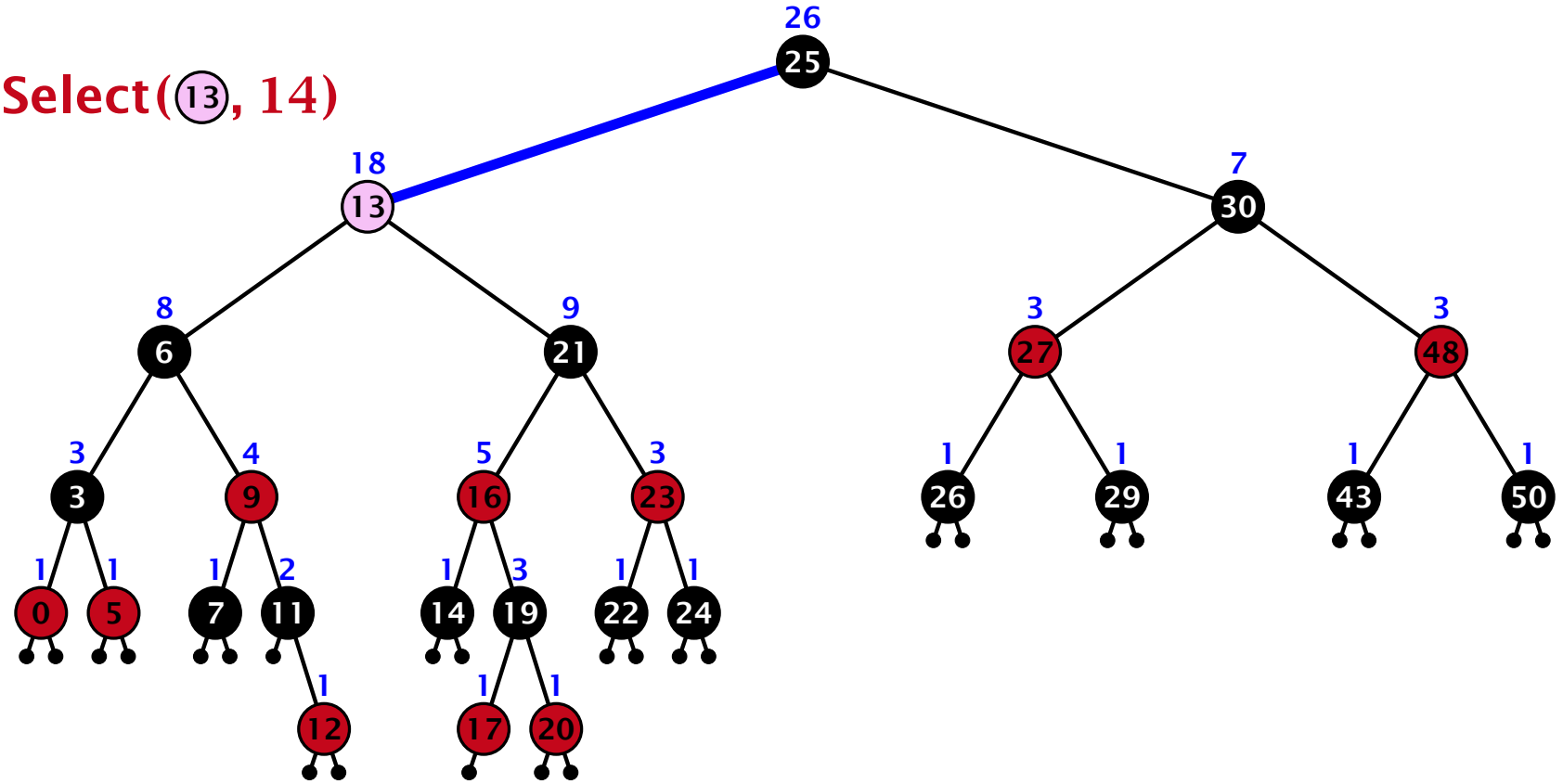

Find-by-rank:

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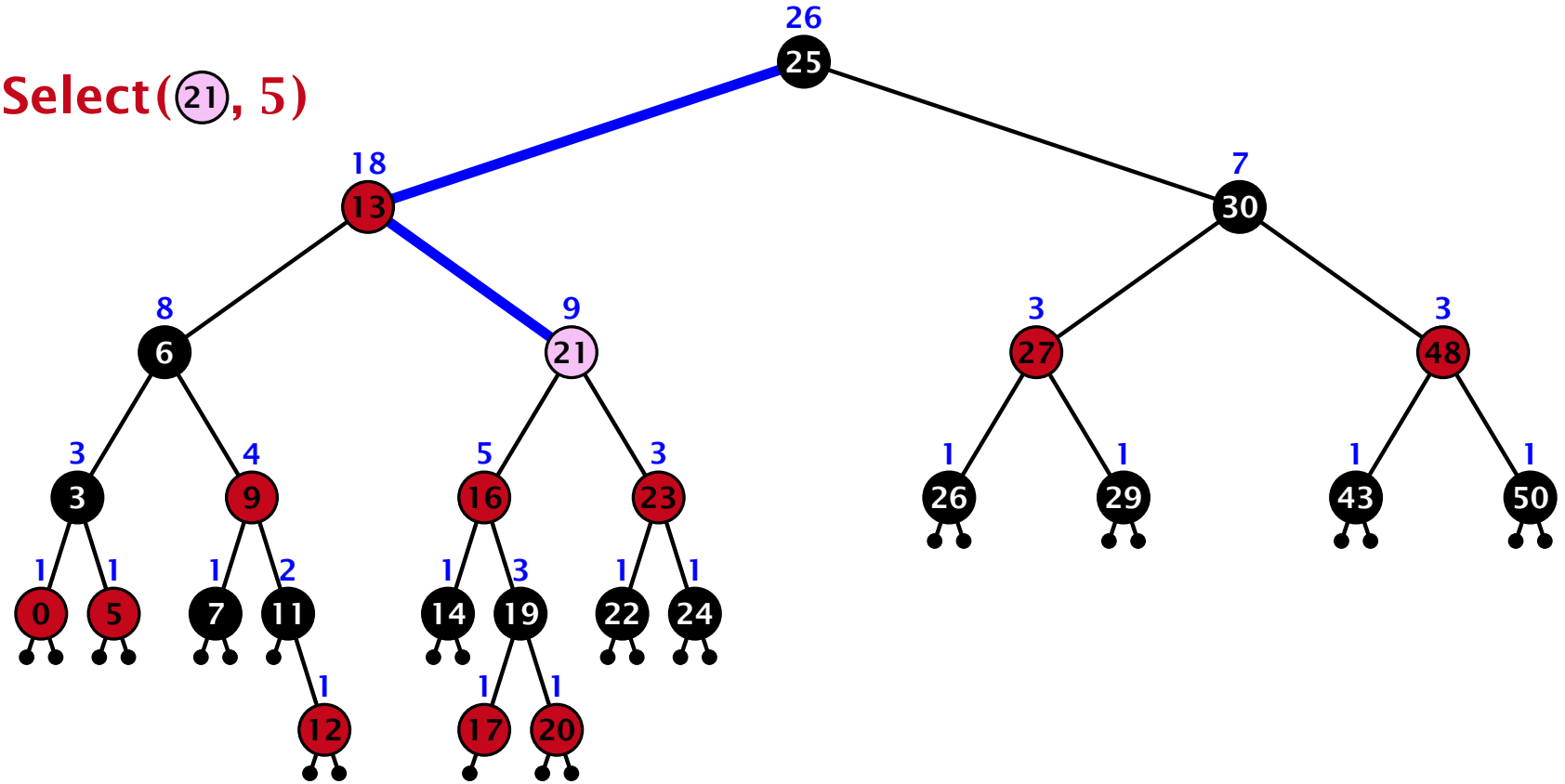
Select(x, i)

Select(21, 5)



Find-by-rank:

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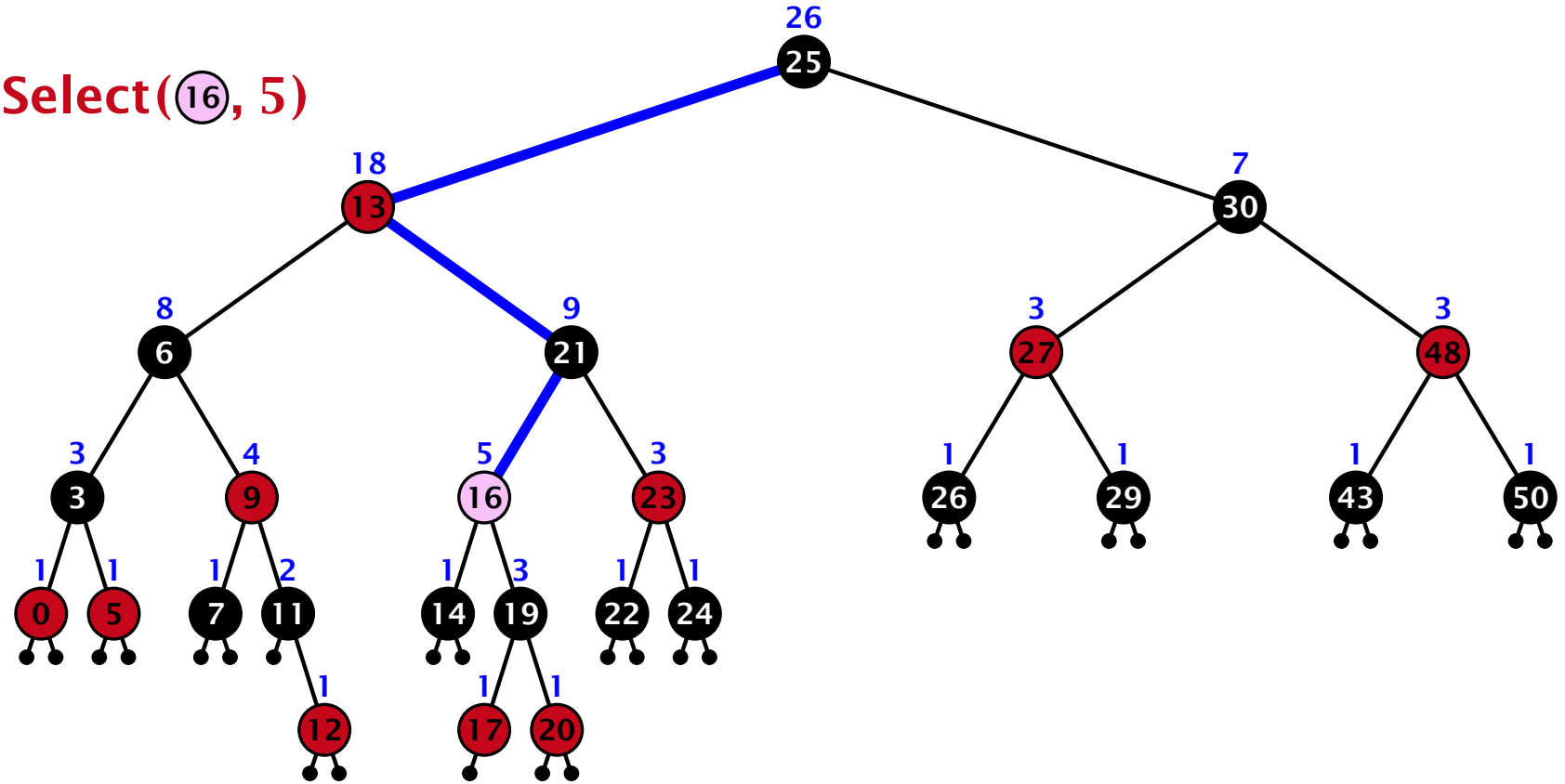
Select(x, i)

Select(16, 5)



Find-by-rank:

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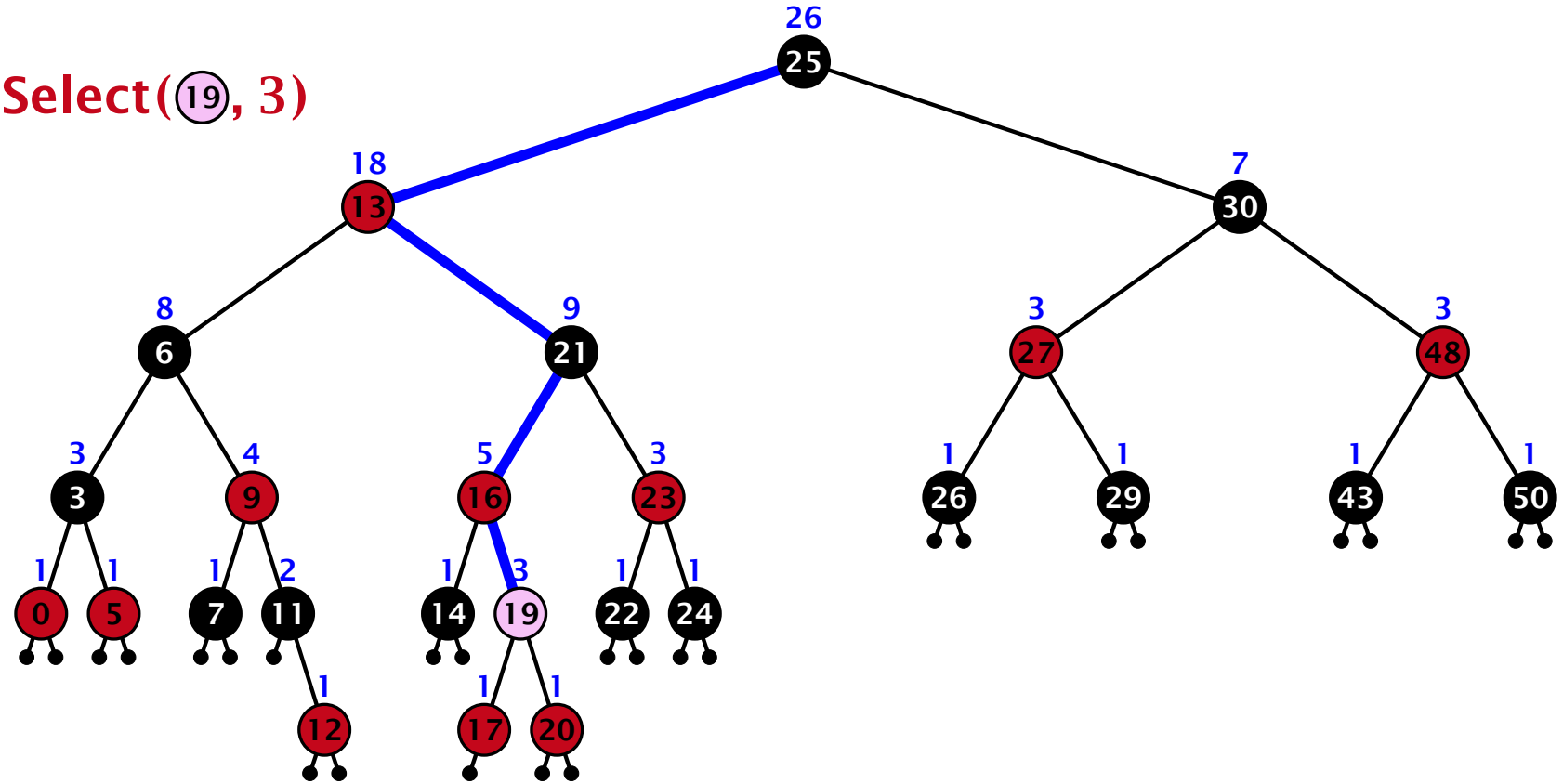
Select(x, i)

Select(19, 3)



Find-by-rank:

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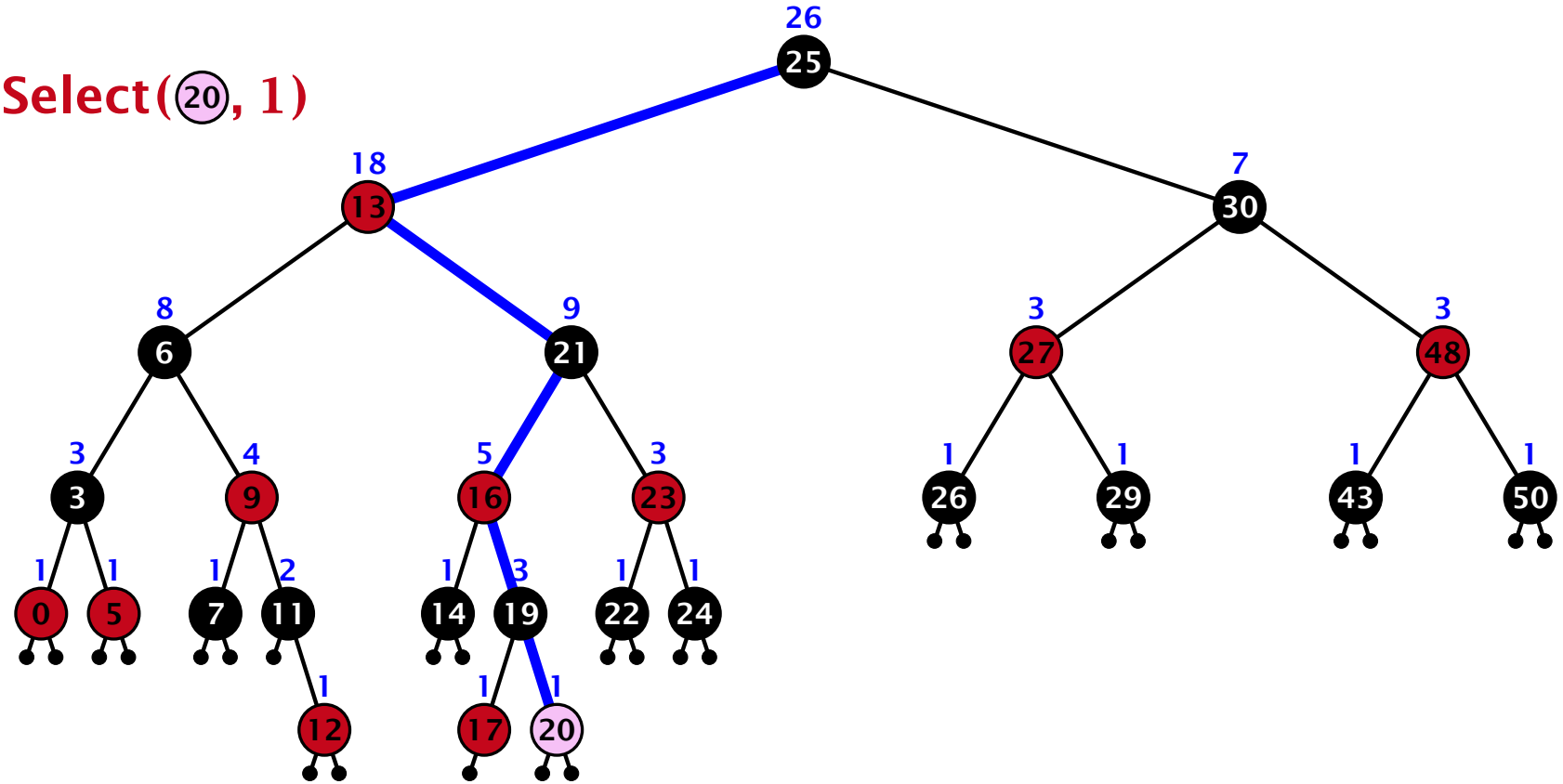
Select(x, i)

Select(20, 1)



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7.4 Augmenting Data Structures

Goal: Design a data-structure that supports insert, delete, search, and find-by-rank in time $\mathcal{O}(\log n)$.

3. How do we maintain information?

$\textbf{Search}(k)$: Nothing to do.

$\textbf{Insert}(x)$: When going down the search path increase the size field for each visited node. Maintain the size field during rotations.

$\textbf{Delete}(x)$: Directly after splicing out a node traverse the path from the spliced out node upwards, and decrease the size counter on every node on this path. Maintain the size field during rotations.

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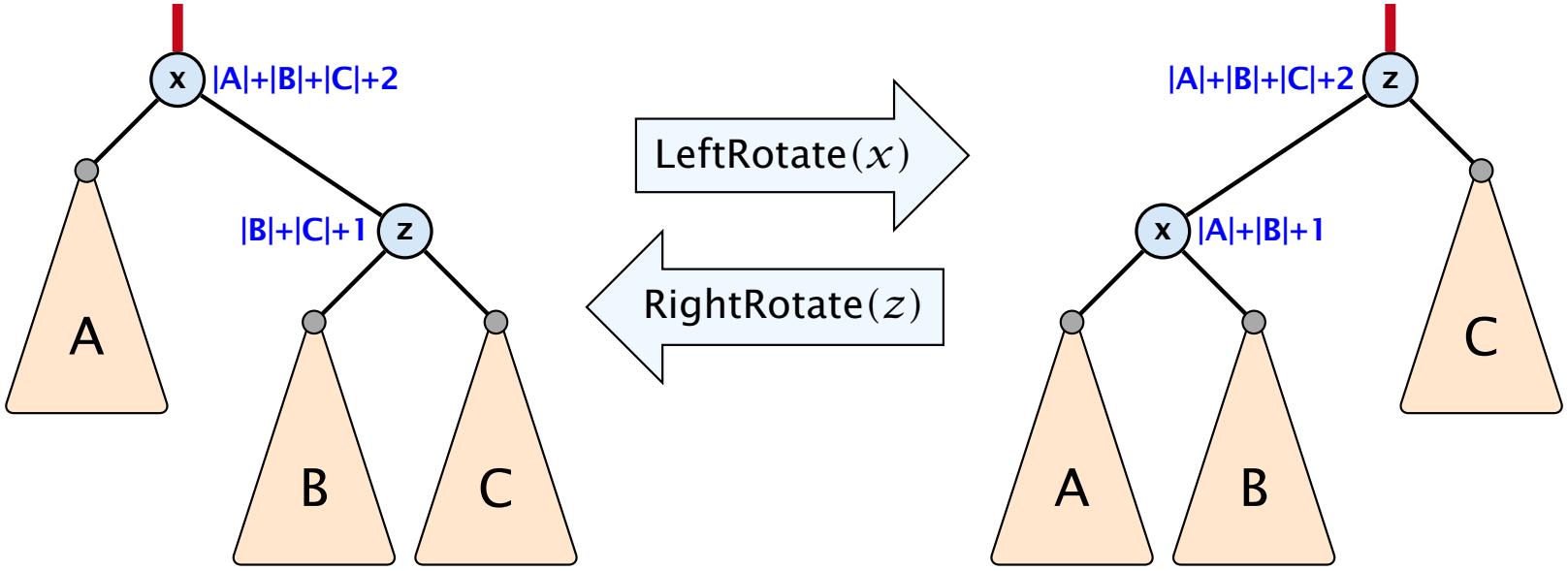
Rotations

The only operation during the fix-up procedure that alters the tree and requires an update of the size-field:



The nodes x and z are the only nodes changing their size-fields.

The new size-fields can be computed locally from the size-fields of the children.

7.5 (a, b)-trees

Definition 17

For $b \geq 2a - 1$ an (a, b)-tree is a search tree with the following properties

1. all leaves have the same distance to the root
2. every internal non-root vertex v has at least a and at most b children
3. the root has degree at least 2 if the tree is non-empty
4. the internal vertices do not contain data, but only keys (external search tree)
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7.5 (a, b)-trees

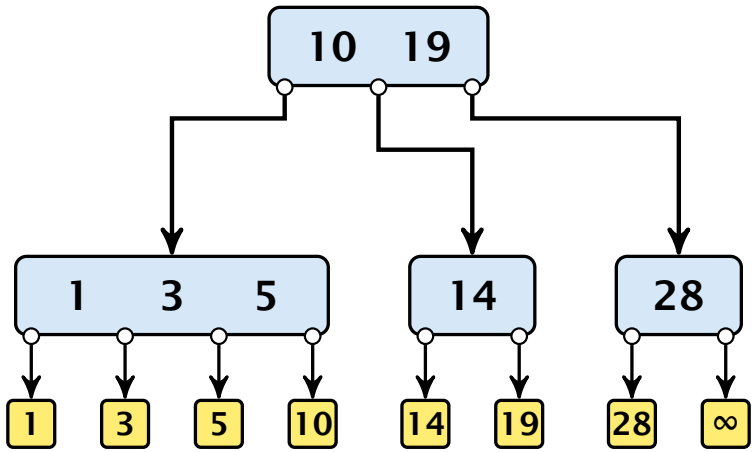
Each internal node v with $d(v)$ children stores $d-1$ keys $k_1, \ldots, k_{d-1}$. The i-th subtree of v fulfills

$$k_{i-1} < \text{ key in } i\text{-th sub-tree } \leq k_i \ ,$$

where we use $k_0 = -\infty$ and $k_d = \infty$.

7.5 (a, b)-trees

Example 18



7.5 (a, b)-trees

Variants

- The dummy leaf element may not exist; it only makes implementation more convenient.
- Variants in which $b = 2a$ are commonly referred to as B-trees.
- A B-tree usually refers to the variant in which keys and data are stored at internal nodes.
- A B^+ tree stores the data only at leaf nodes as in our definition. Sometimes the leaf nodes are also connected in a linear list data structure to speed up the computation of successors and predecessors.
- A B^* tree requires that a node is at least $2/3$-full as opposed to $1/2$-full (the requirement of a B-tree).

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Lemma 19

Let T be an (a,b)-tree for $n > 0$ elements (i.e., $n+1$ leaf nodes) and height h (number of edges from root to a leaf vertex). Then

1. $2a^{h-1} \leq n+1 \leq b^h$
2. $\log_b(n+1) \leq h \leq 1 + \log_a(\frac{n+1}{2})$

Proof.

- If $r \neq t$ the root has degree at least 2 and all other nodes have degree at least a. This gives that the number of leaf nodes is at least $2a^{h-1}$.
- Analogously, the degree of any node is at most b, and, hence, the number of leaf nodes at most b^h.

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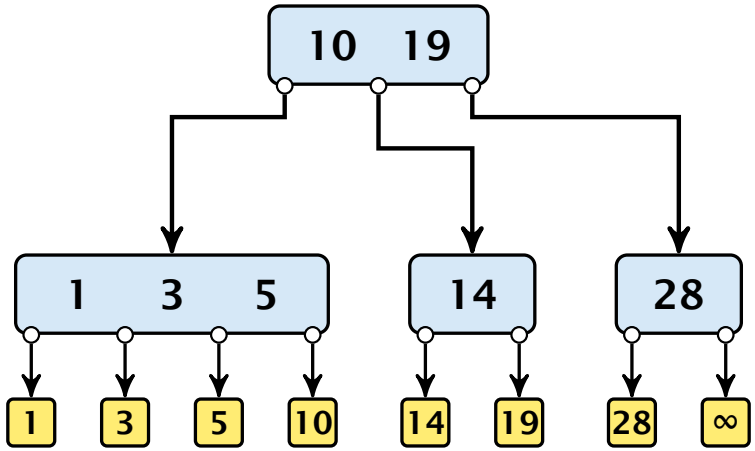
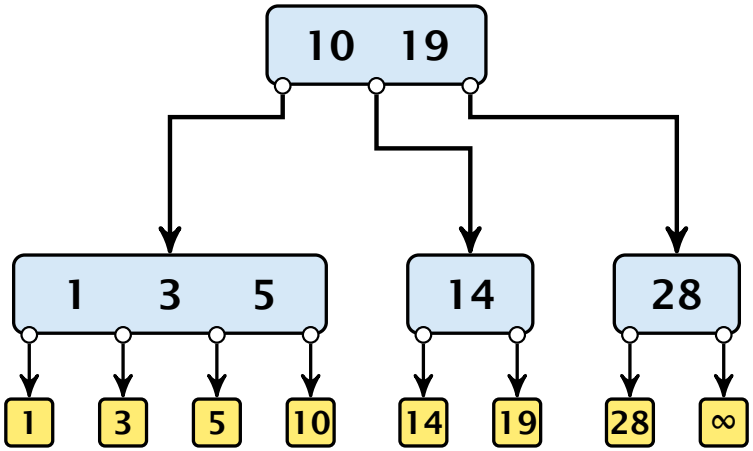
Search



Search

Search(8)



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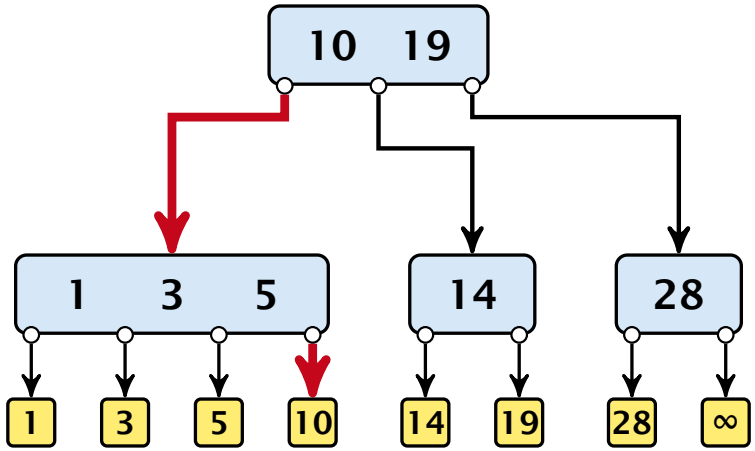

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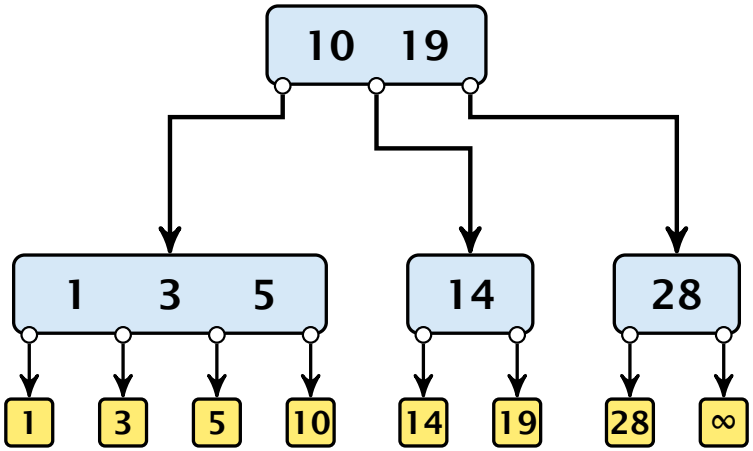

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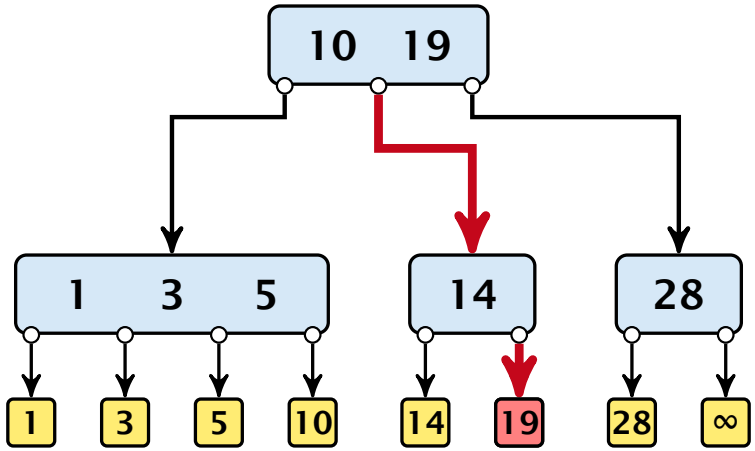

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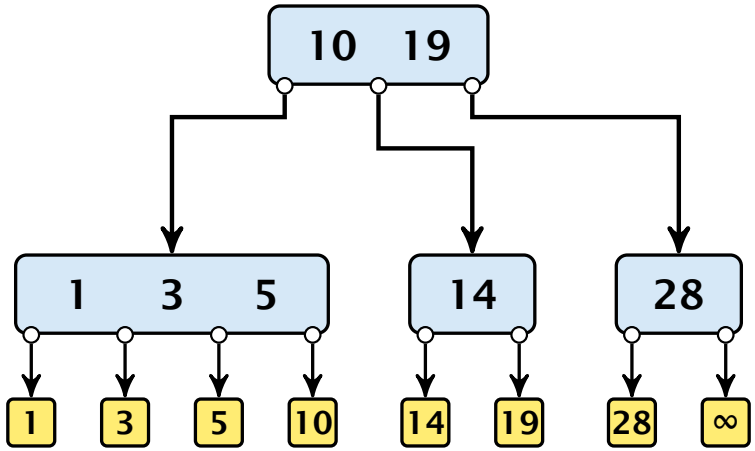

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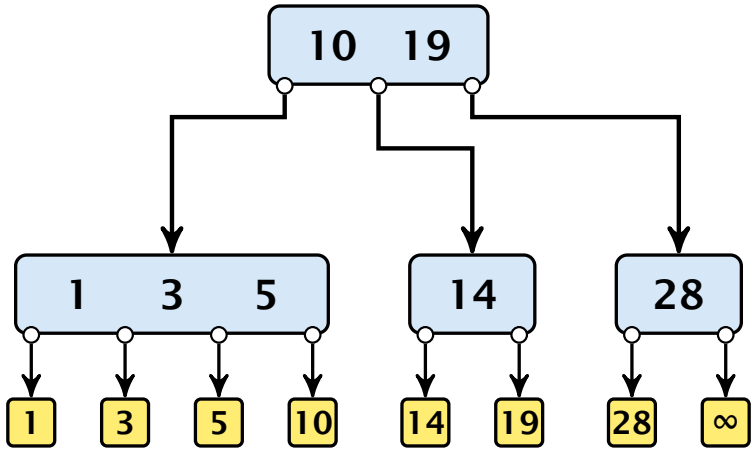

The search is straightforward. It is only important that you need to go all the way to the leaf.

Time: $\mathcal{O}(b \cdot h) = \mathcal{O}(b \cdot \log n)$, if the individual nodes are organized as linear lists.

Insert

Insert element x:

- Follow the path as if searching for $\text{key}[x]$.
- If this search ends in leaf ℓ, insert x before this leaf.
- For this add $\text{key}[x]$ to the key-list of the last internal node v on the path.
- If after the insert v contains b nodes, do Rebalance(v).

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Rebalance(v):

- Let k_i, $i = 1, \ldots, b$ denote the keys stored in v.
- Let $j := \lfloor \frac{b+1}{2} \rfloor$ be the middle element.
- Create two nodes v_1, and v_2. v_1 gets all keys $k_1, \ldots, k_{j-1}$ and v_2 gets keys $k_{j+1}, \ldots, k_b$.
- Both nodes get at least $\lfloor \frac{b-1}{2} \rfloor$ keys, and have therefore degree at least $\lfloor \frac{b-1}{2} \rfloor + 1 \geq a$ since $b \geq 2a-1$.
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- Let $j := \lfloor \frac{b+1}{2} \rfloor$ be the middle element.
- Create two nodes v_1, and v_2. v_1 gets all keys $k_1, \ldots, k_{j-1}$ and v_2 gets keys $k_{j+1}, \ldots, k_b$.
- Both nodes get at least $\lfloor \frac{b-1}{2} \rfloor$ keys, and have therefore degree at least $\lfloor \frac{b-1}{2} \rfloor + 1 \geq a$ since $b \geq 2a - 1$.
- They get at most $\lceil \frac{b-1}{2} \rceil$ keys, and have therefore degree at most $\lceil \frac{b-1}{2} \rceil + 1 \leq b$ (since $b \geq 2$).
- The key k_j is promoted to the parent of v. The current pointer to v is altered to point to v_1, and a new pointer (to the right of k_j) in the parent is added to point to v_2.
- Then, re-balance the parent.

Insert

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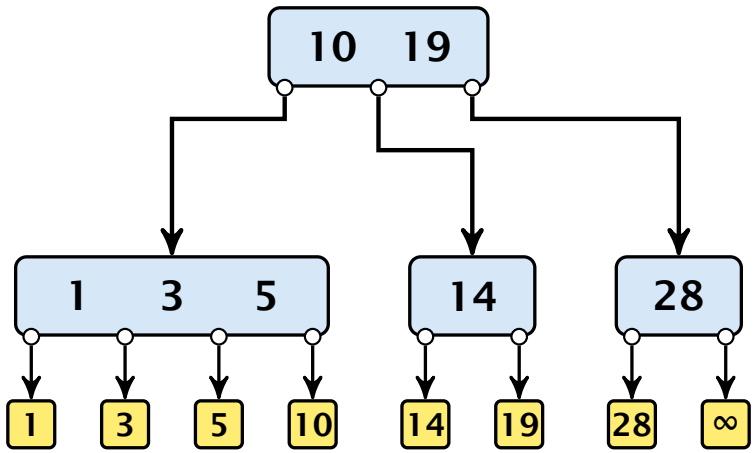
Insert



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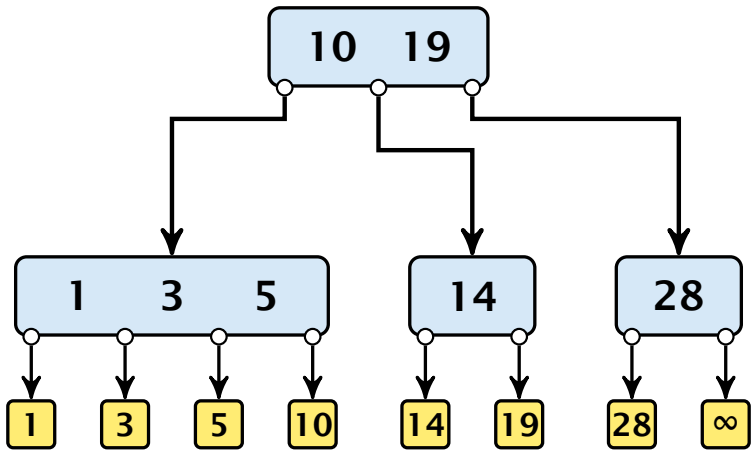
Insert(8)



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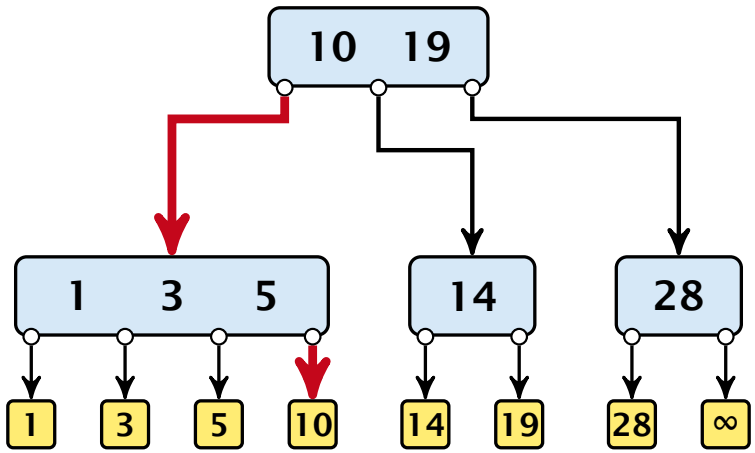

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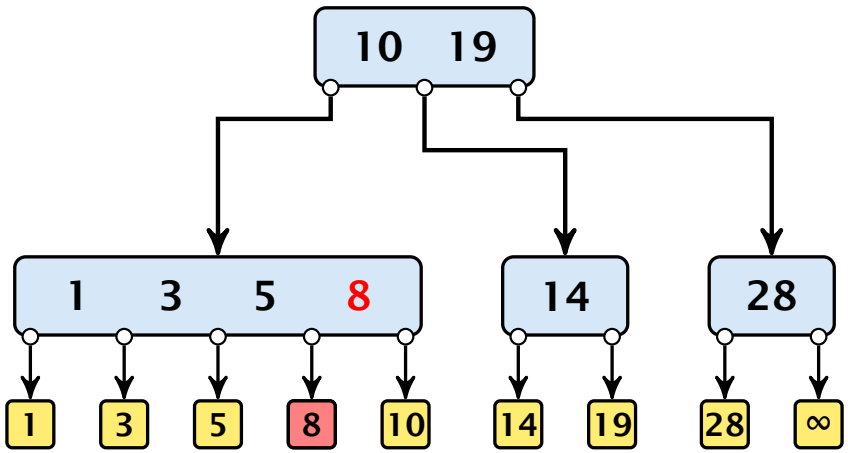

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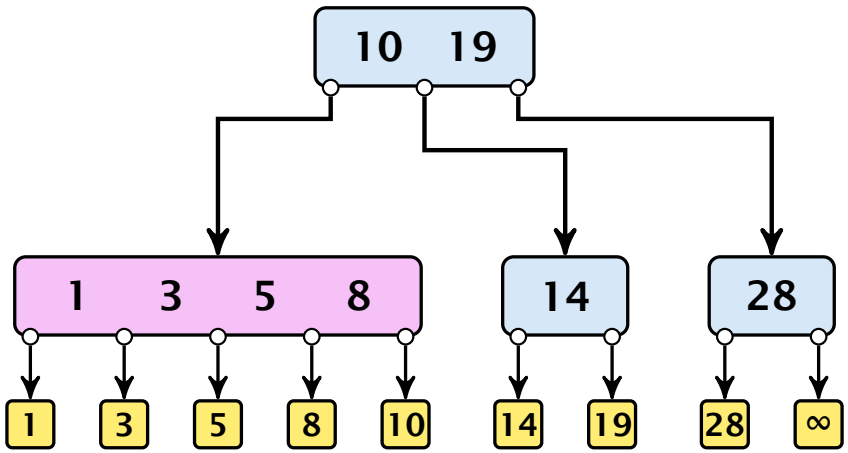

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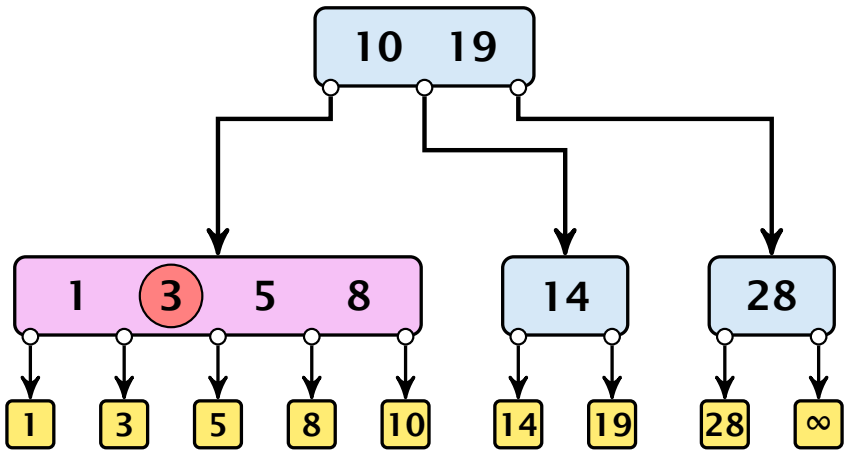

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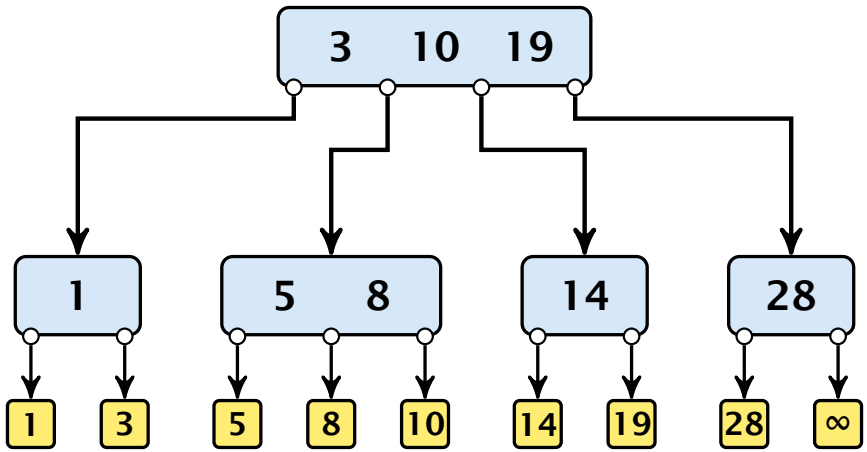

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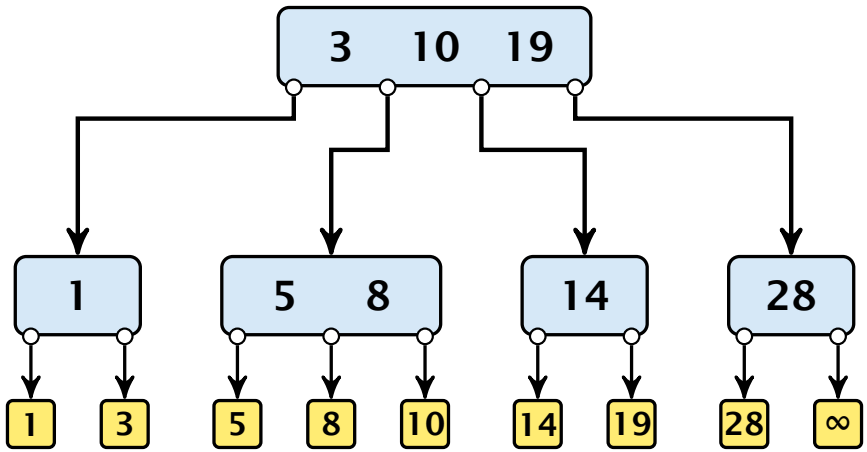

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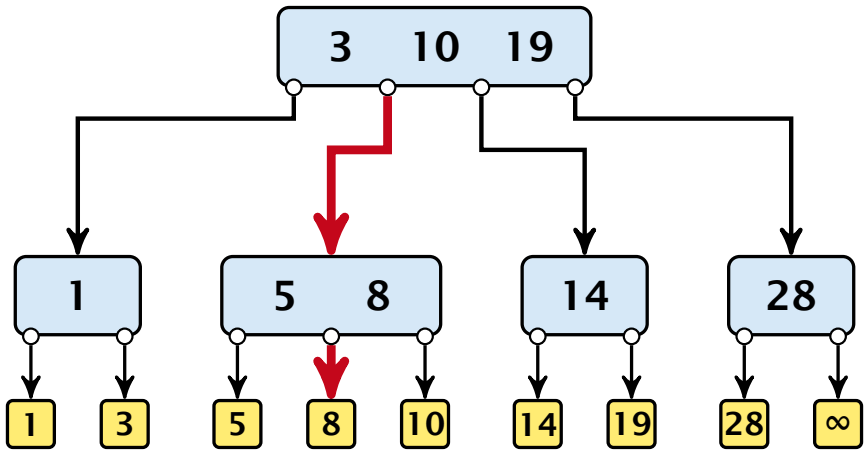

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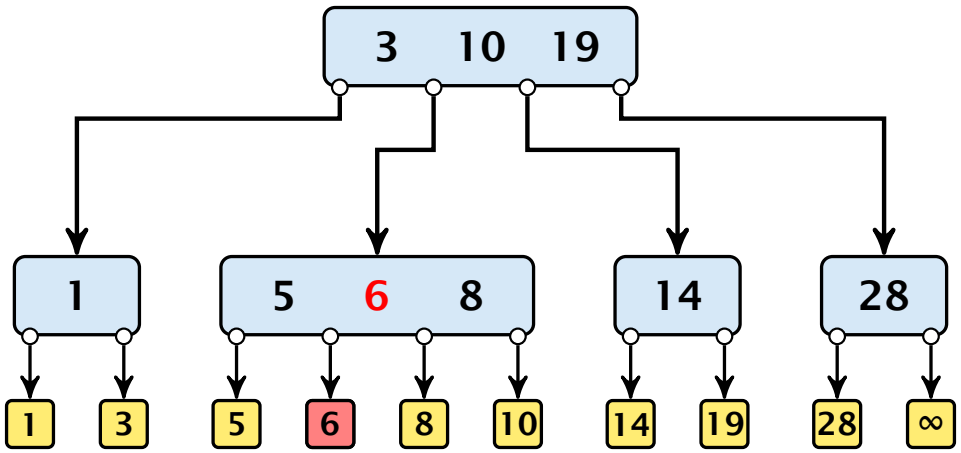

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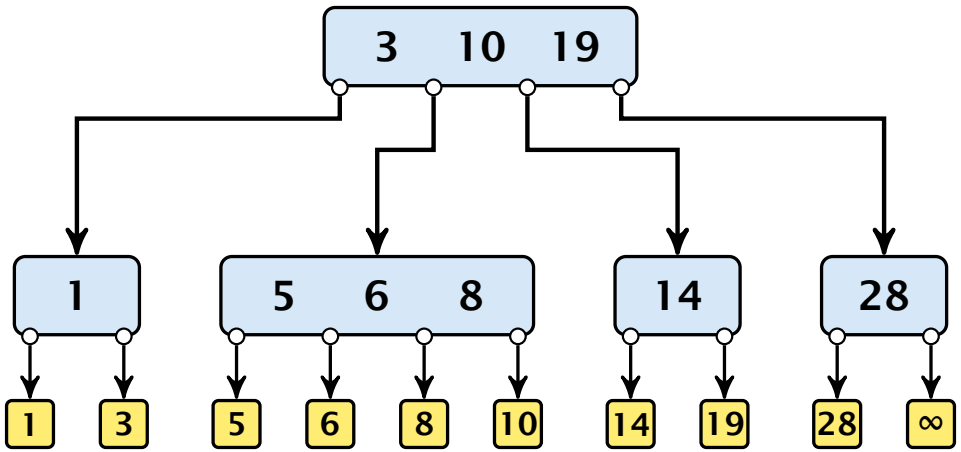

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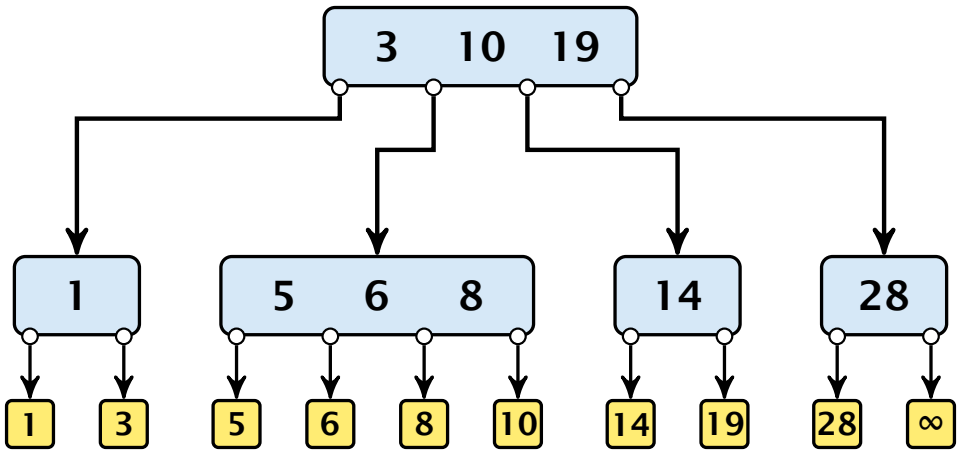

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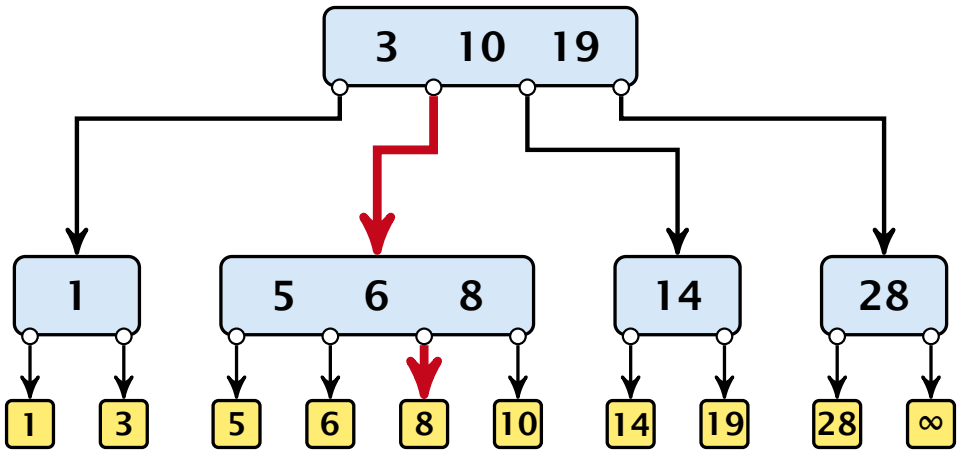

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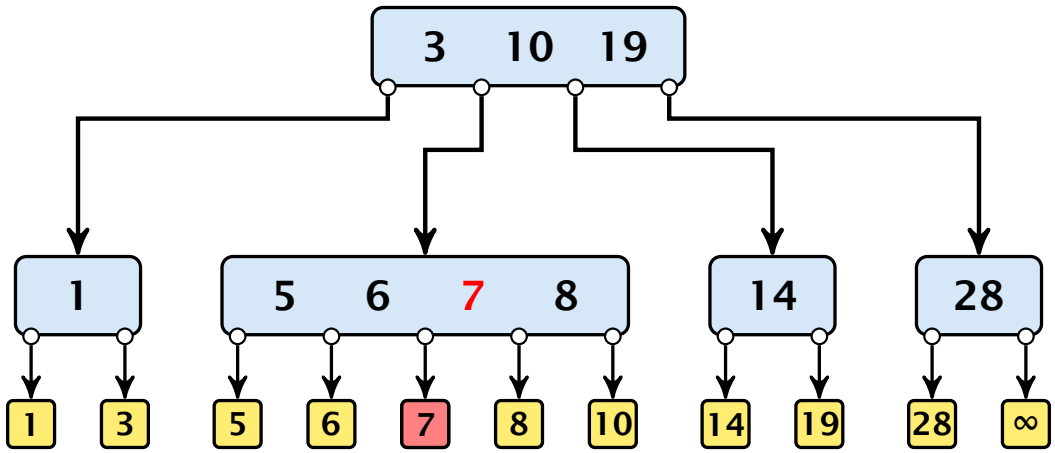

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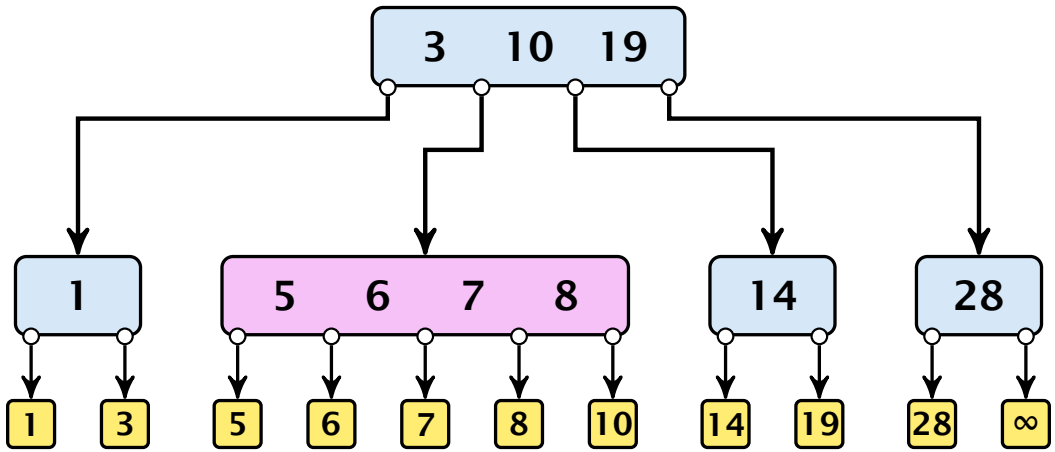

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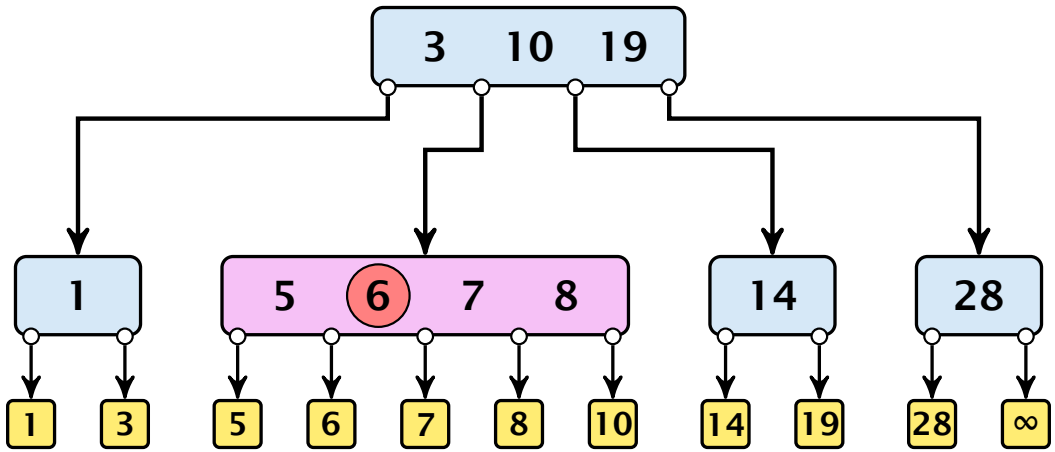

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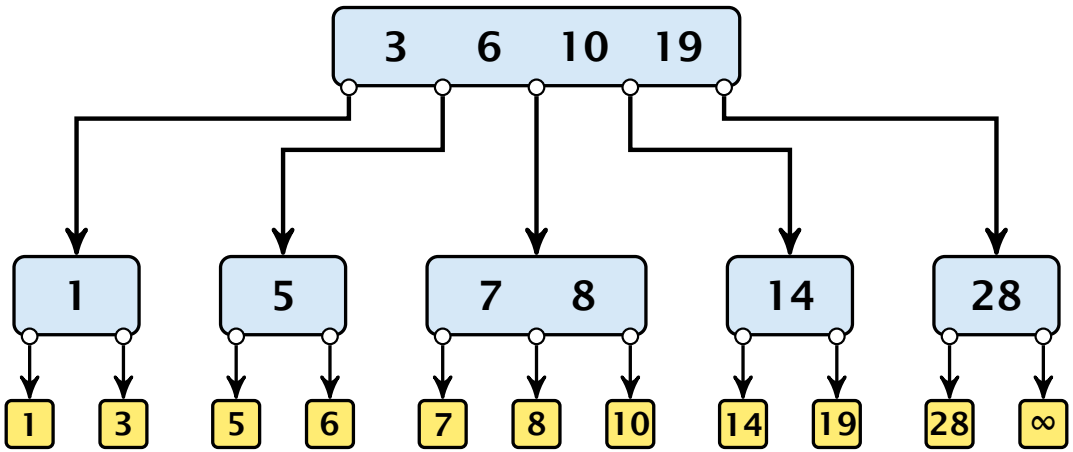

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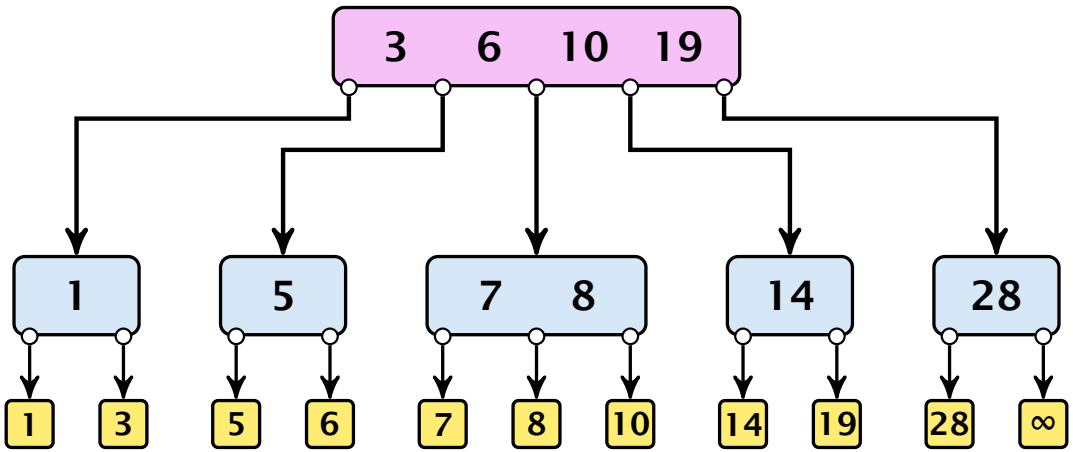

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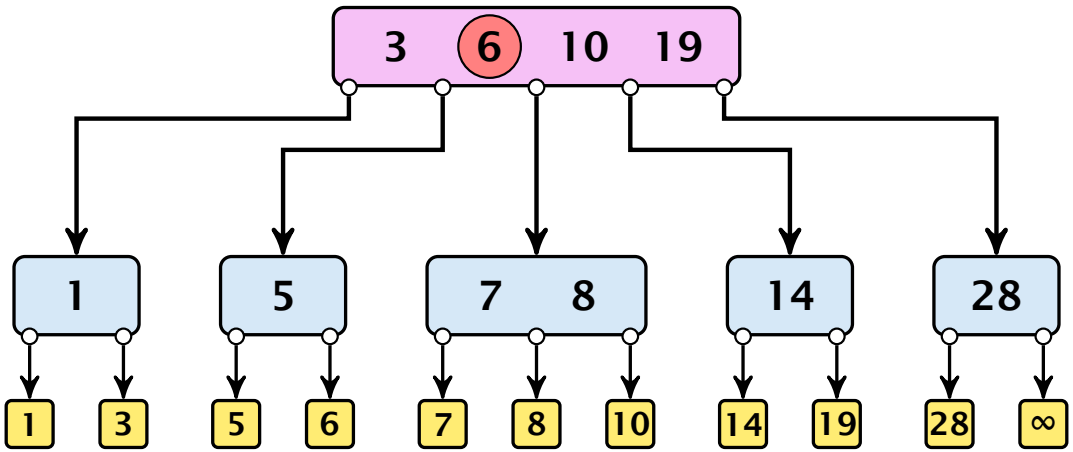

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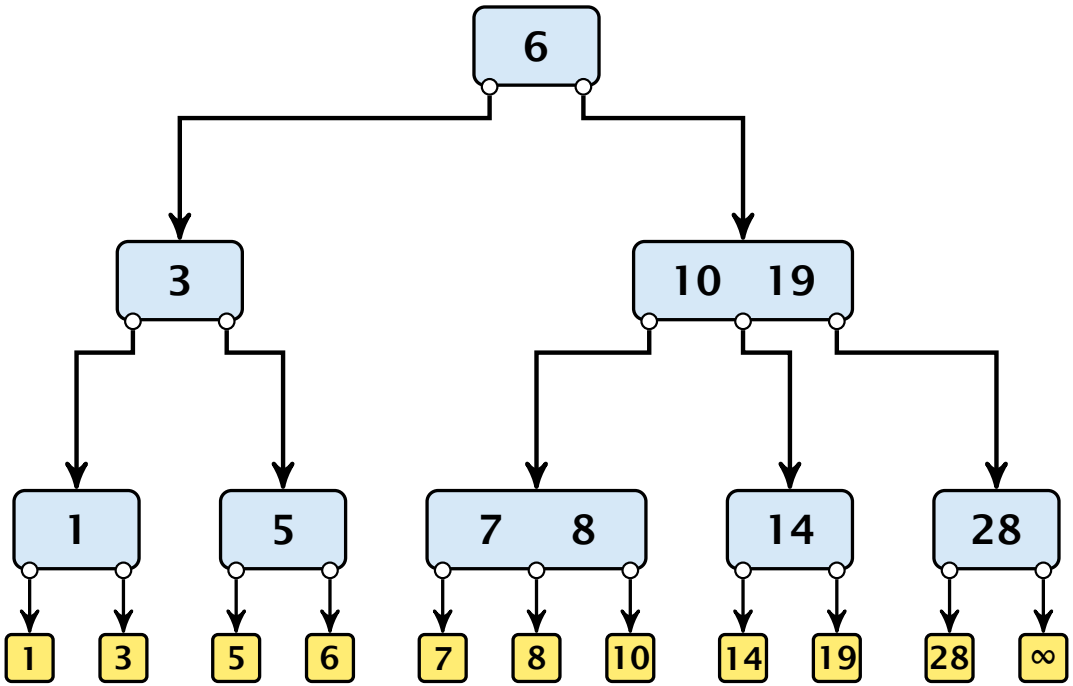

Delete

Delete element x (pointer to leaf vertex):

- Let v denote the parent of x. If $\text{key}[x]$ is contained in v, remove the key from v, and delete the leaf vertex.
- Otherwise delete the key of the predecessor of x from v; delete the leaf vertex; and replace the occurrence of $\text{key}[x]$ in internal nodes by the predecessor key. (Note that it appears in exactly one internal vertex).
- If now the number of keys in v is below $a-1$ perform Rebalance$'(v)$.

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Delete

Rebalance′(v):

- If there is a neighbour of v that has at least a keys take over the largest (if right neighbor) or smallest (if left neighbour) and the corresponding sub-tree.
- If not: merge v with one of its neighbours.
- The merged node contains at most $(a-2)+(a-1)+1$ keys, and has therefore at most $2a-1 \leq b$ successors.
- Then rebalance the parent.
- During this process the root may become empty. In this case the root is deleted and the height of the tree decreases.

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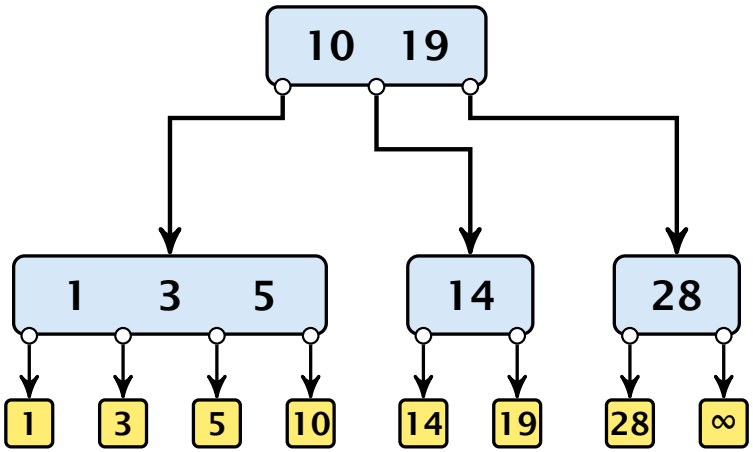
Delete



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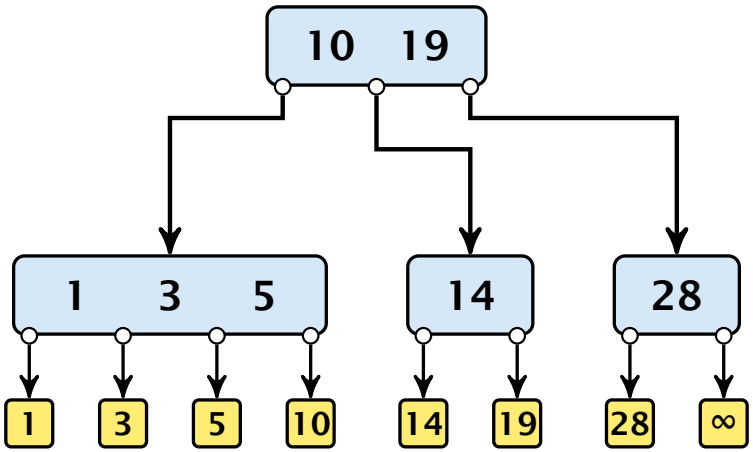

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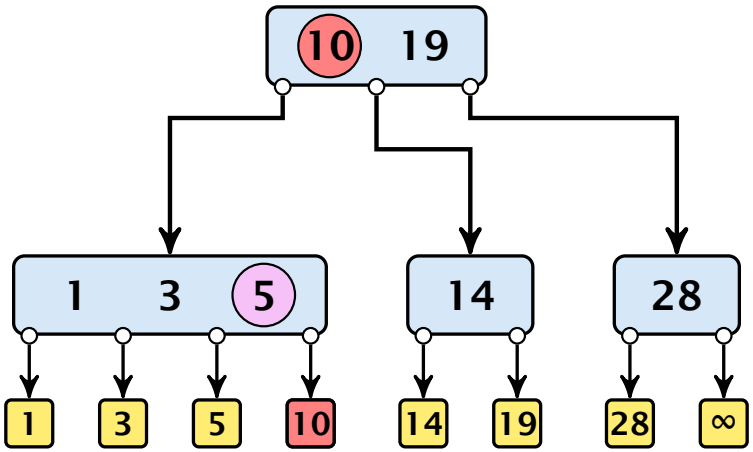

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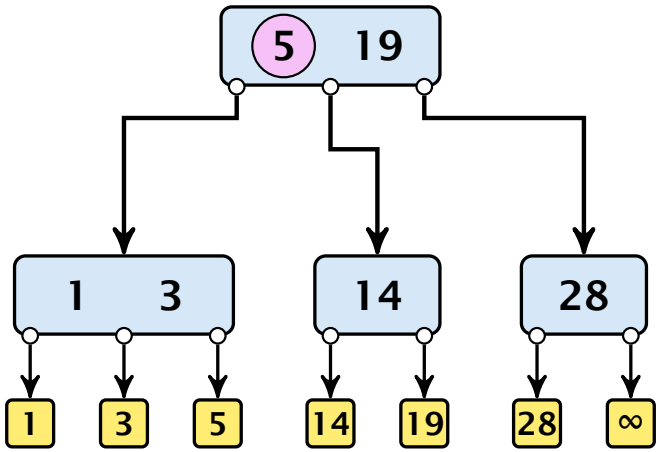

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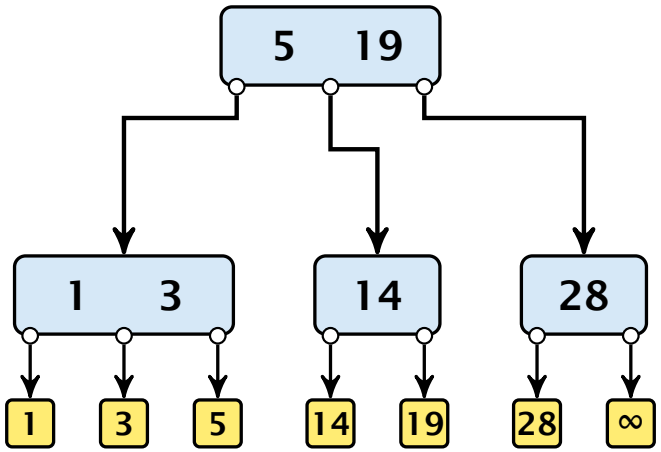

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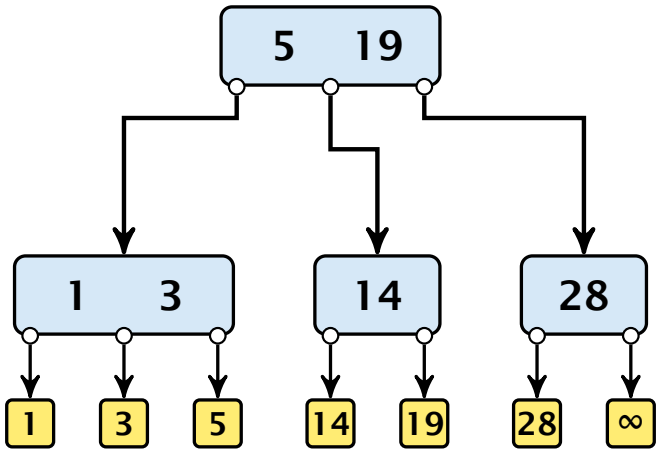

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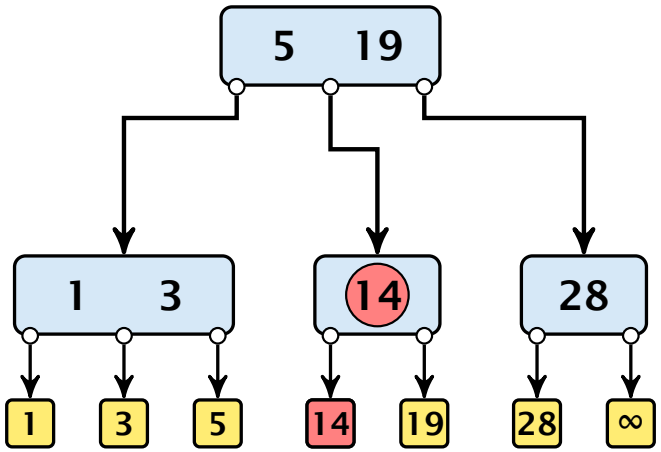

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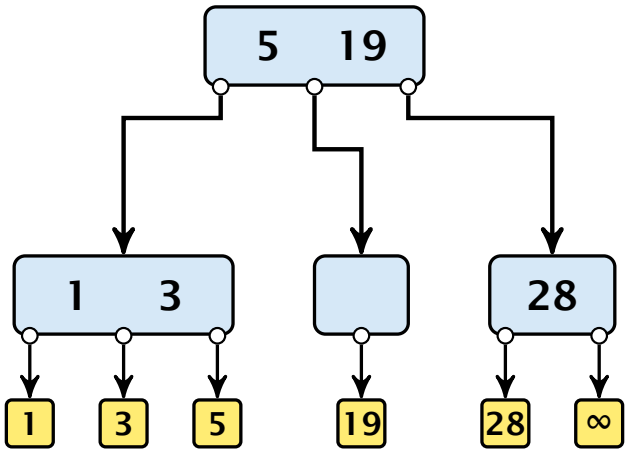

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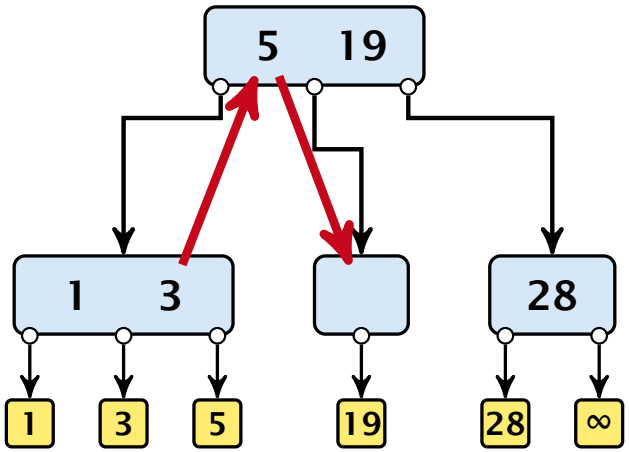

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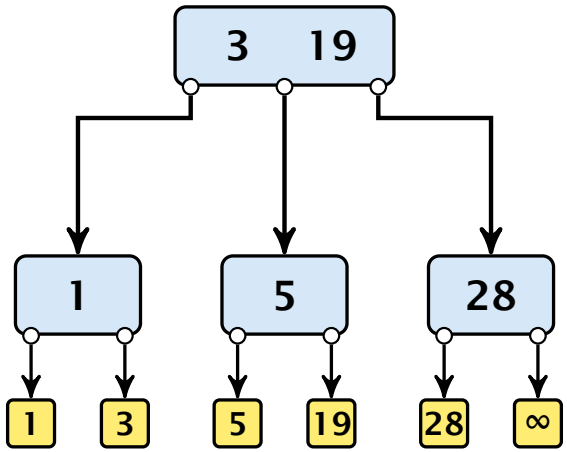

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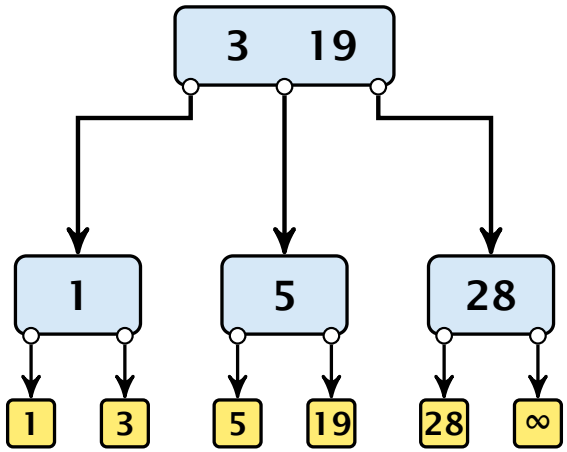

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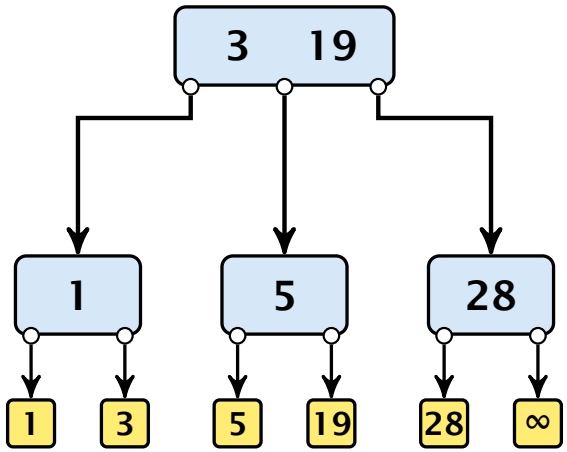

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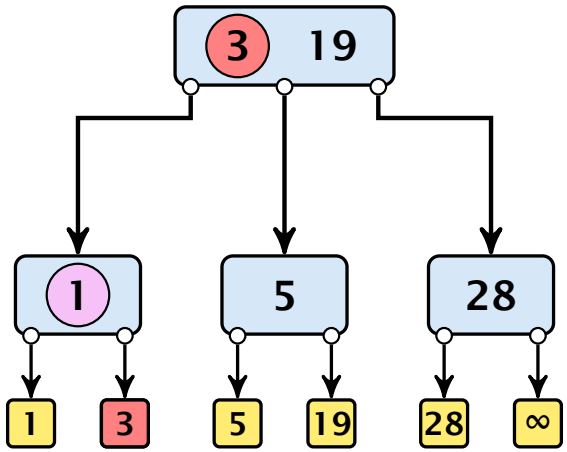

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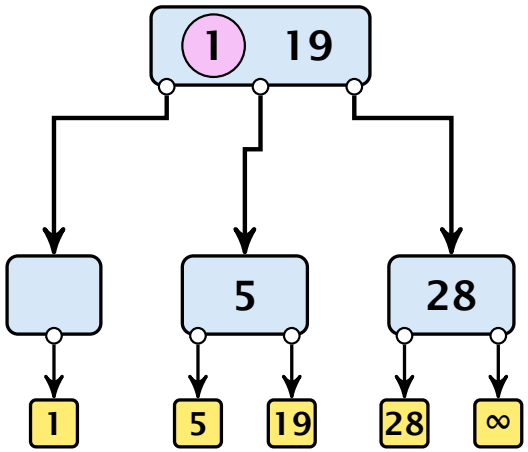

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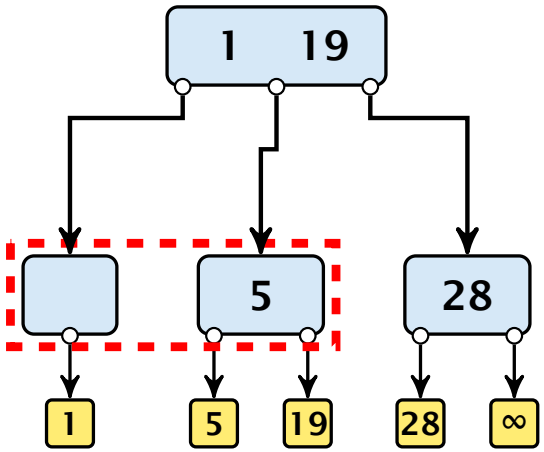

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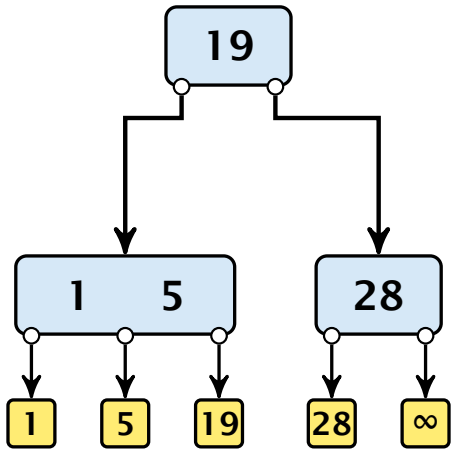

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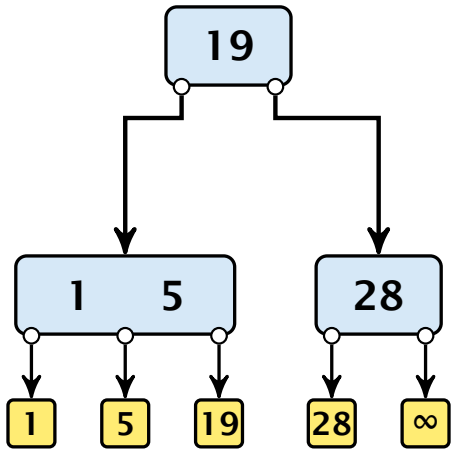

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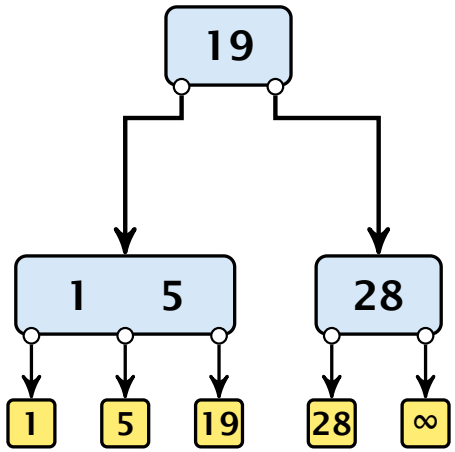

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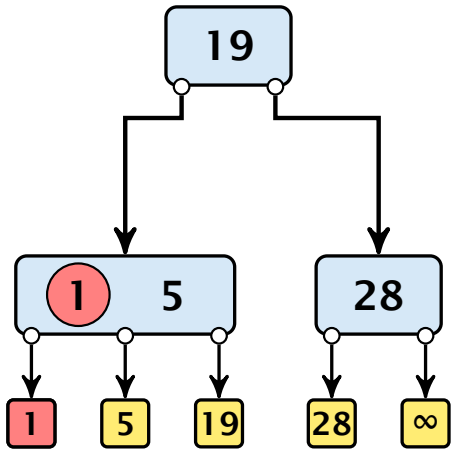
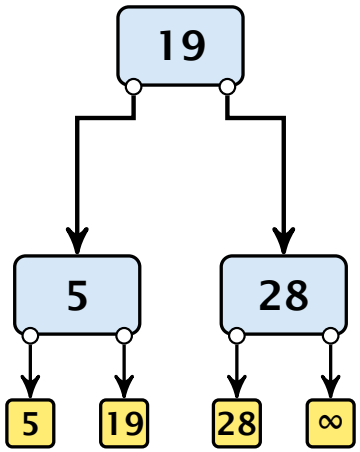
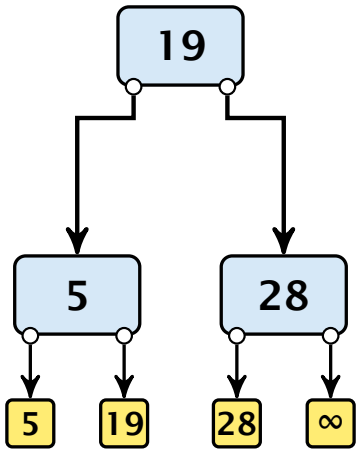


Delete

Delete(1)

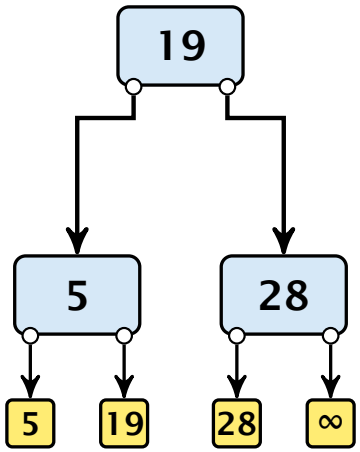


Delete



Delete

Delete(19)



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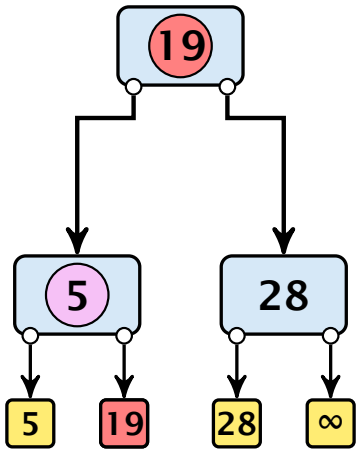

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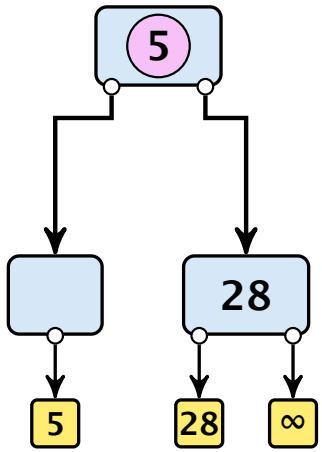

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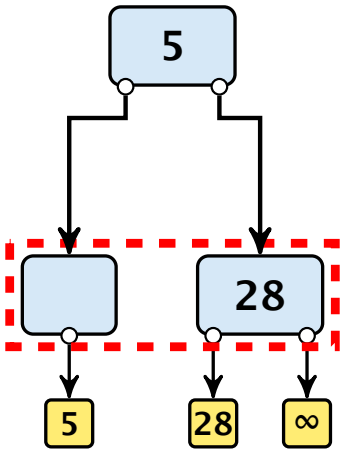

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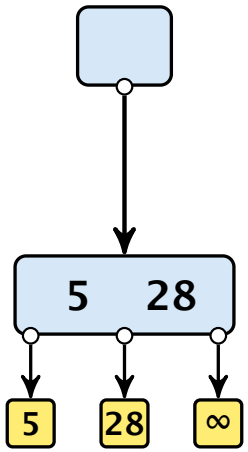

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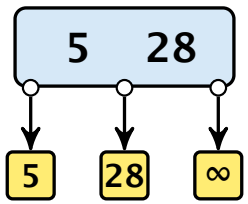
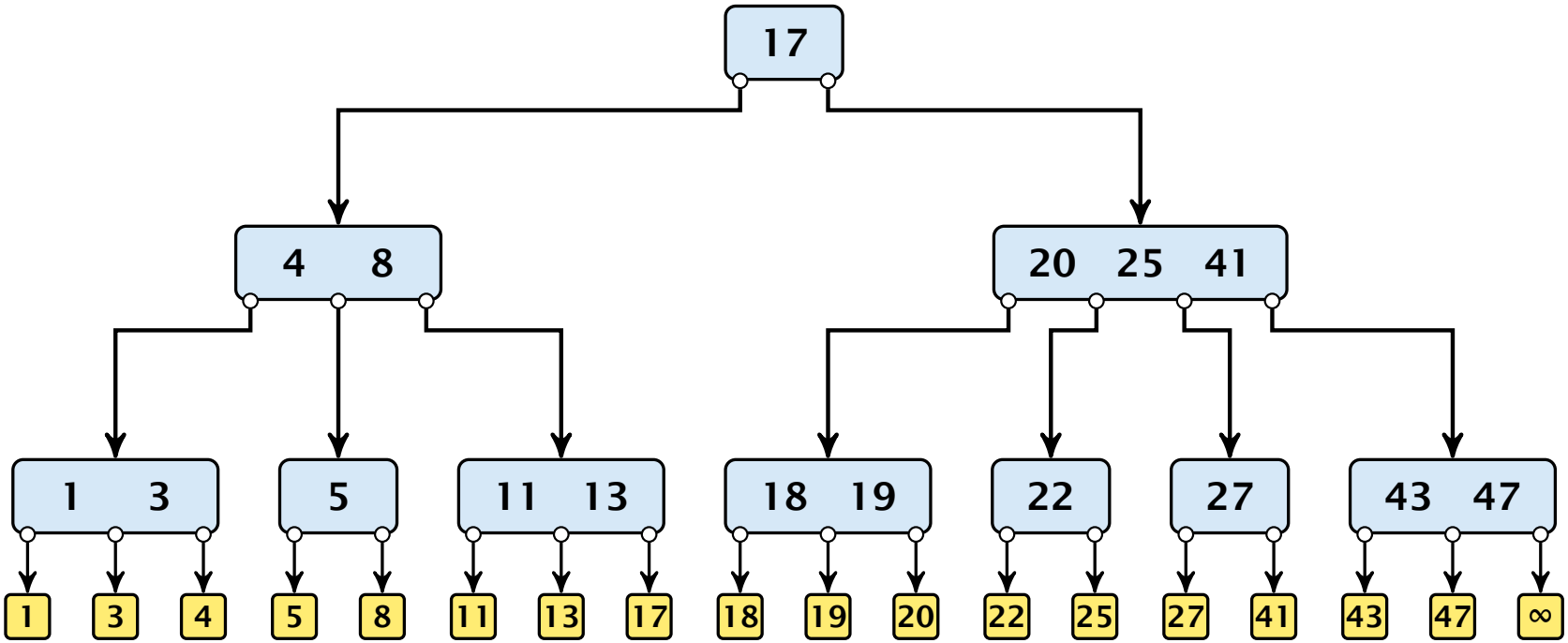


$(2,4)$-trees and red black trees

There is a close relation between red-black trees and $(2,4)$-trees:



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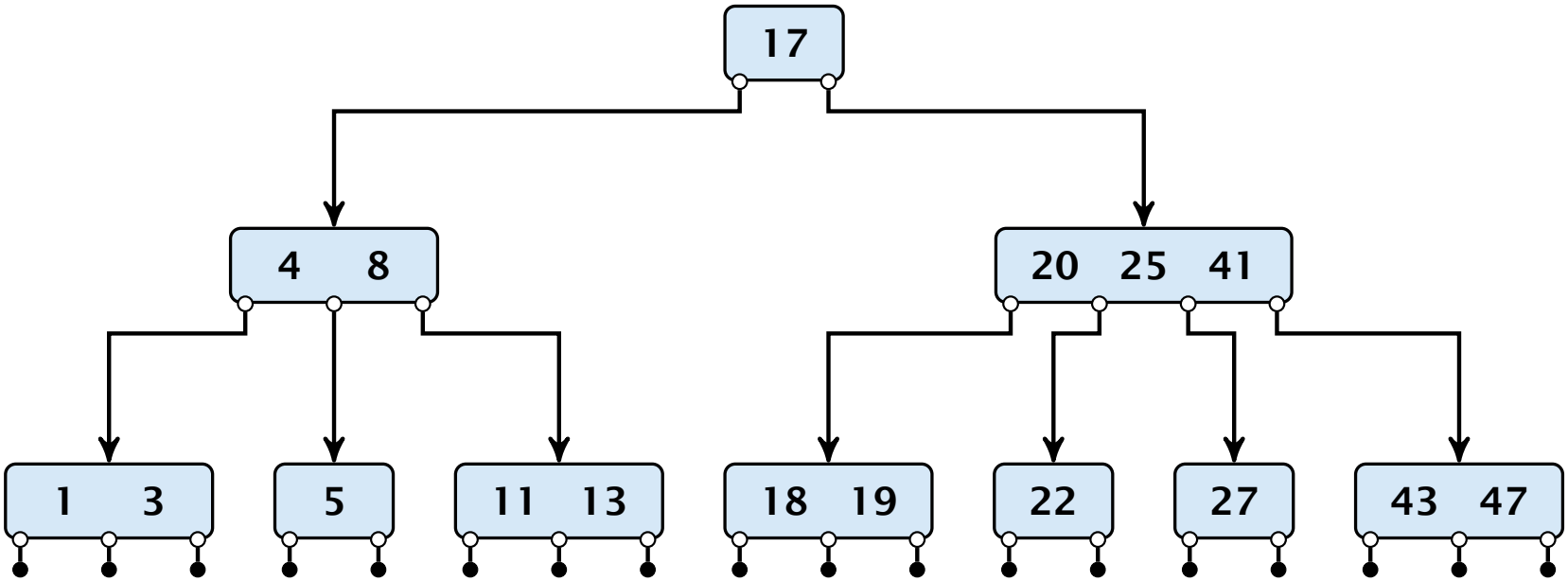

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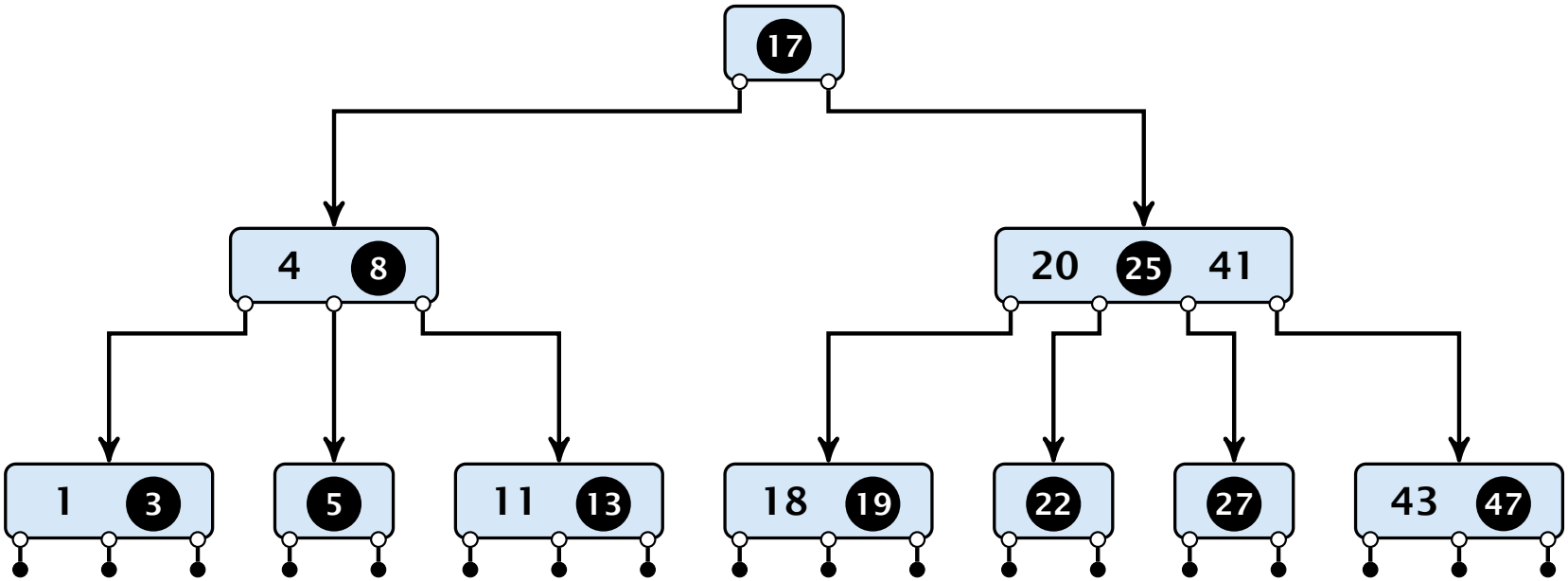

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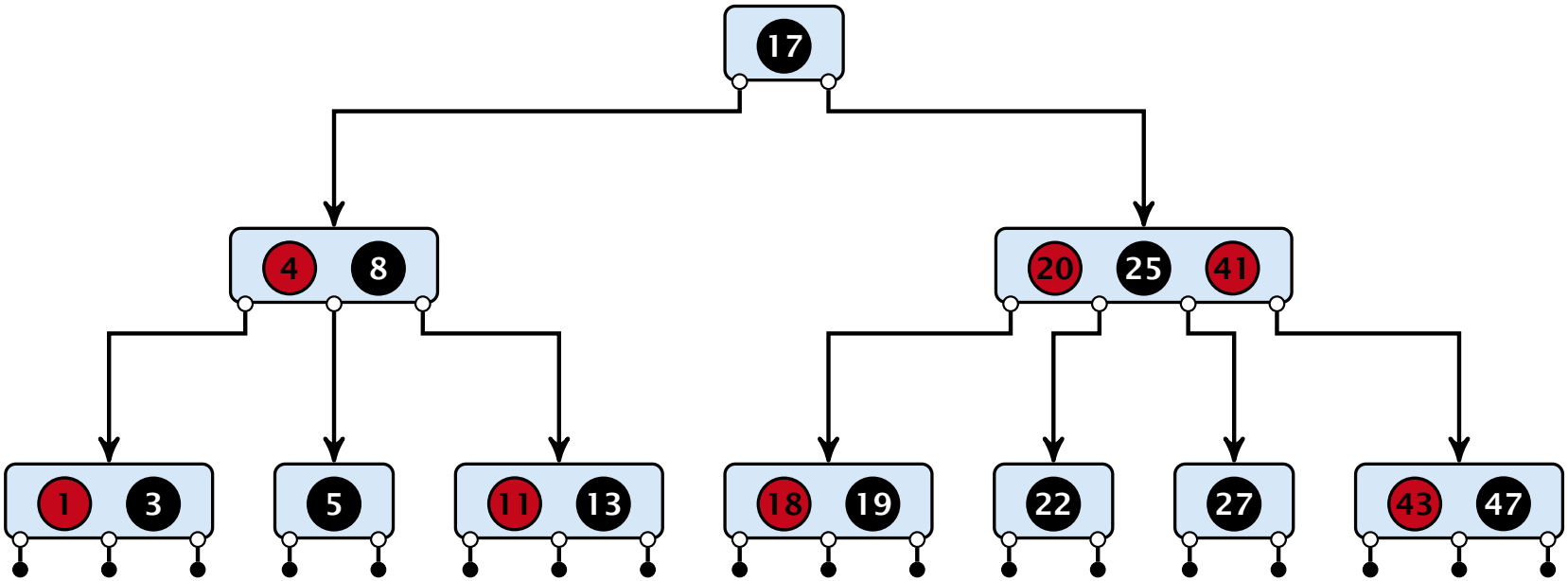

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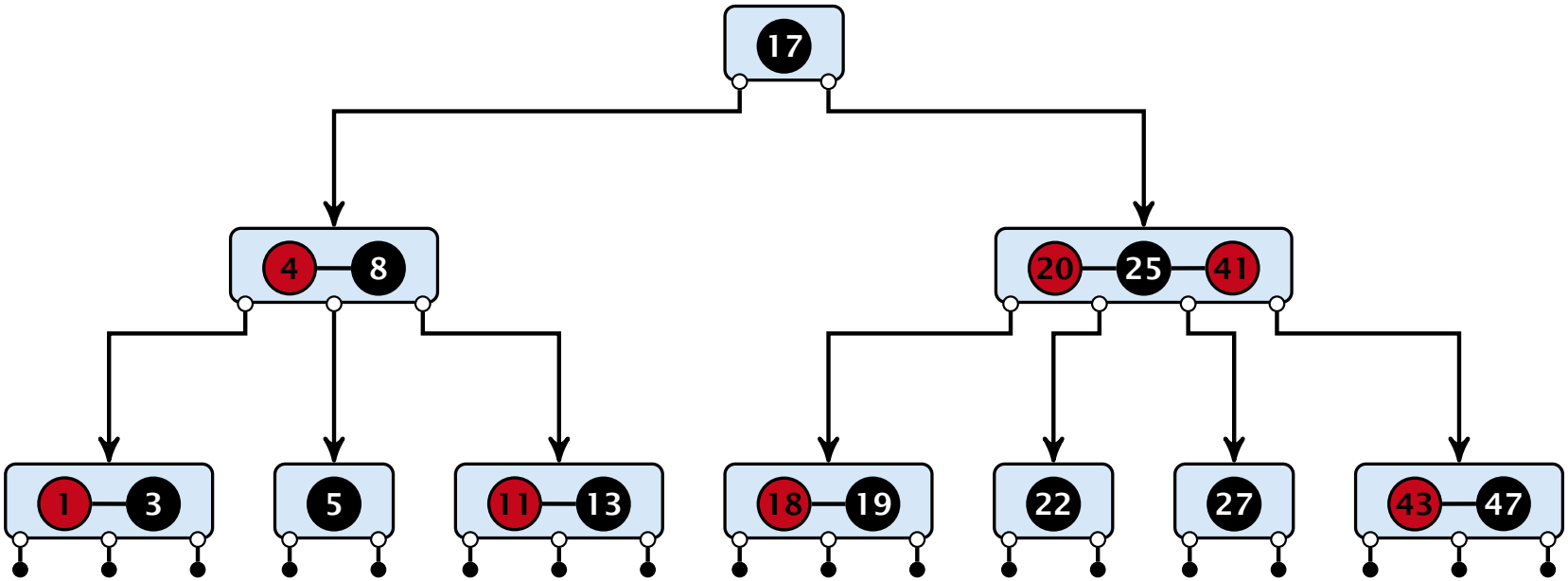

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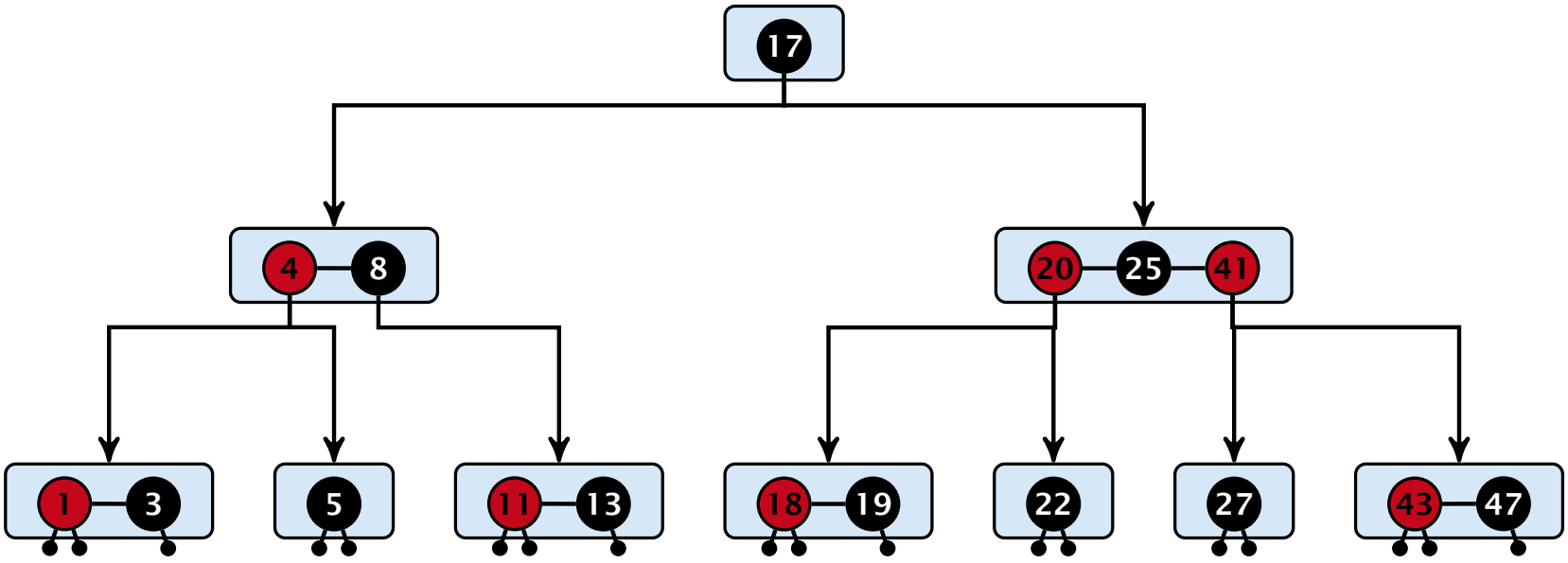

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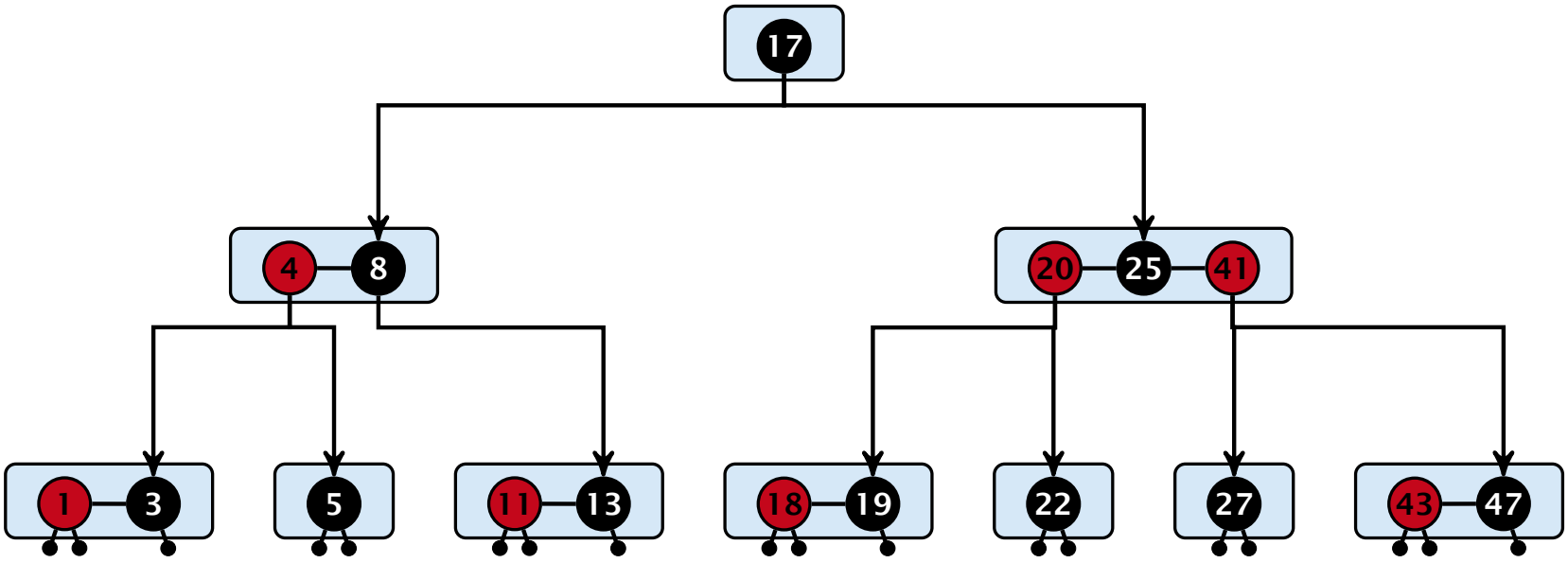

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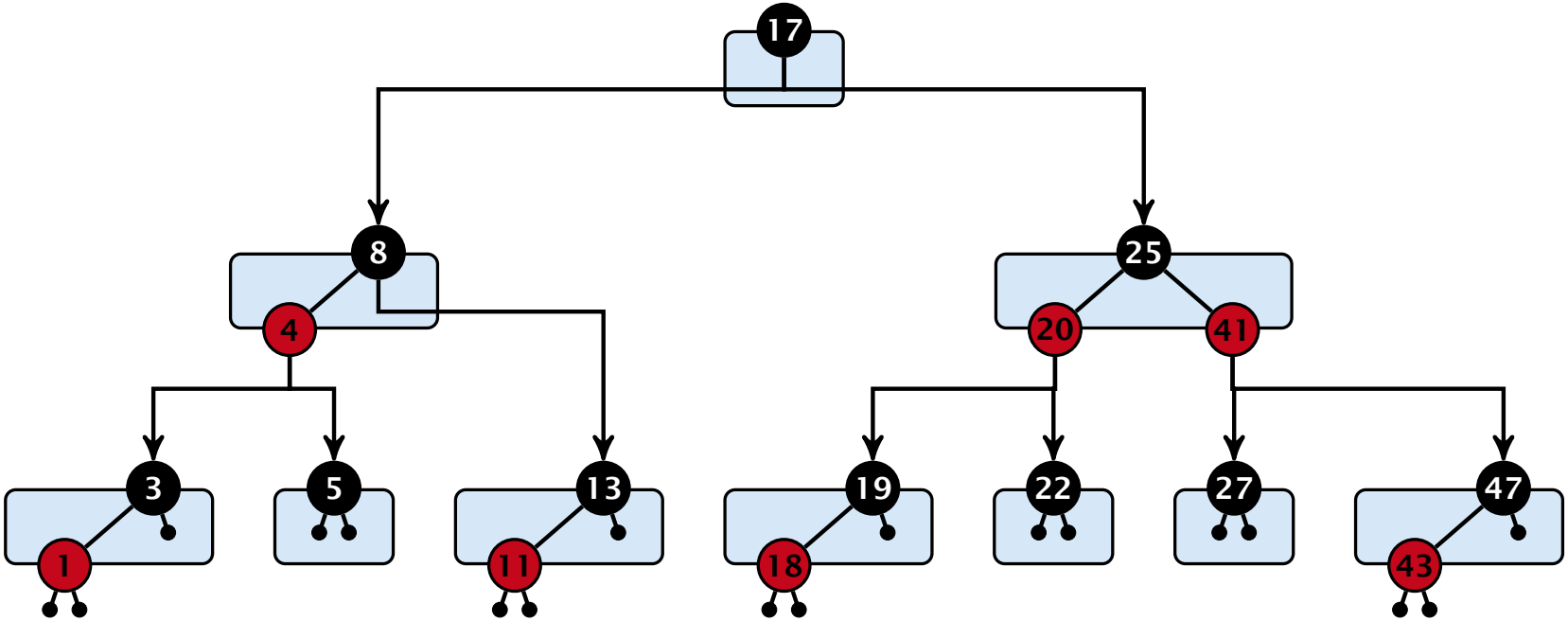

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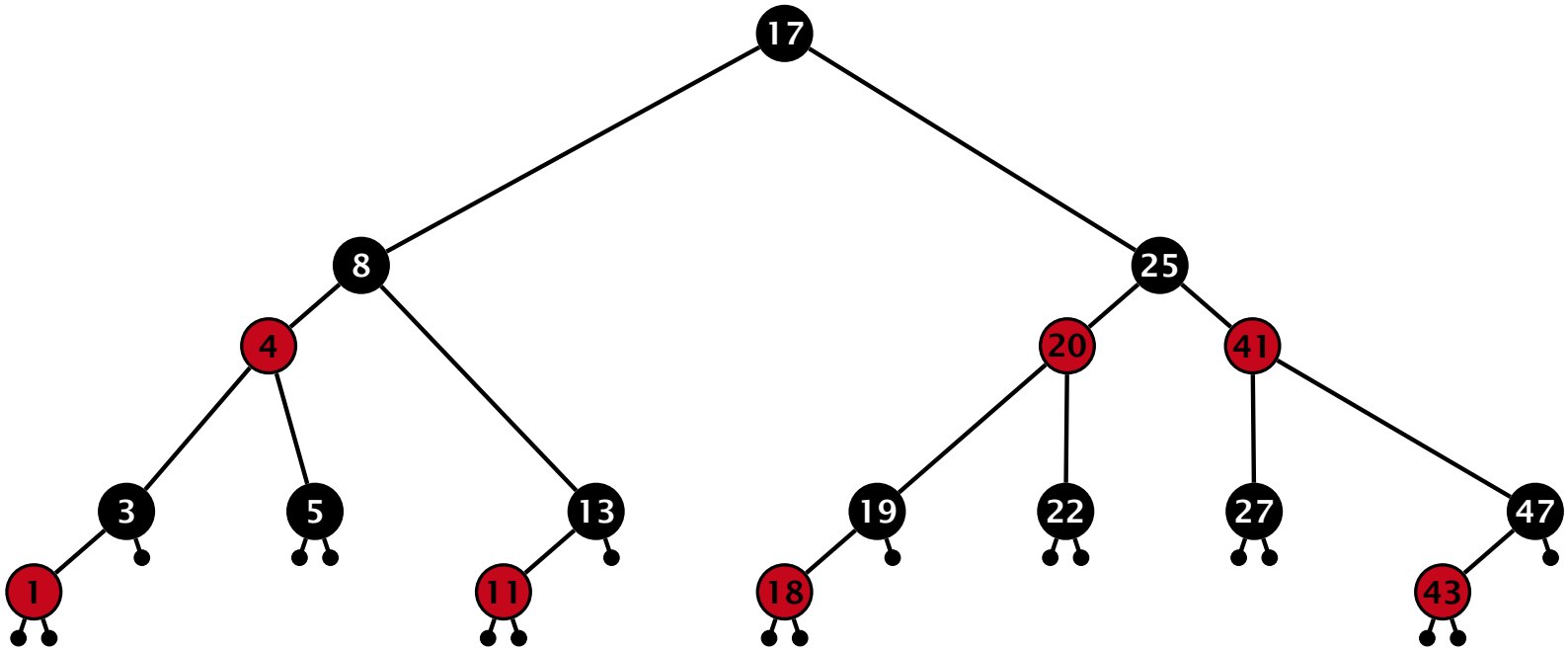

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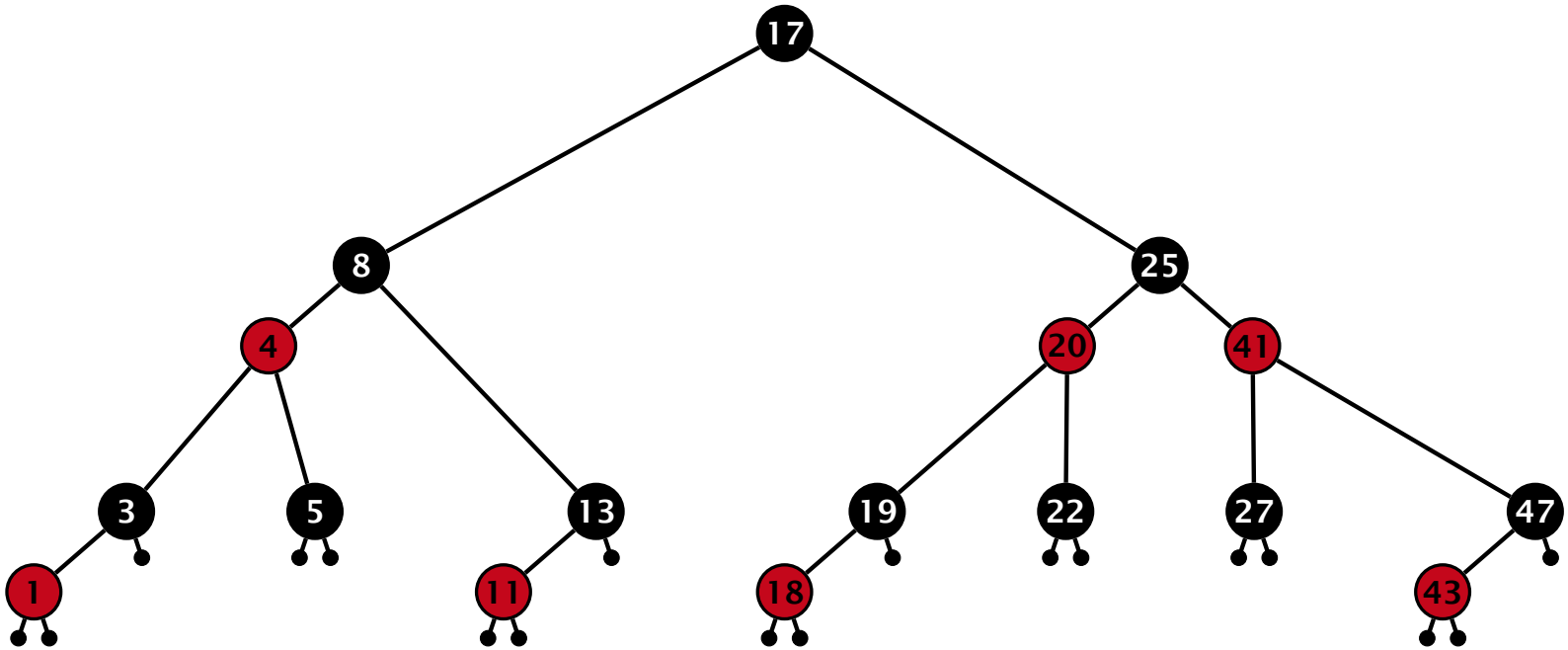

Note that this correspondence is not unique. In particular, there are different red-black trees that correspond to the same $(2,4)$-tree.

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Why do we not use a list for implementing the ADT Dynamic Set?

- time for search $\Theta(n)$
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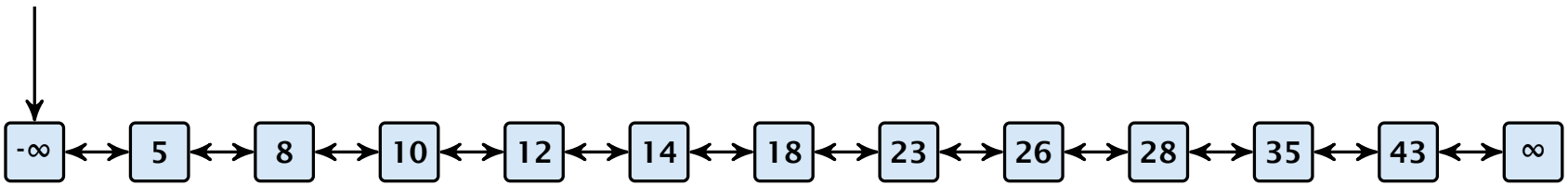

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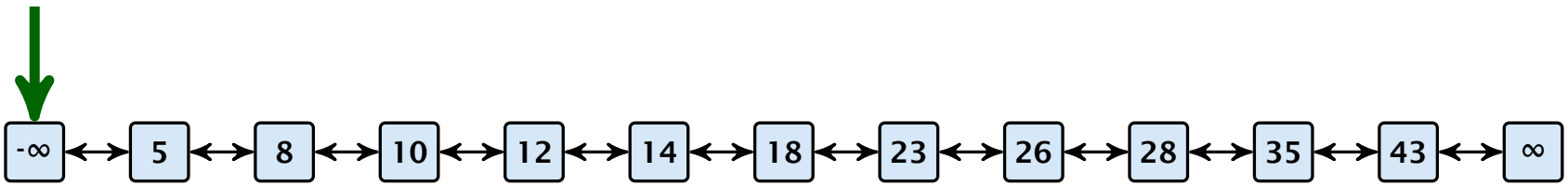

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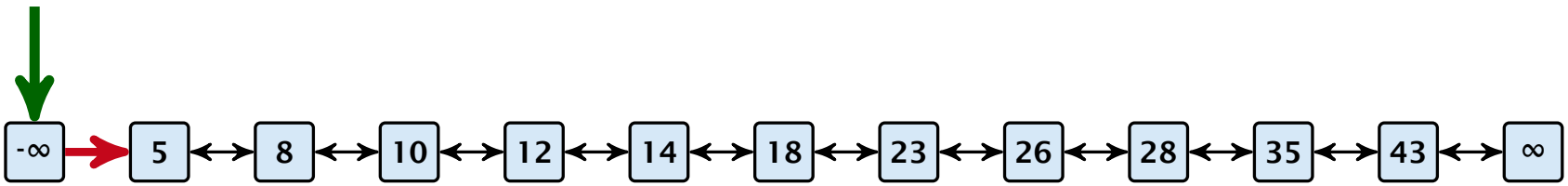

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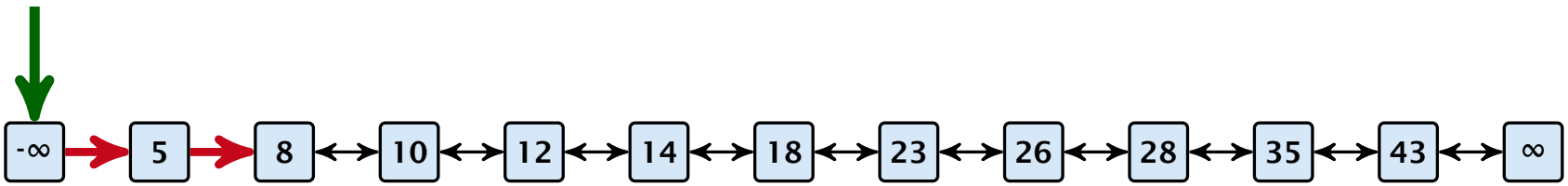

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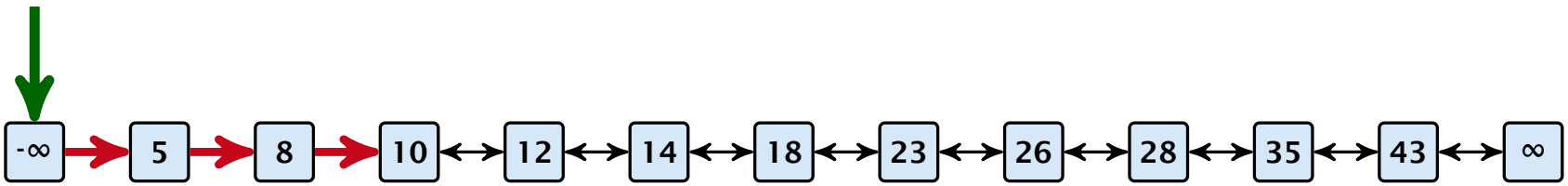

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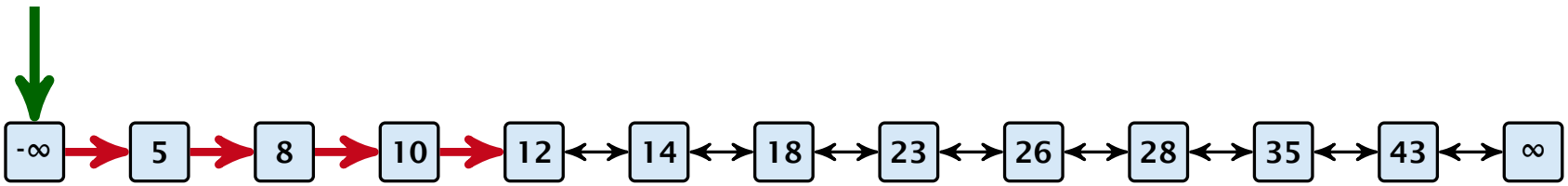

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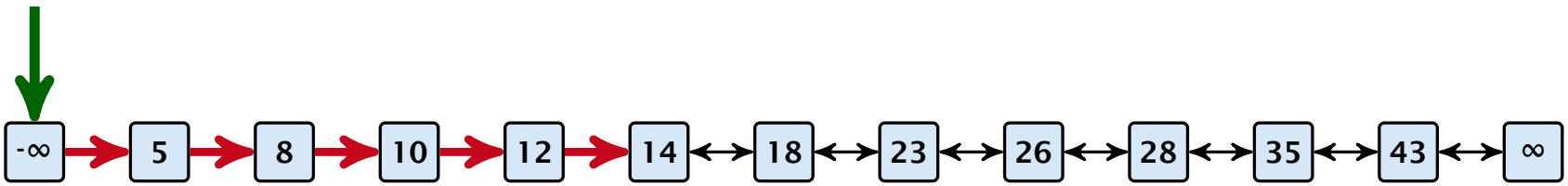

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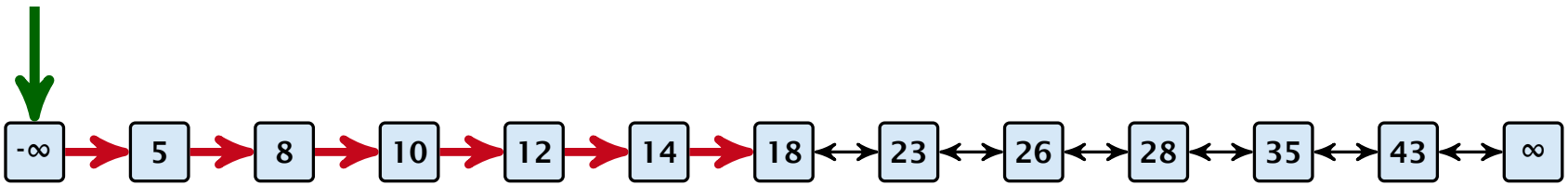

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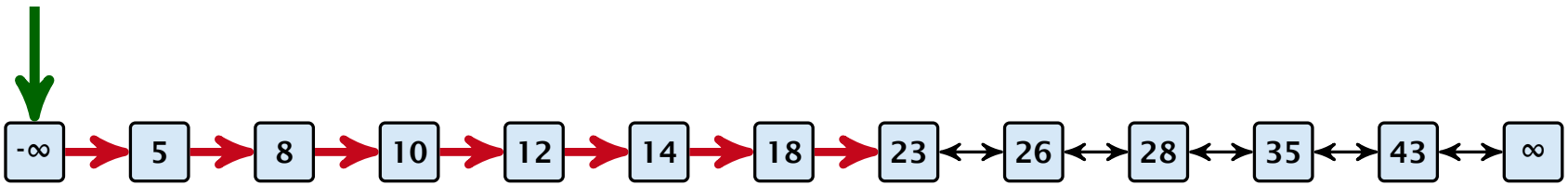

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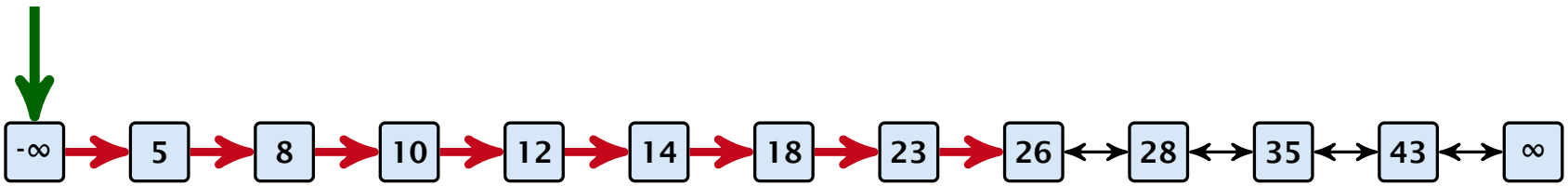

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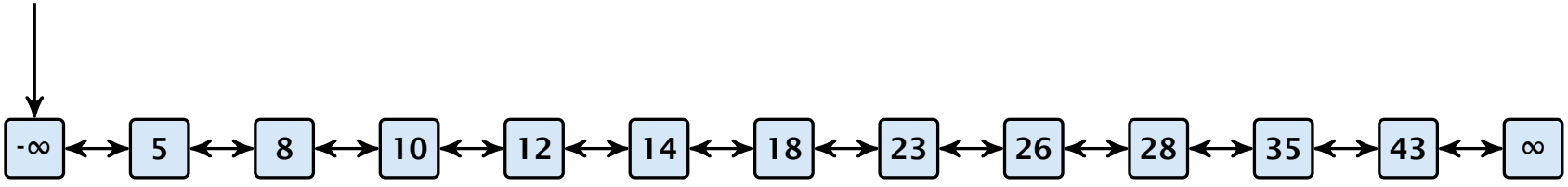

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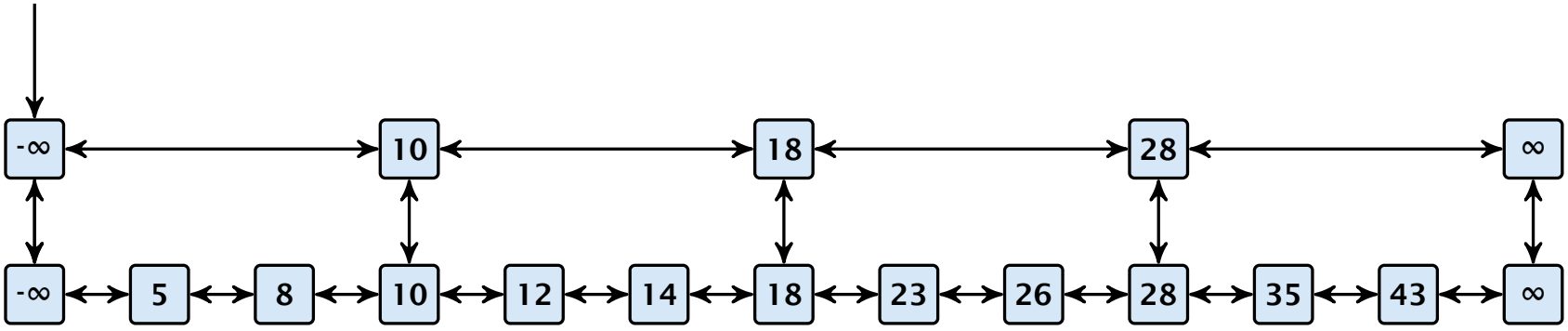

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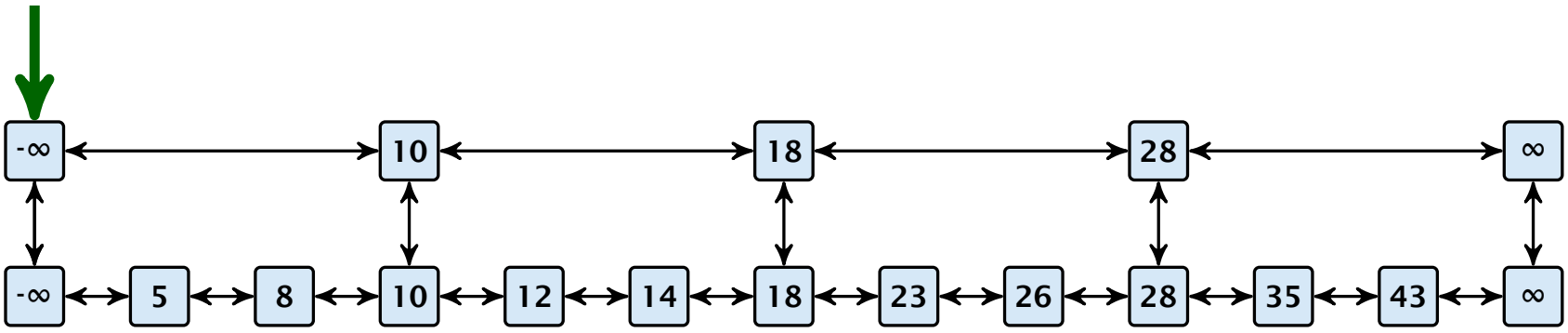

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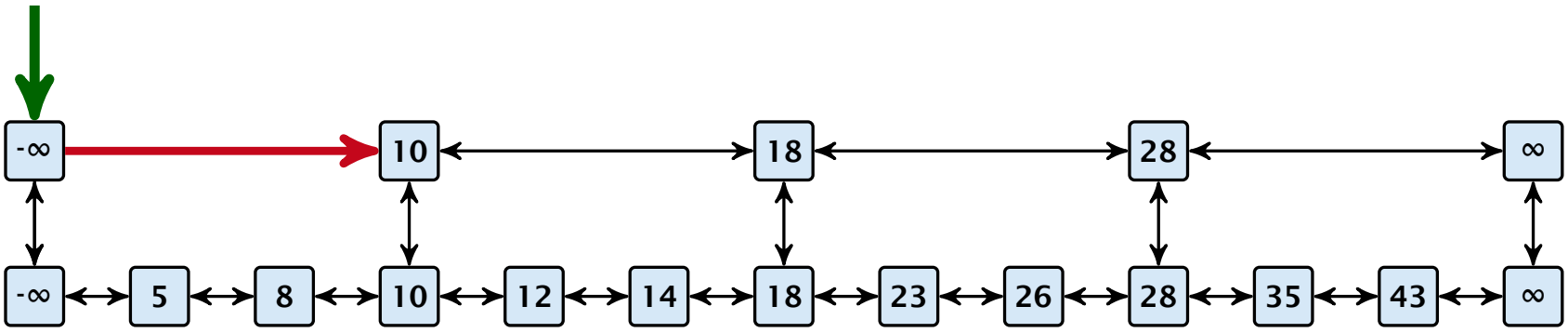

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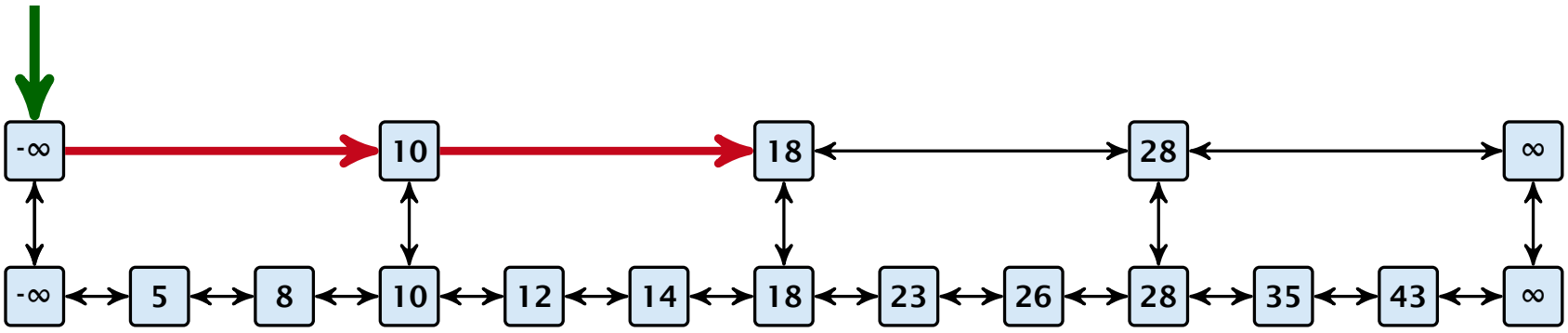

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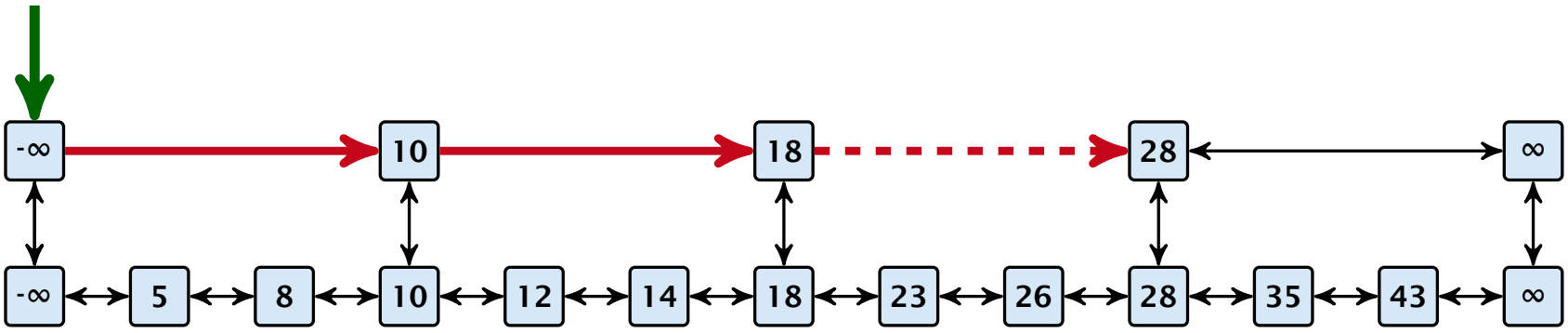

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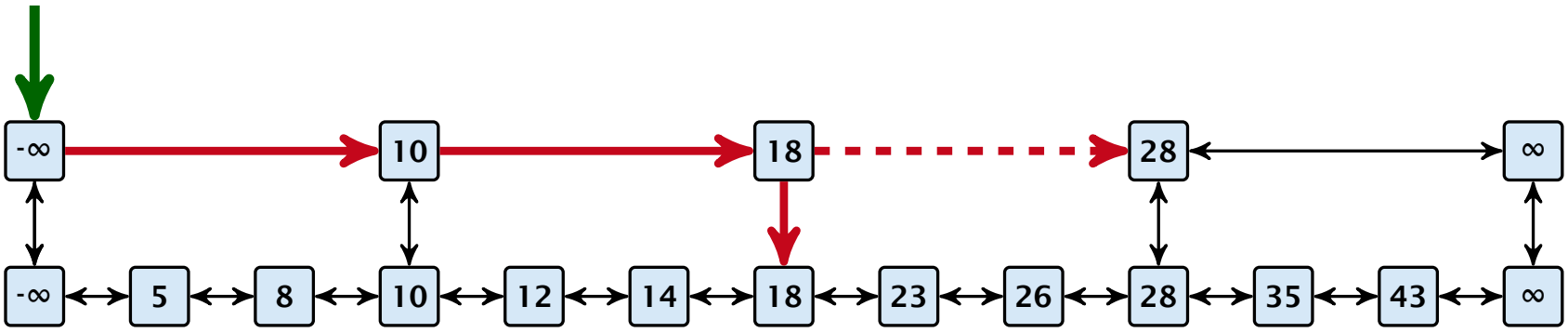

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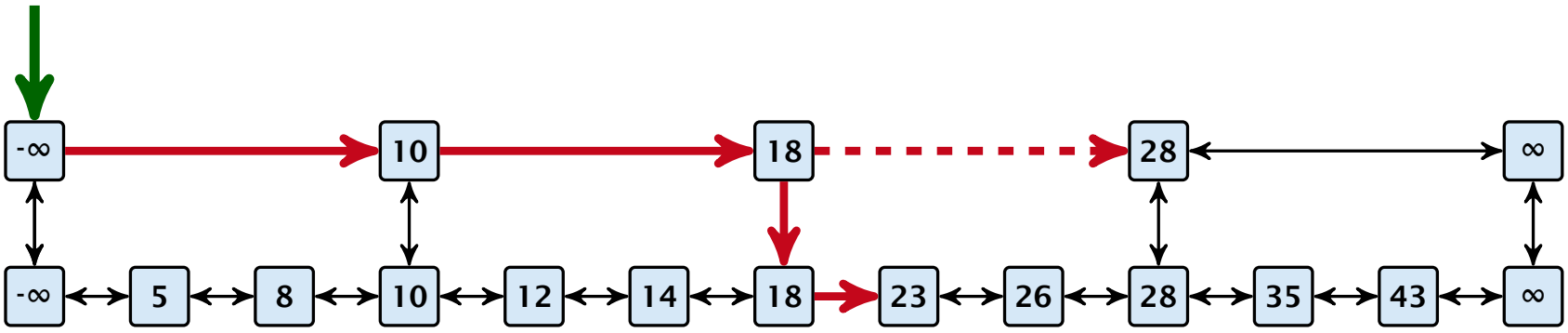

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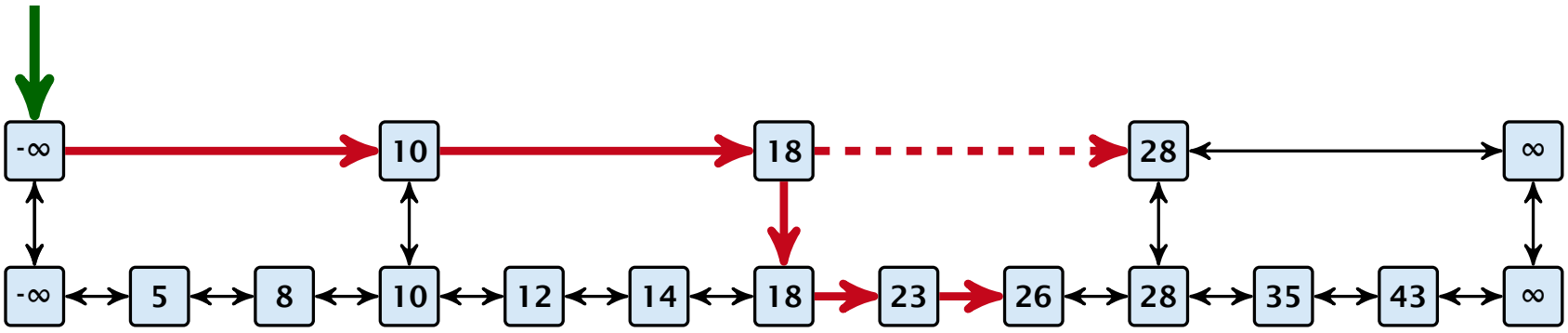

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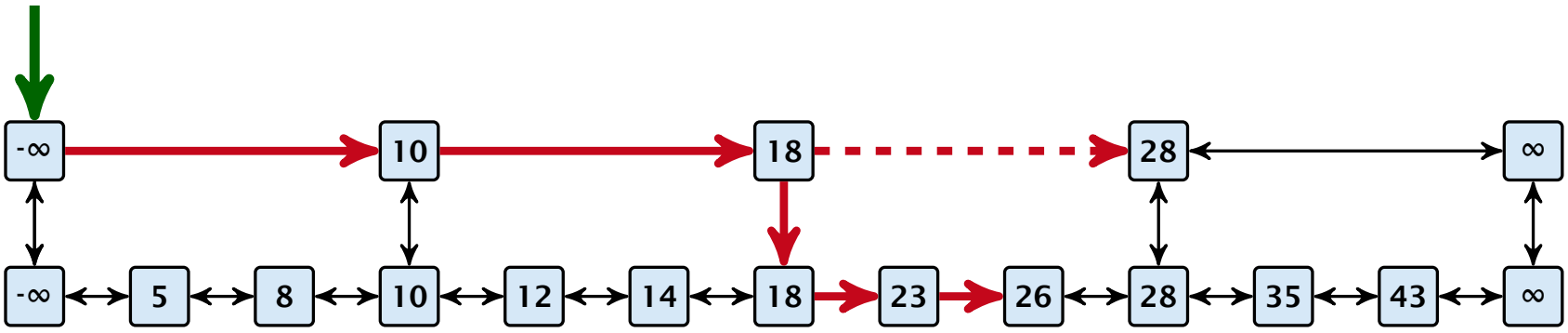

Let $|L_1|$ denote the number of elements in the "express lane", and $|L_0| = n$ the number of all elements (ignoring dummy elements).

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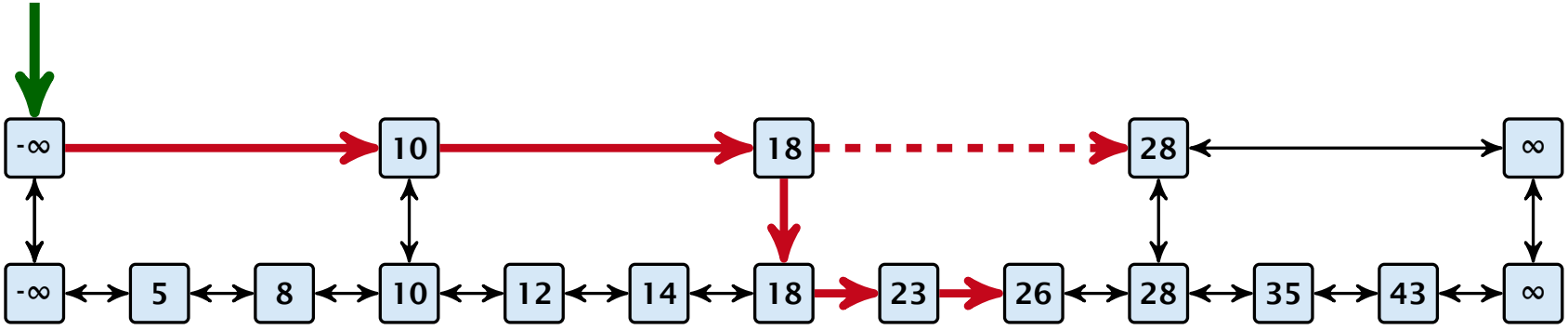

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Worst case search time: $|L_1| + \frac{|L_0|}{|L_1|}$ (ignoring additive constants)

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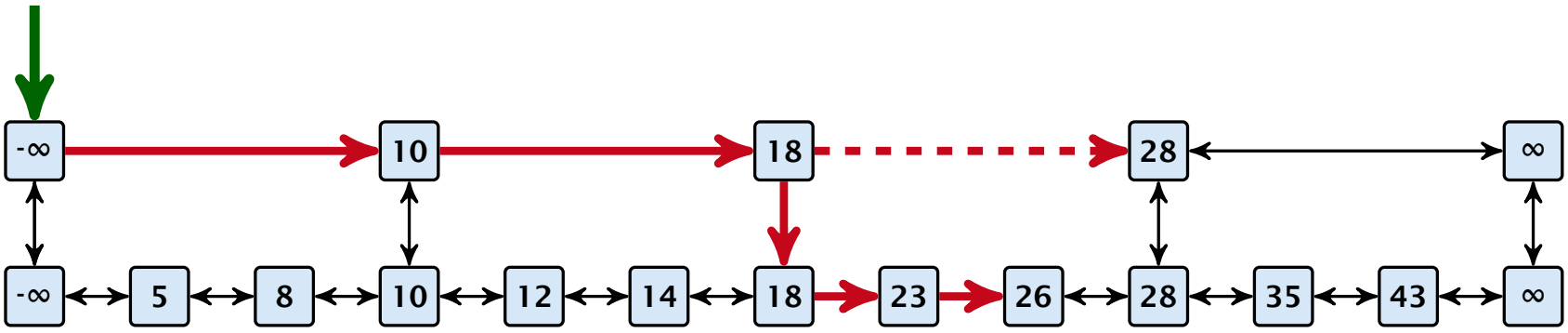

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Worst case search time: $|L_1| + \frac{|L_0|}{|L_1|}$ (ignoring additive constants)

Choose $|L_1| = \sqrt{n}$. Then search time $\Theta(\sqrt{n})$.

7.6 Skip Lists

Add more express lanes. Lane L_i contains roughly every $\frac{L_{i-1}}{L_i}$-th item from list L_{i-1}.

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Choose ratios between list-lengths evenly, i.e., $\frac{|L_{i-1}|}{|L_i|} = r$, and, hence, $L_k \approx r^{-k} n$.

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Choosing $k = \Theta(\log n)$ gives a logarithmic running time.

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How to do insert and delete?

- If we want that in L_i we always skip over roughly the same number of elements in L_{i-1}, an insert or delete may require a lot of re-organisation.

Use randomization instead!

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- Flip a coin until it shows head, and record the number $t \in \{1, 2, \ldots\}$ of trials needed.
- Insert x into lists $L_0, \ldots, L_{t-1}$.

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- You get all predecessors via backward pointers.
- Delete x in all lists it actually appears in.

The time for both operations is dominated by the search time.

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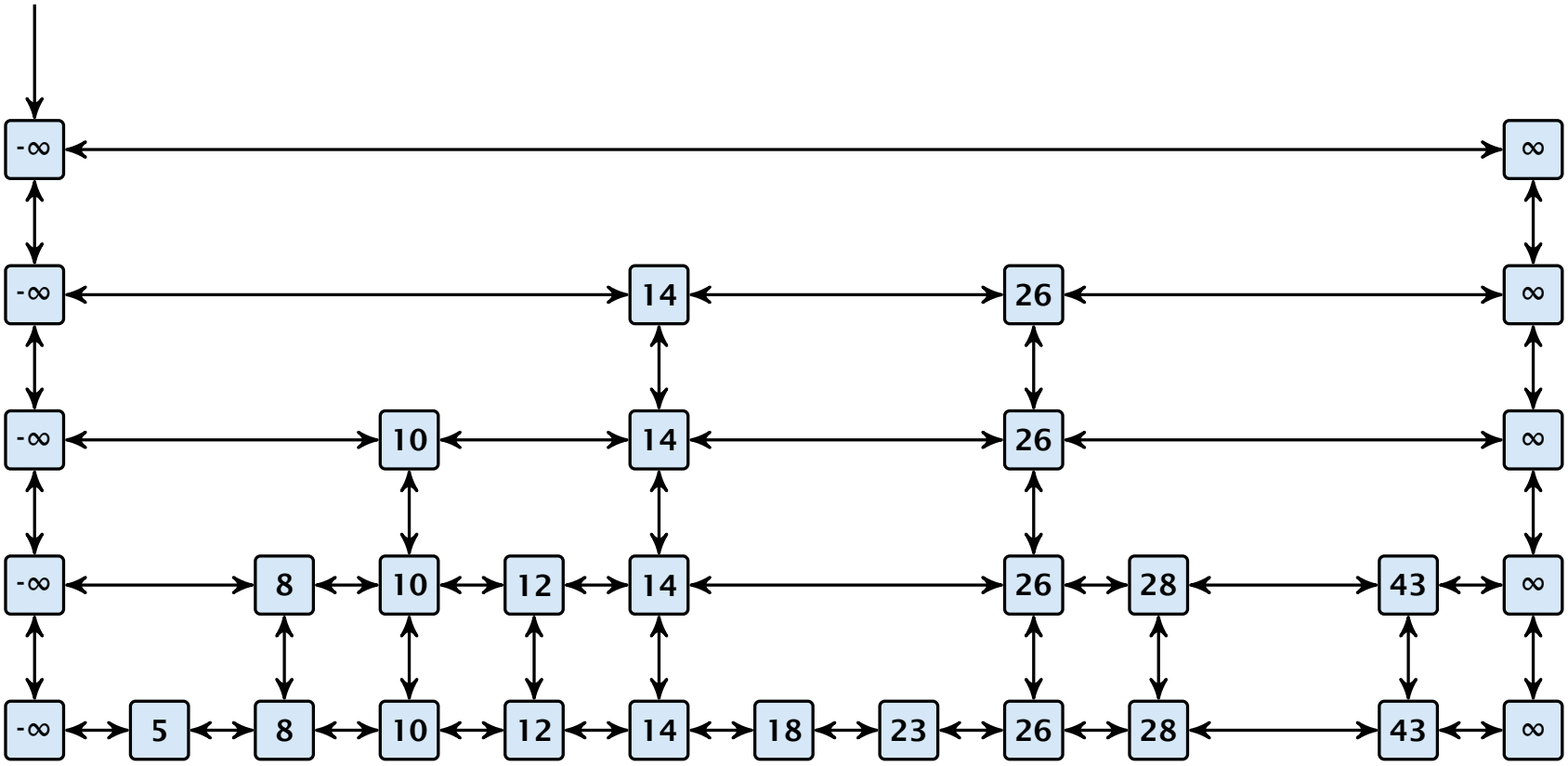

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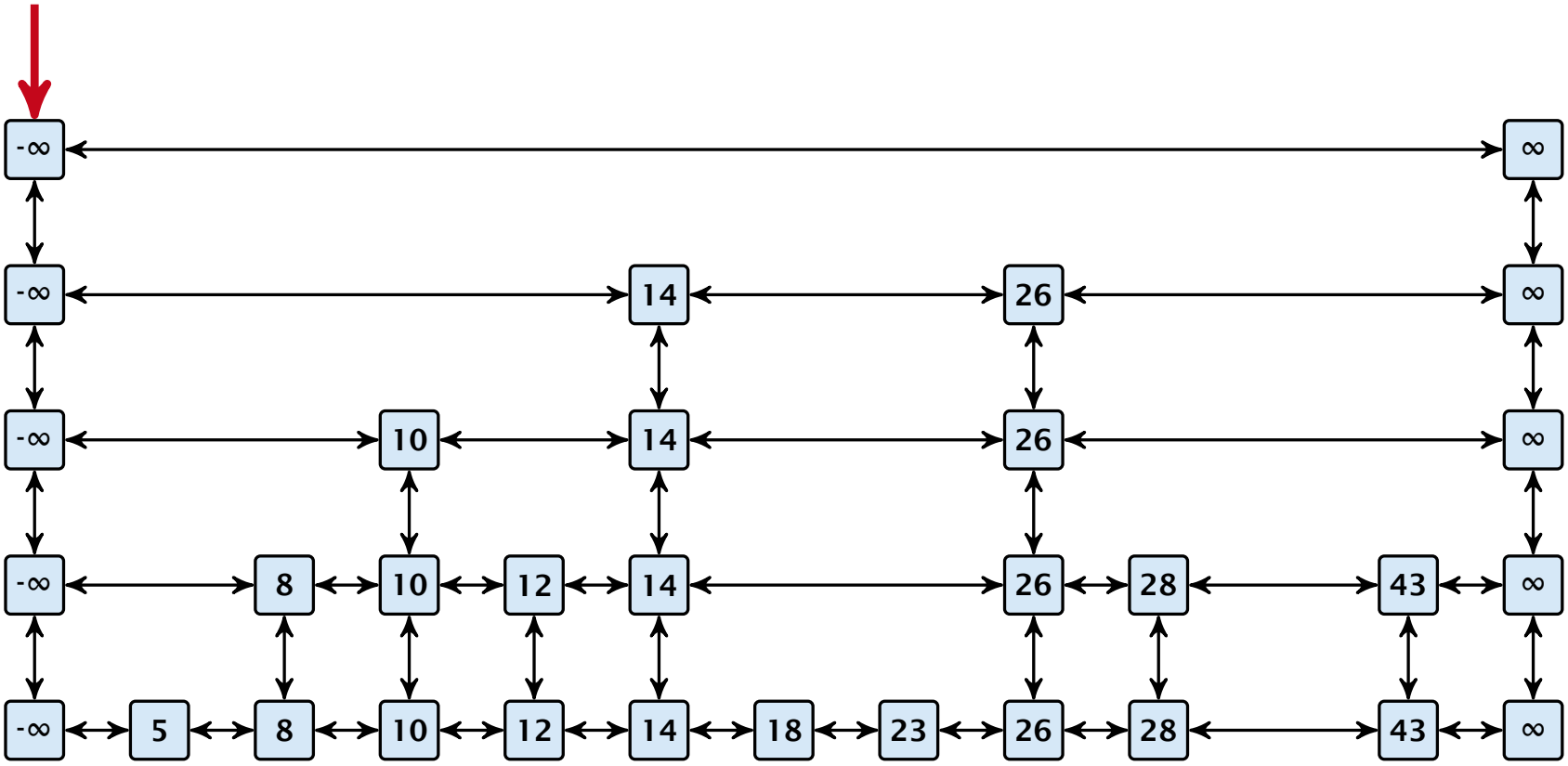

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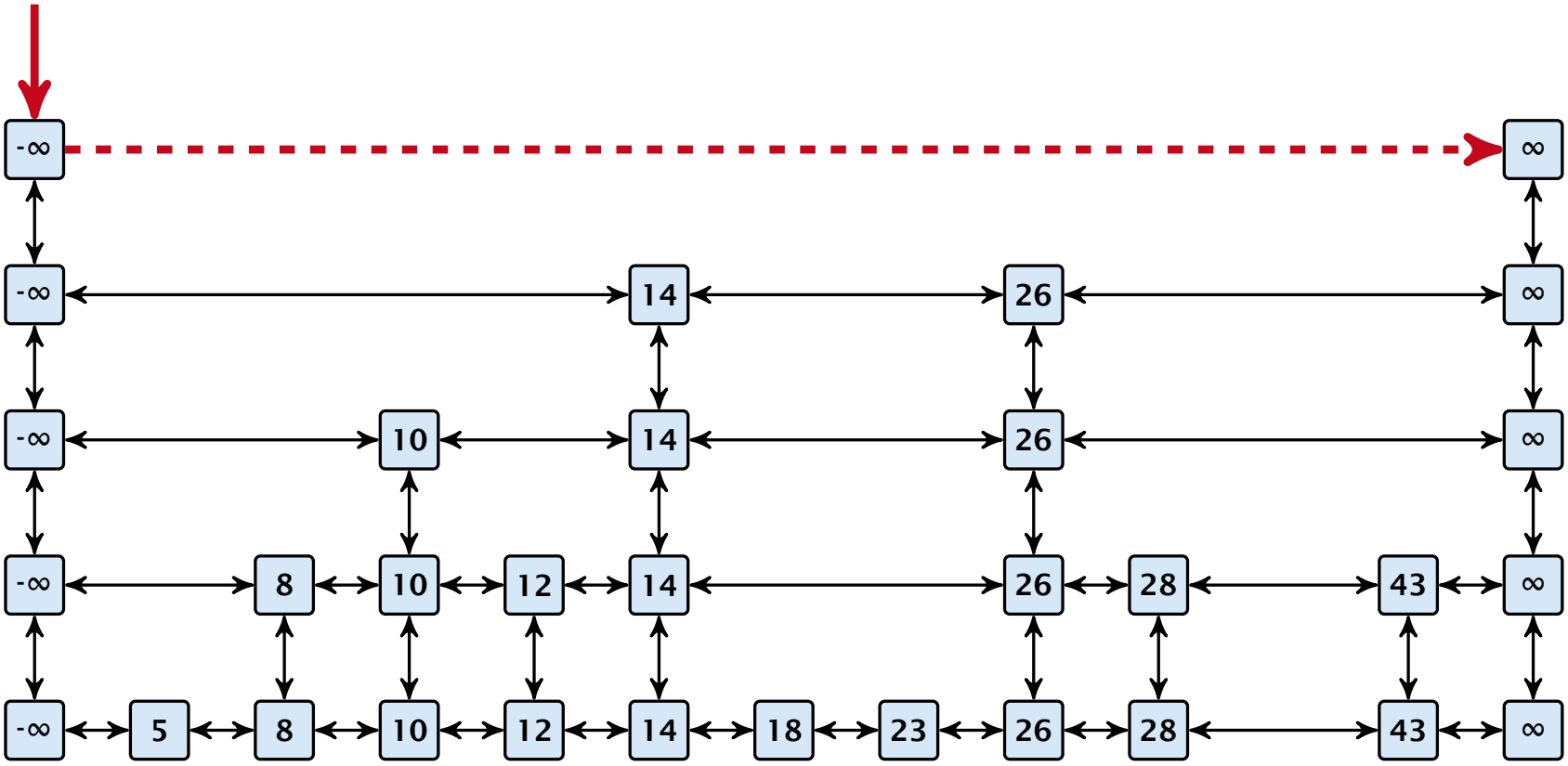

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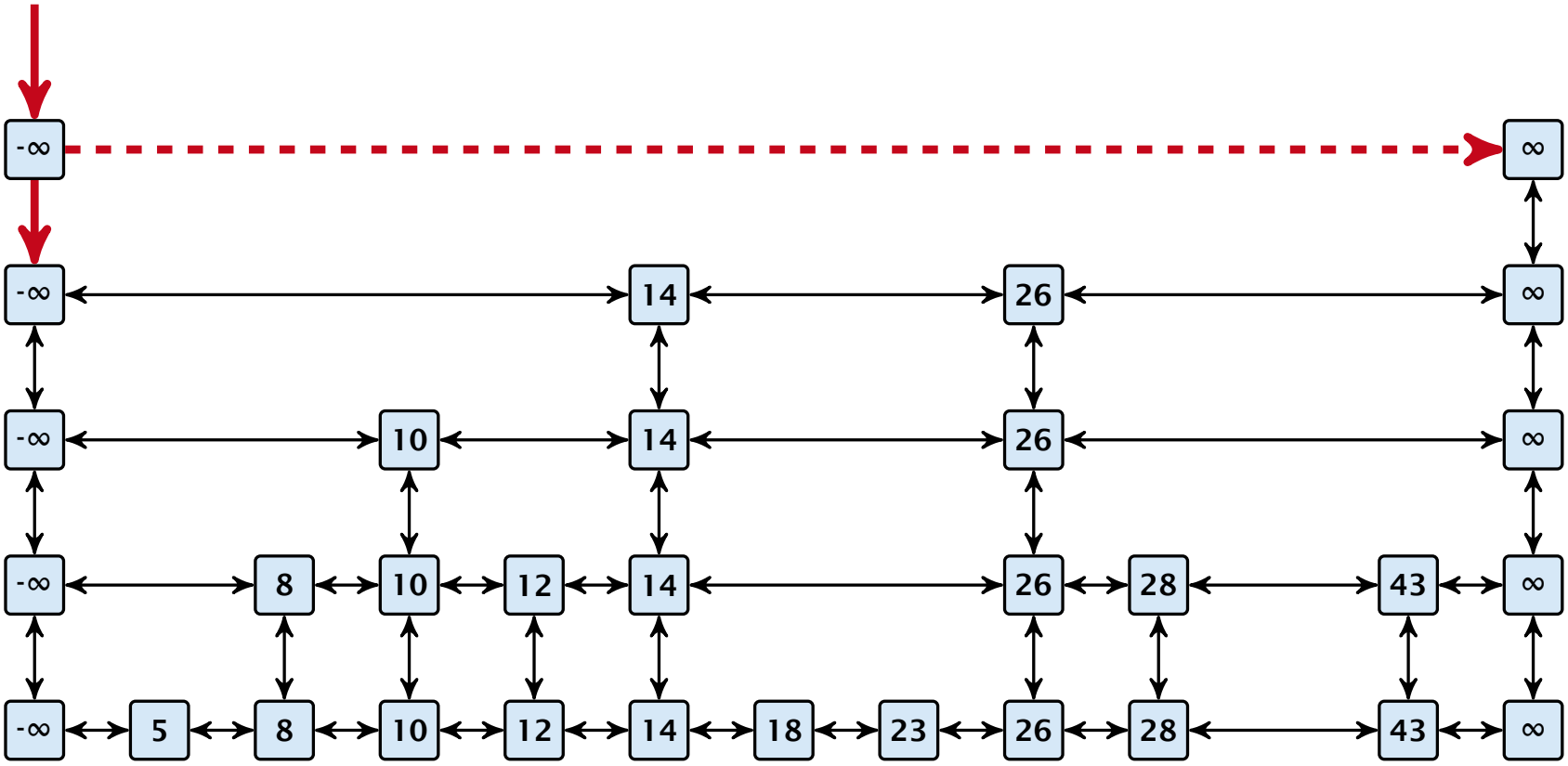

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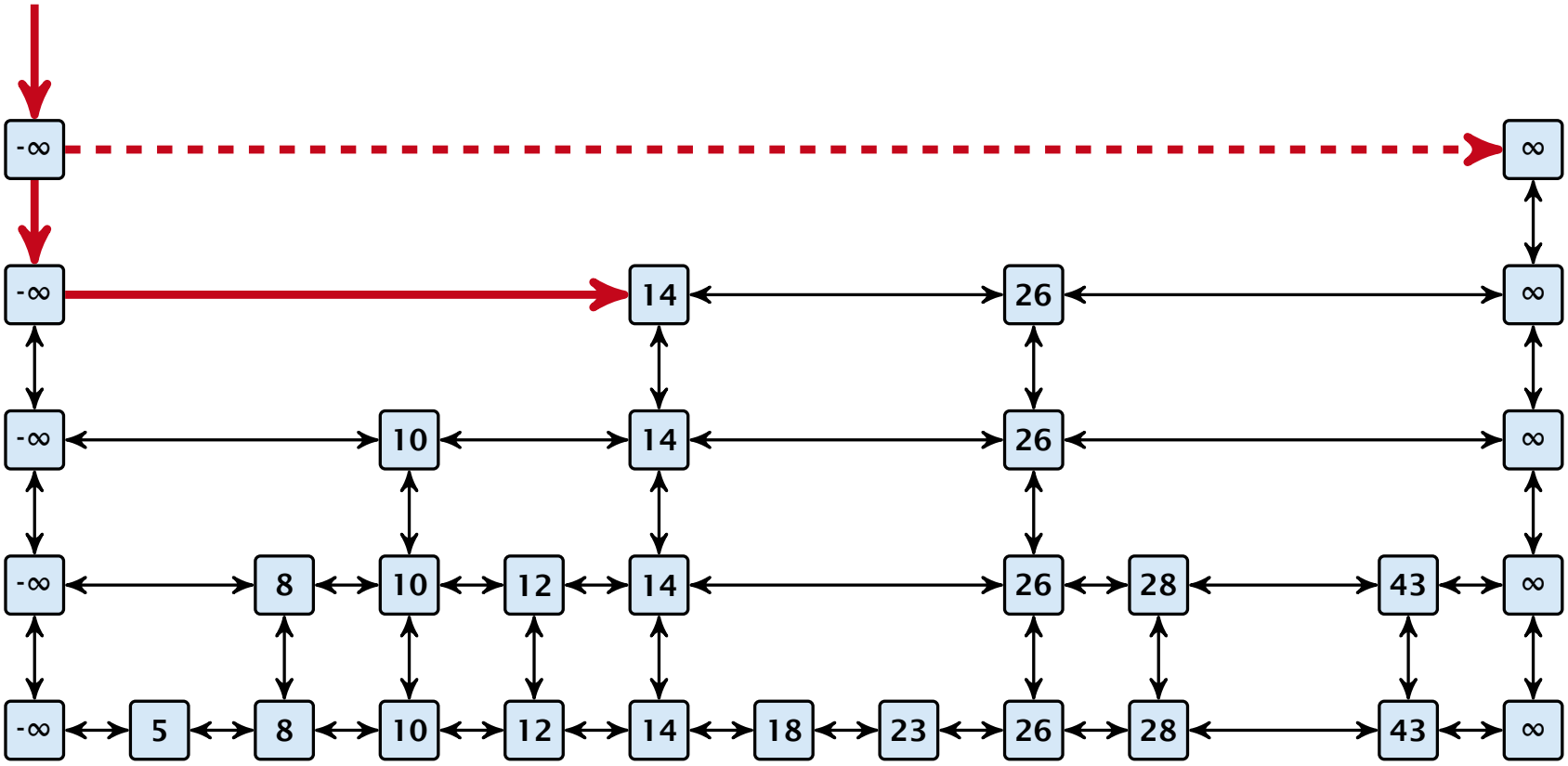

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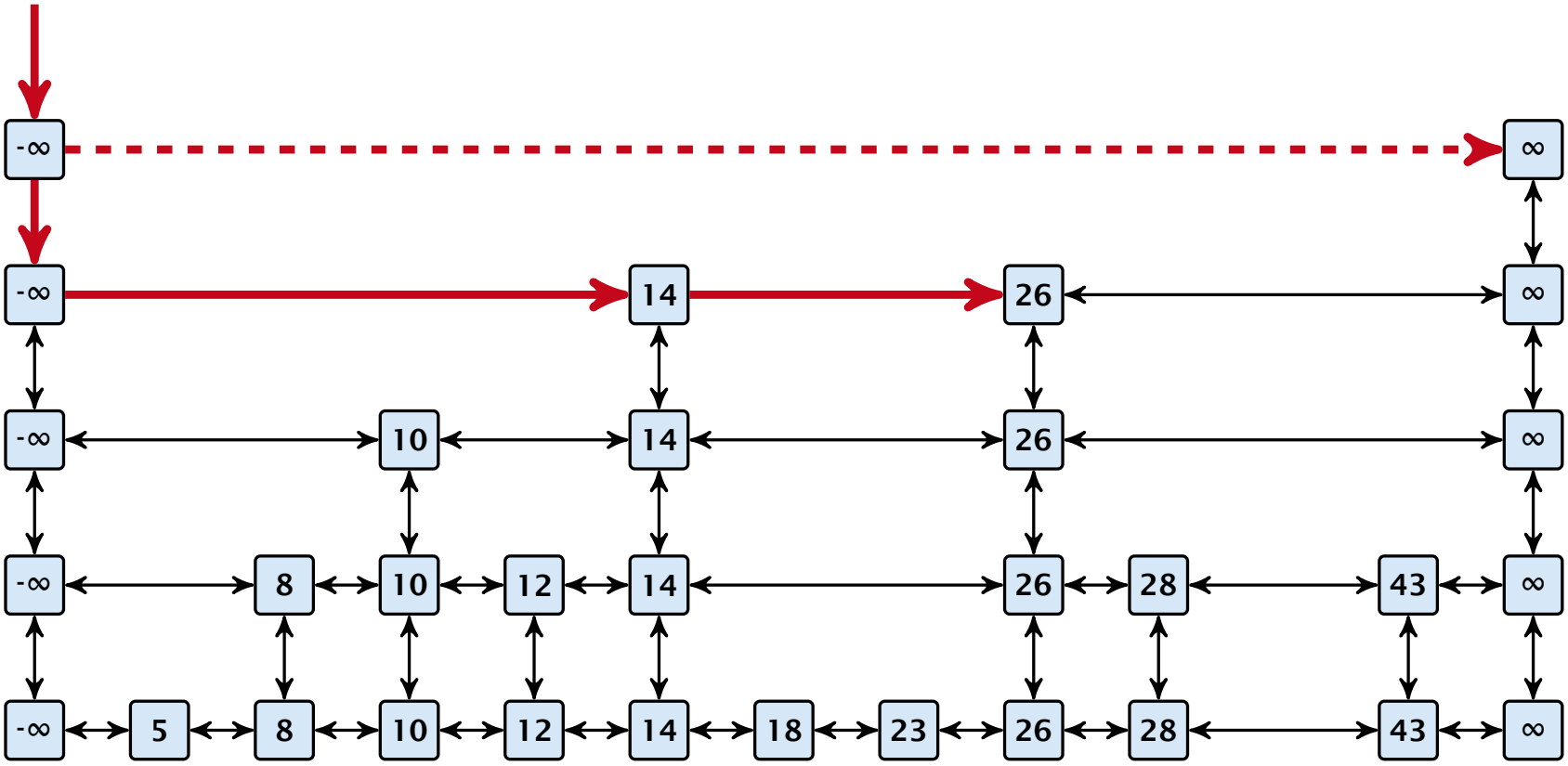

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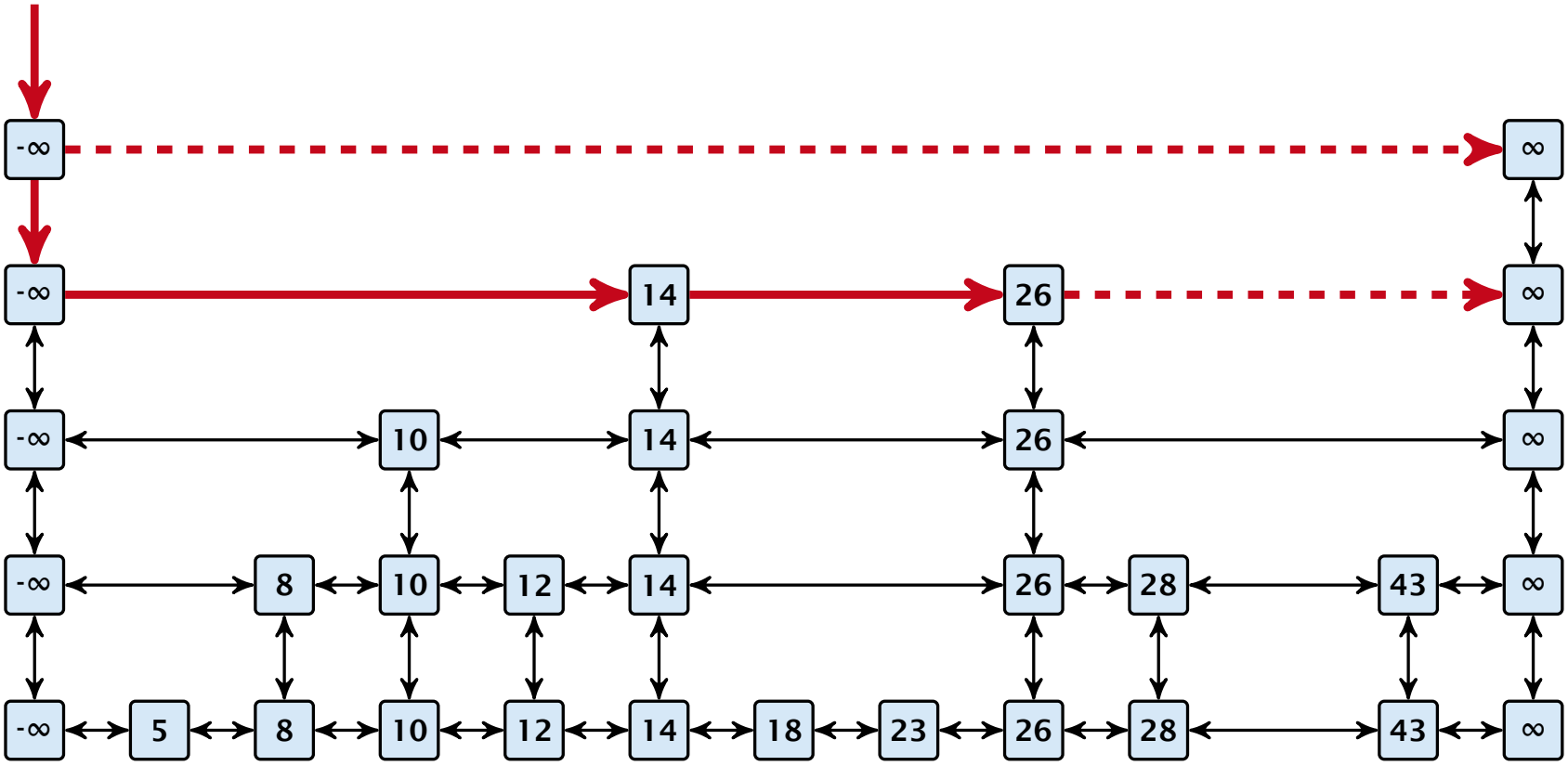

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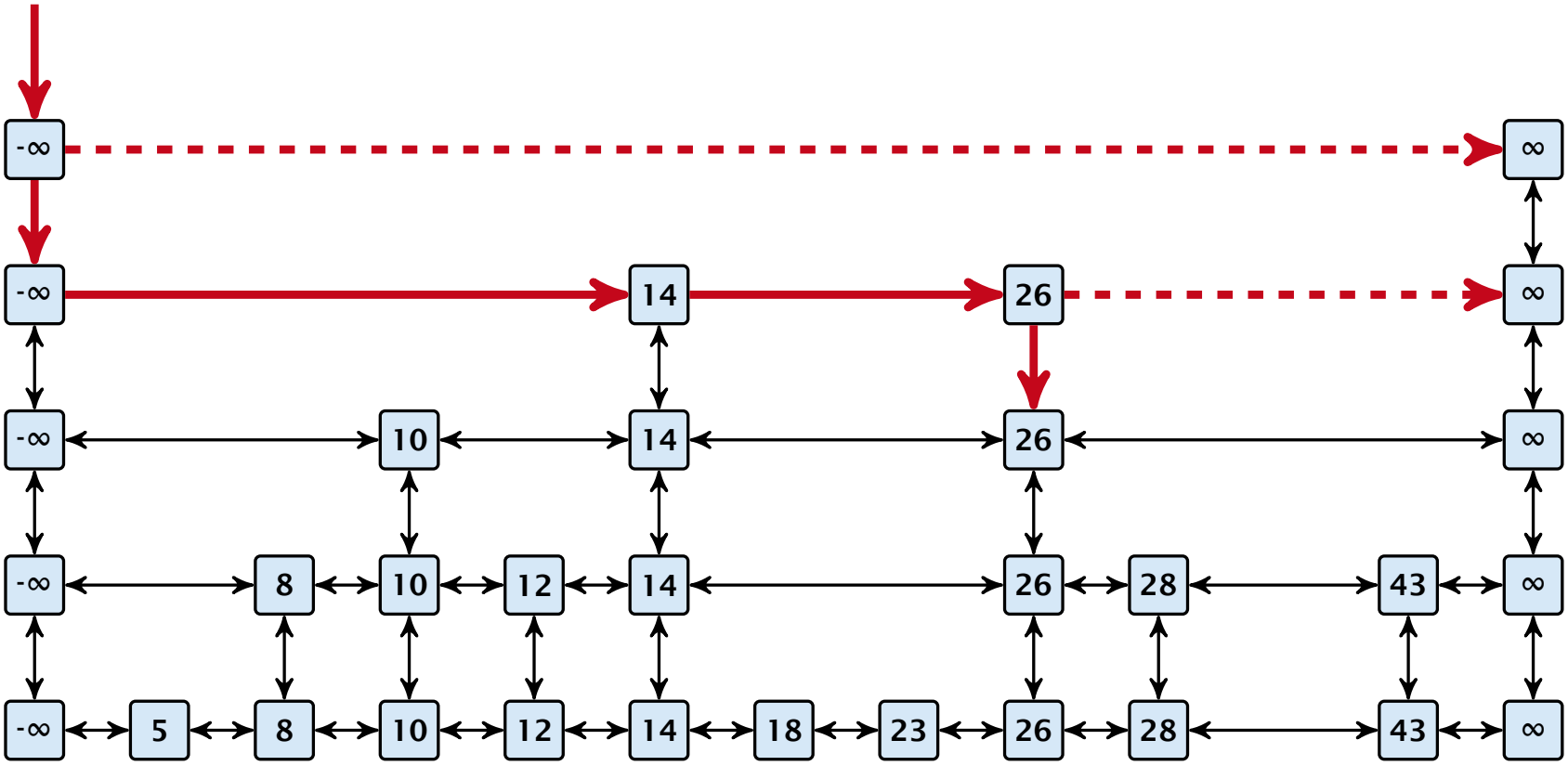

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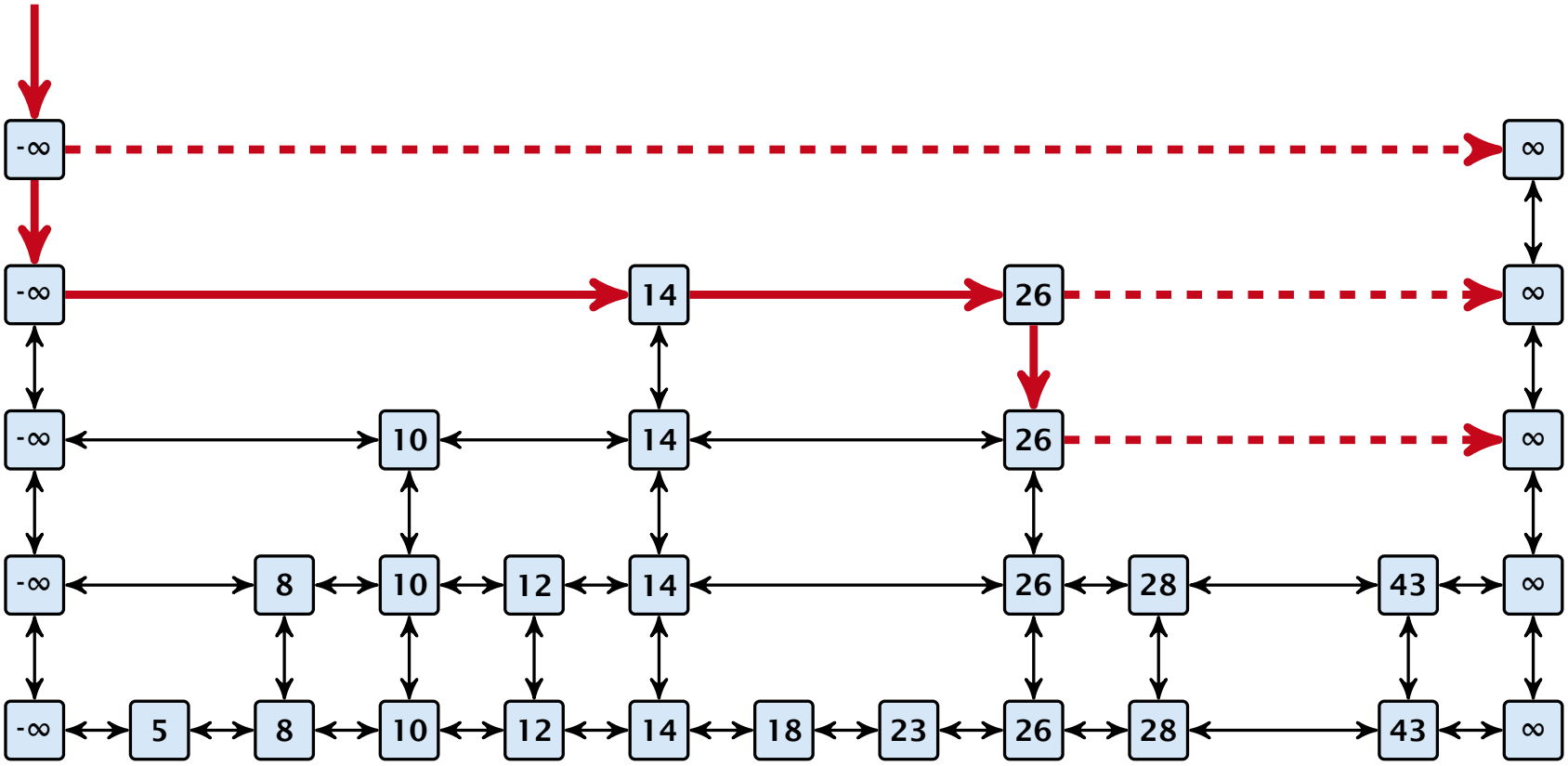

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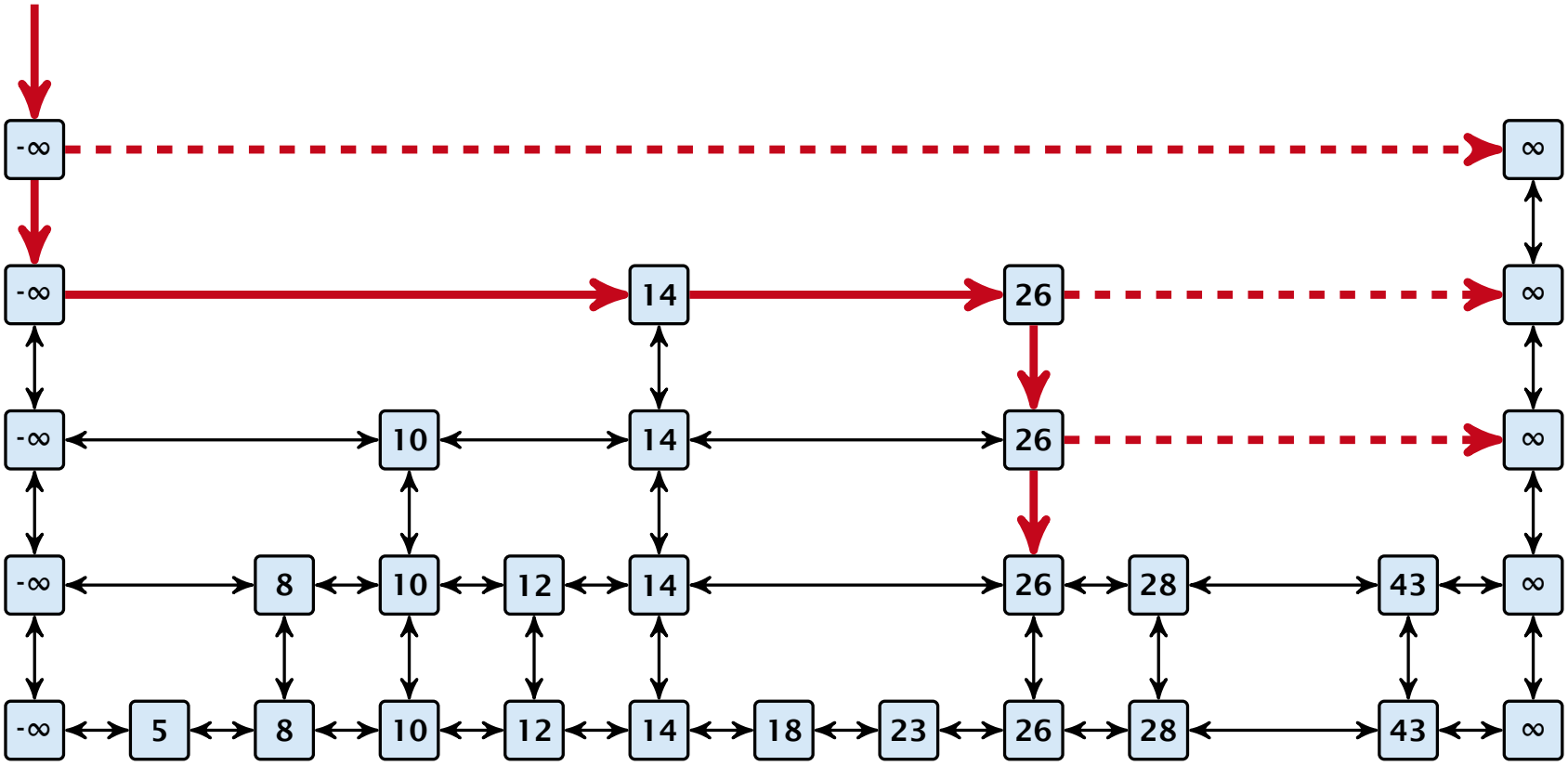

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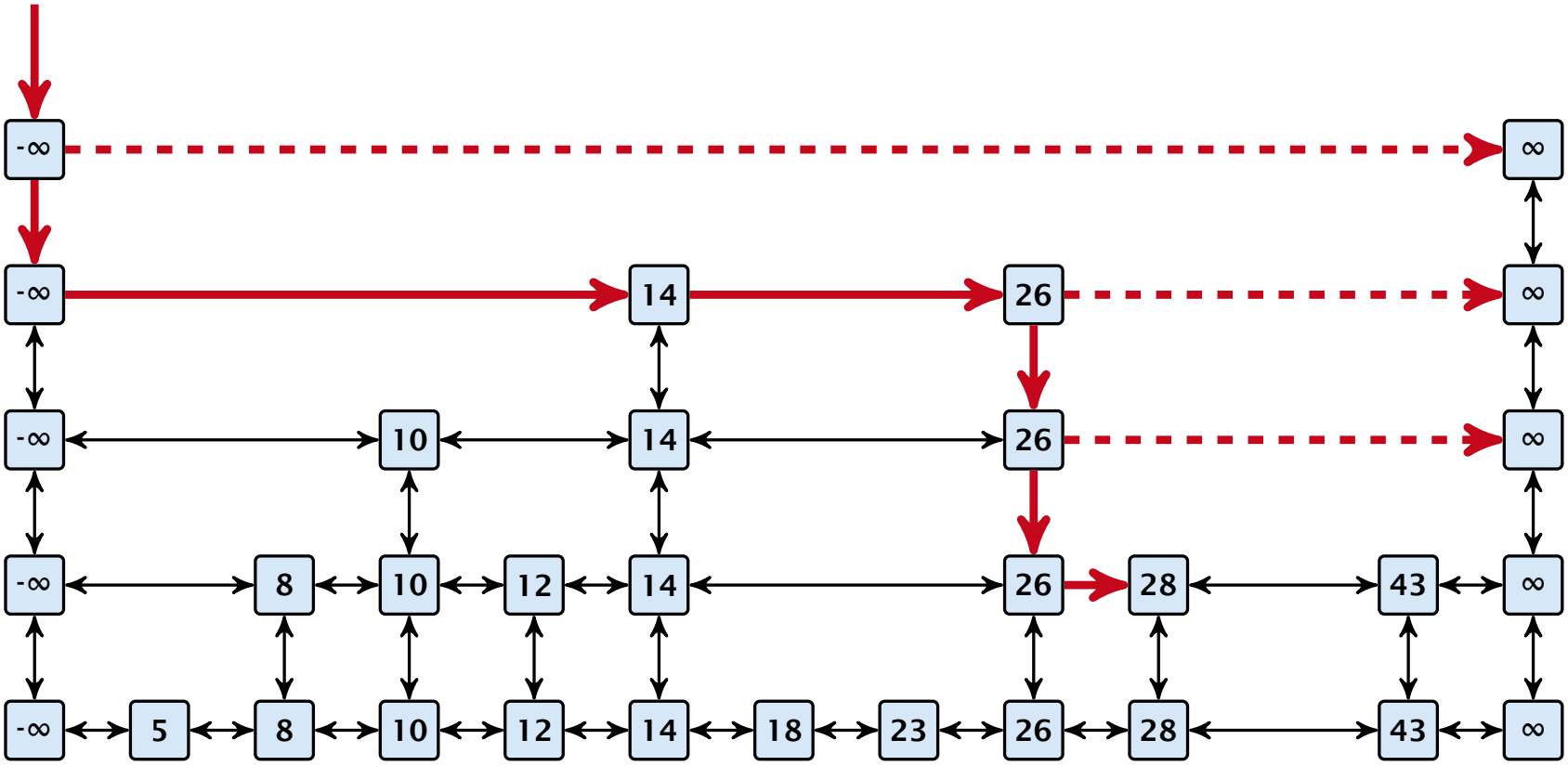

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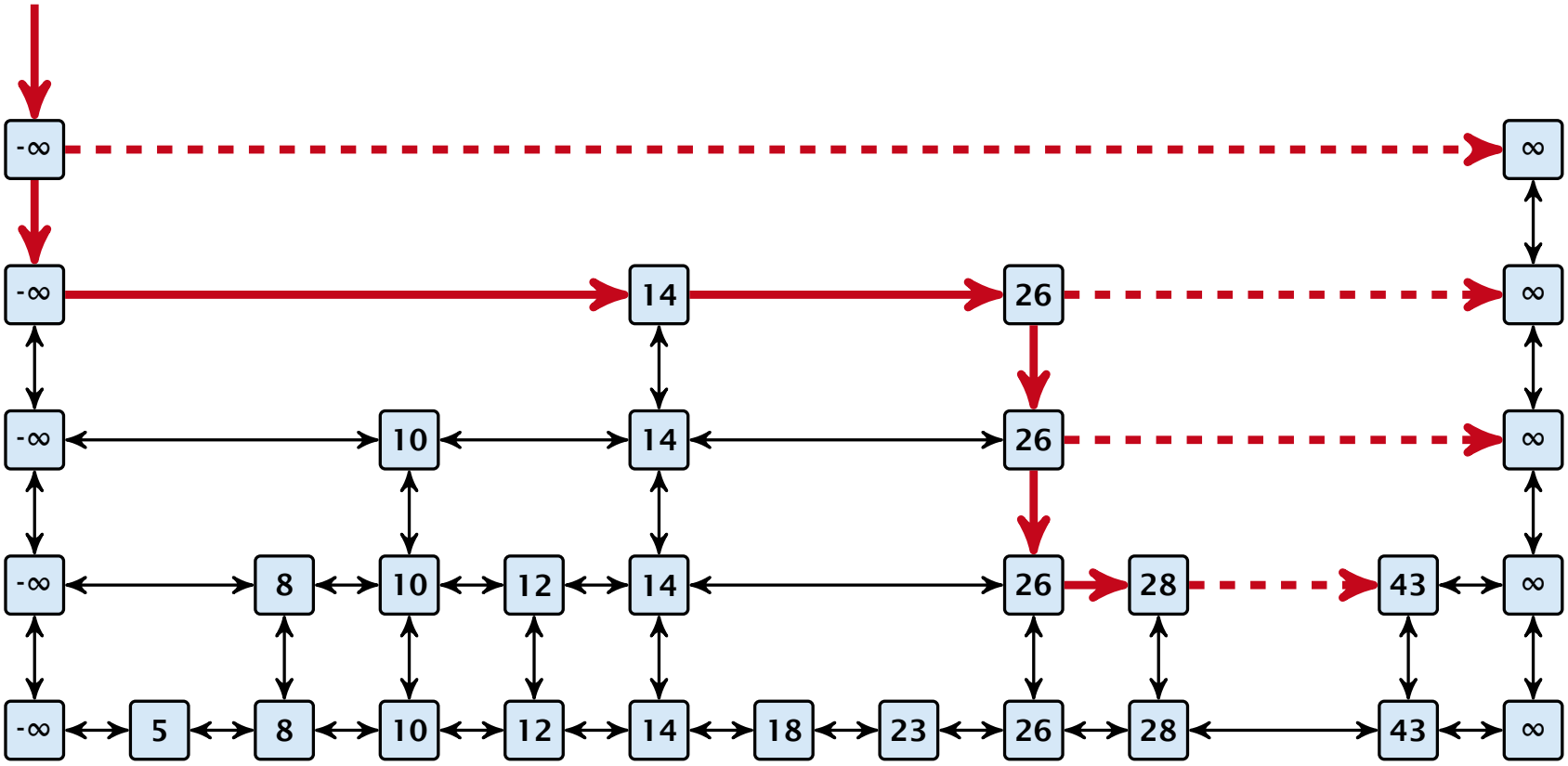

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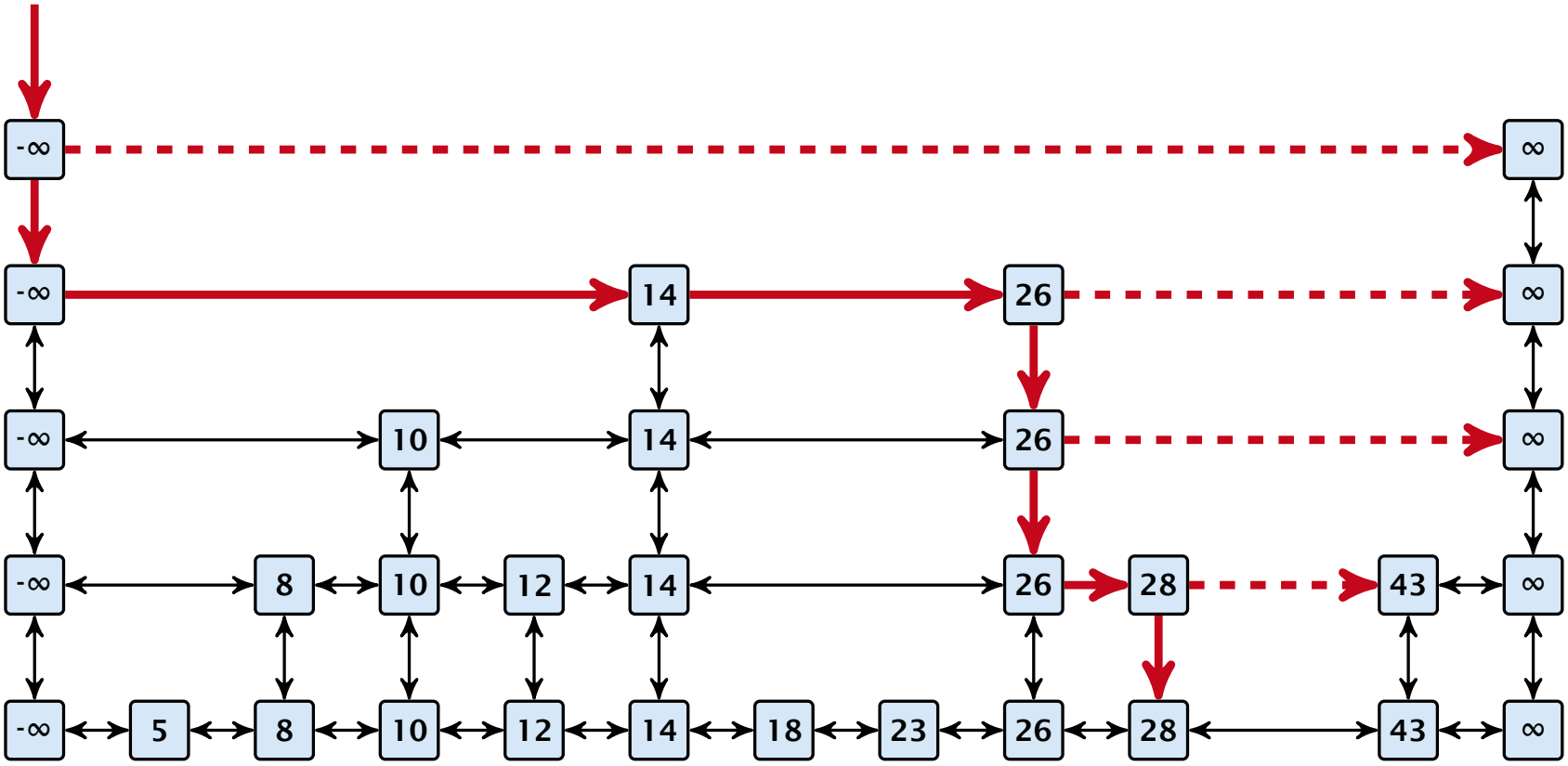

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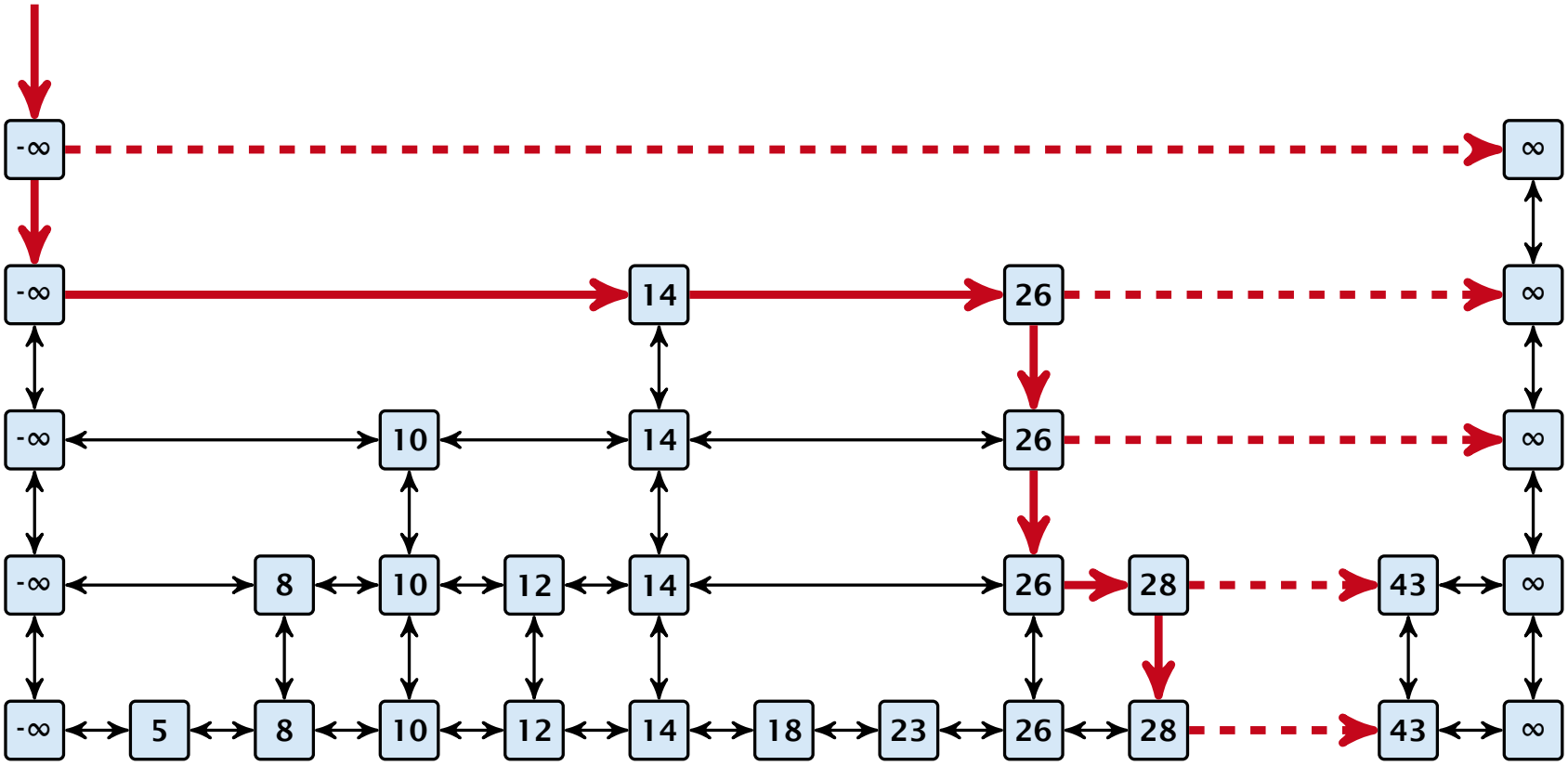

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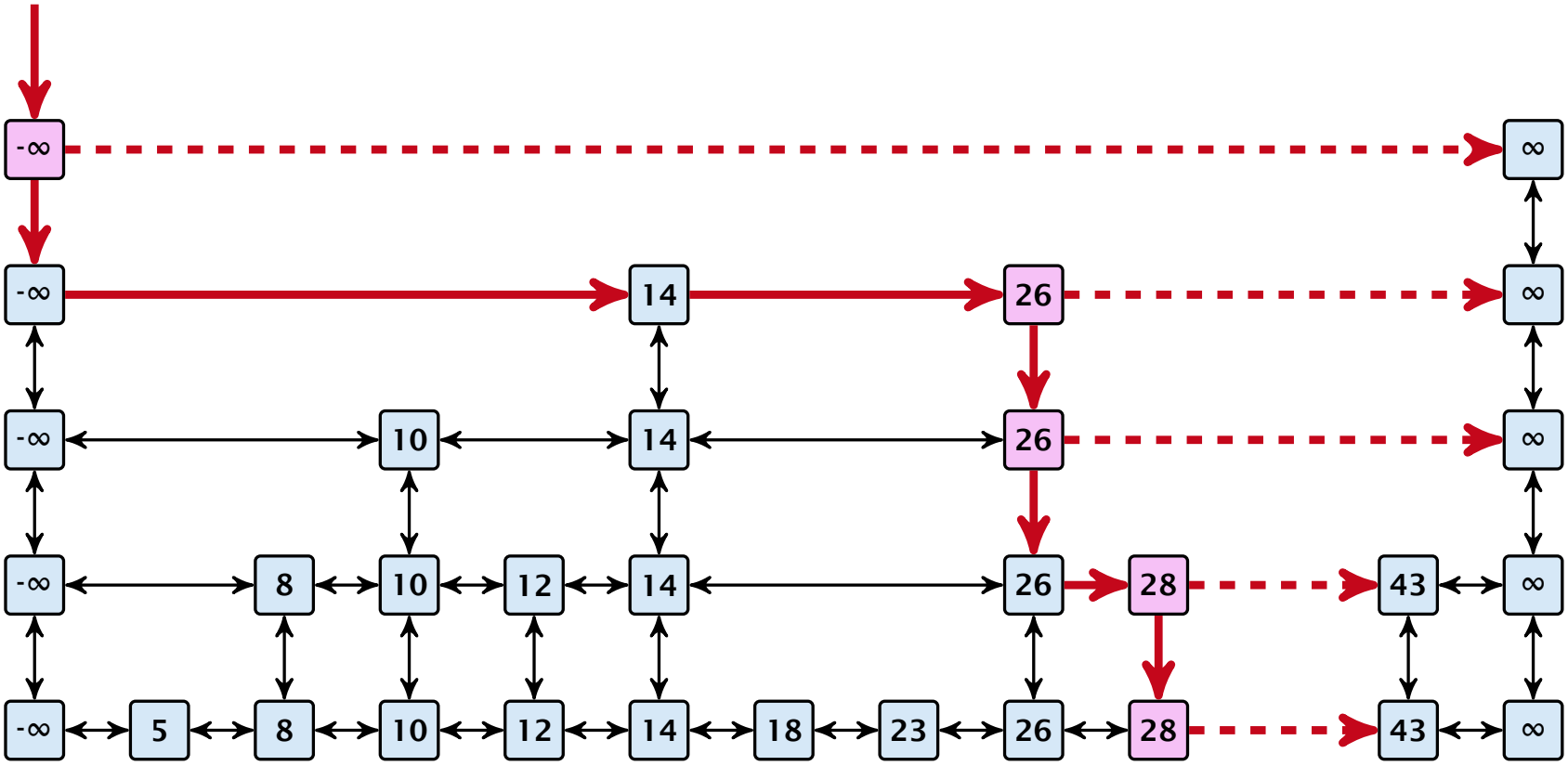

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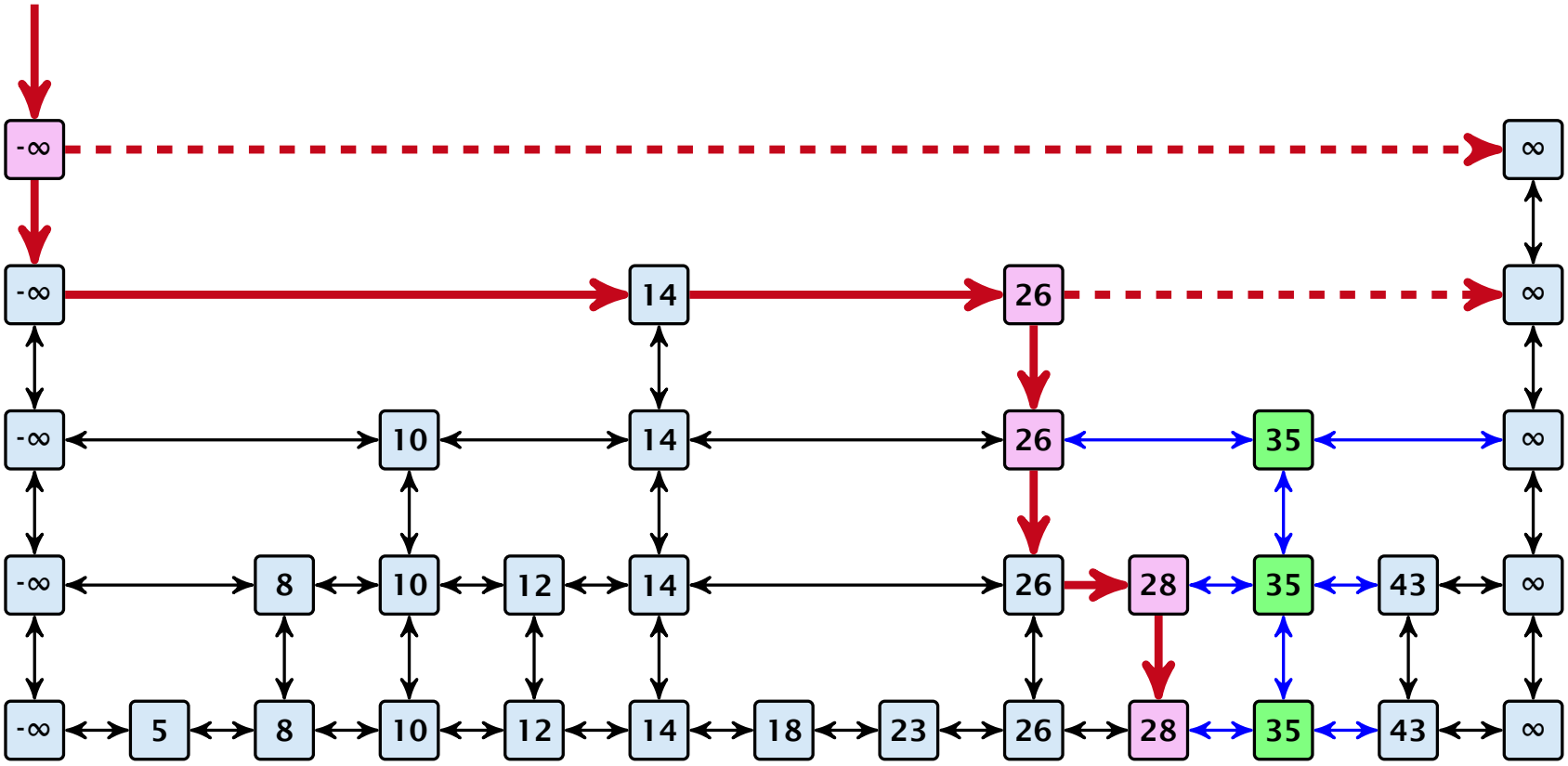

High Probability

Definition 20 (High Probability)

We say a **randomized** algorithm has running time $\mathcal{O}(\log n)$ with high probability if for any constant α the running time is at most $\mathcal{O}(\log n)$ with probability at least $1 - \frac{1}{n^\alpha}$.

Here the $\mathcal{O}$-notation hides a constant that may depend on α.

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High Probability

Suppose there are a polynomially many events $E_1, E_2, \ldots, E_\ell$, $\ell = n^c$ each holding with high probability (e.g. E_i may be the event that the i-th search in a skip list takes time at most $\mathcal{O}(\log n)$).

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This means $\Pr[E_1 \wedge \cdots \wedge E_\ell]$ holds with high probability.

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Lemma 21

A search (and, hence, also insert and delete) in a skip list with n elements takes time $\mathcal{O}(log n)$ with high probability (w. h. p.).

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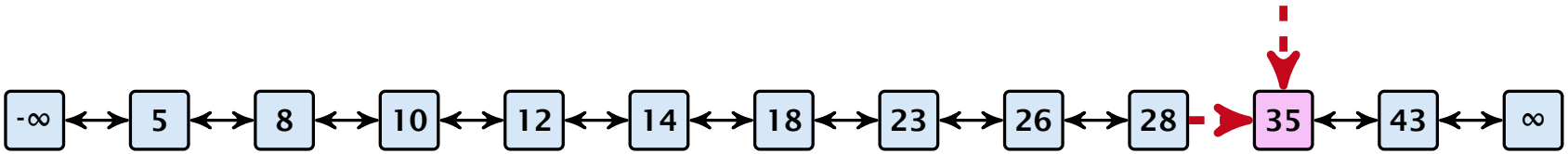

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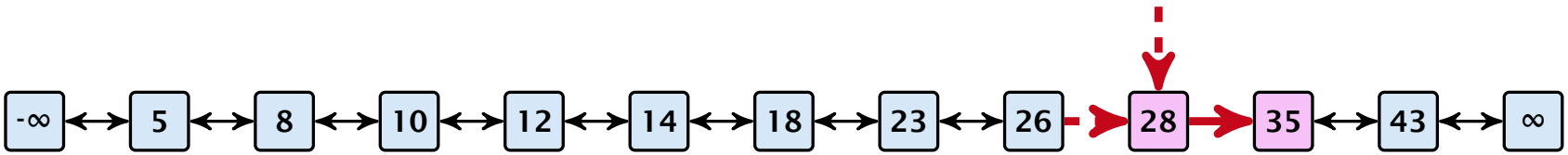

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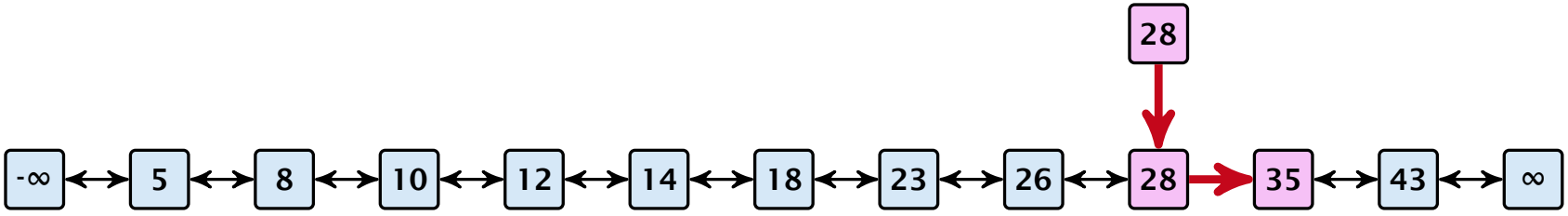

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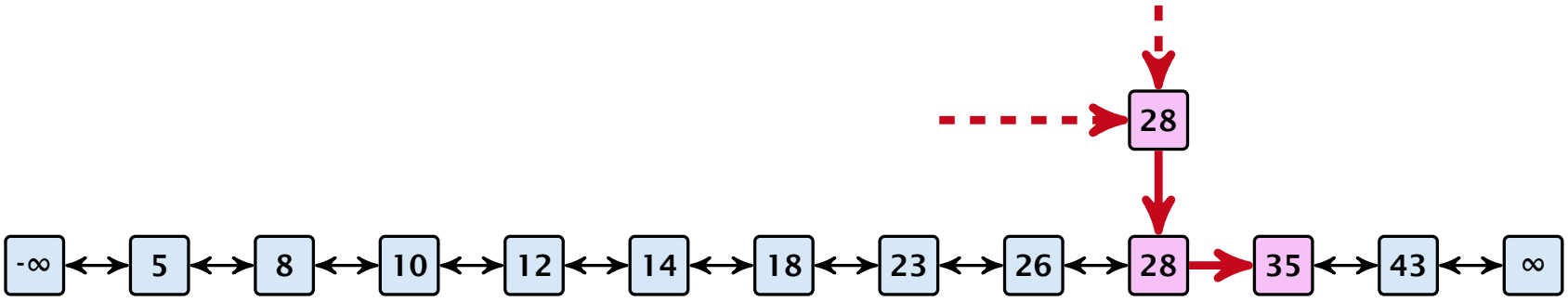

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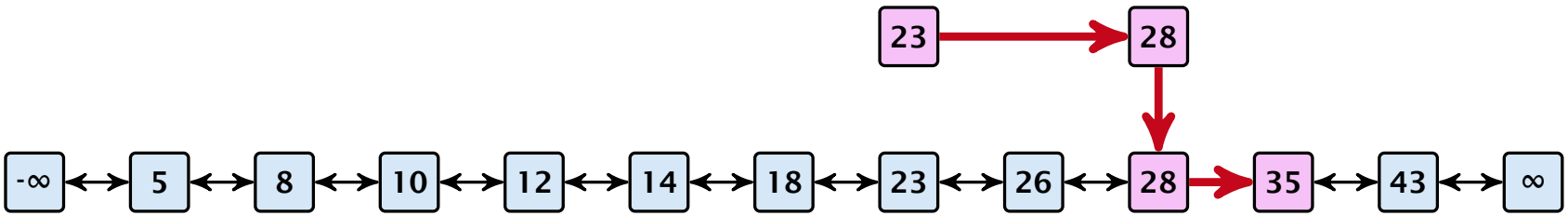

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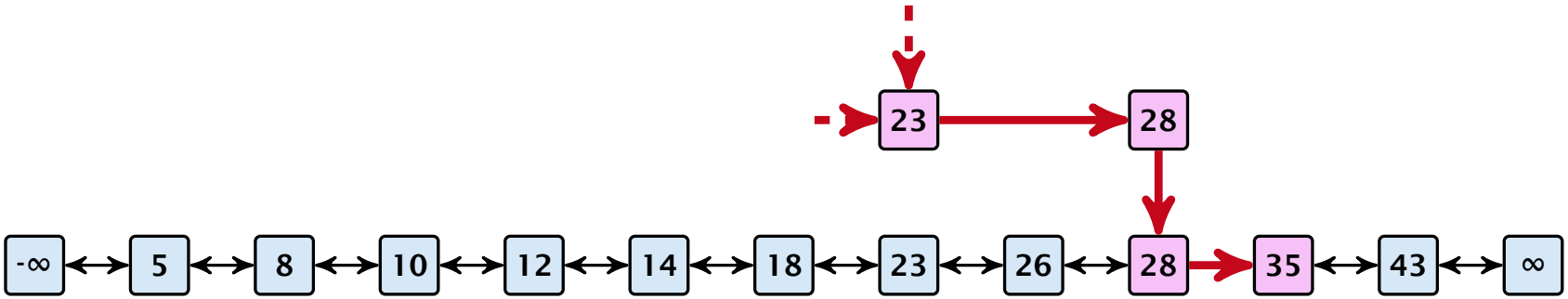

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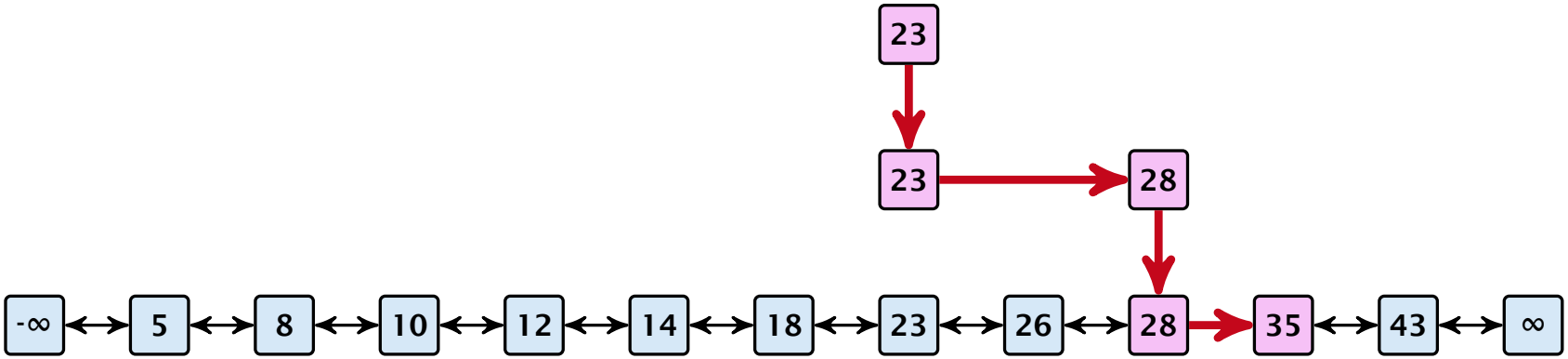

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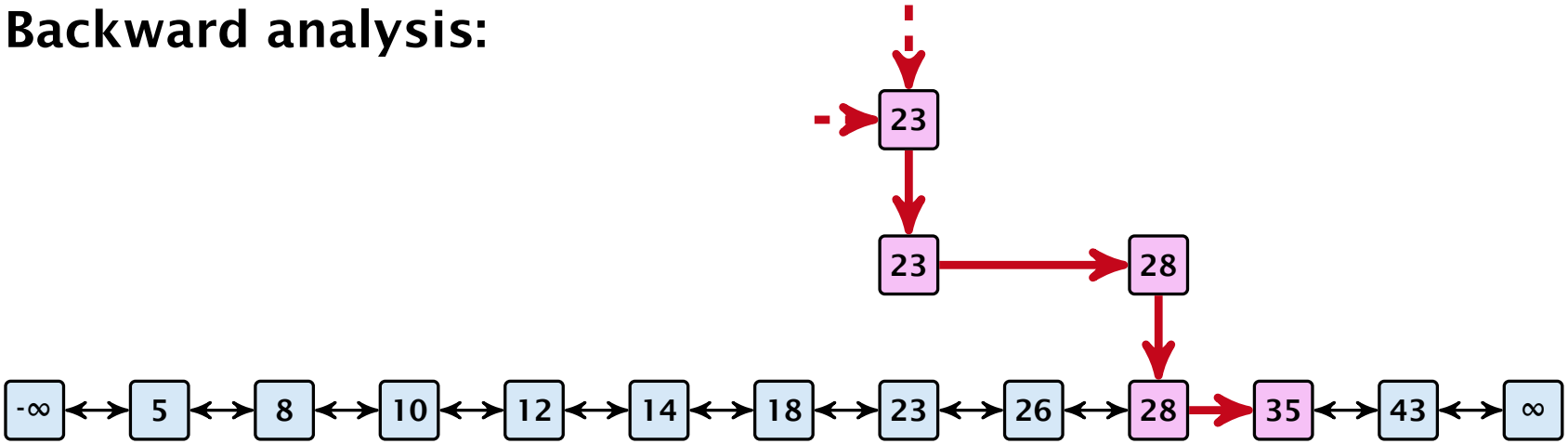

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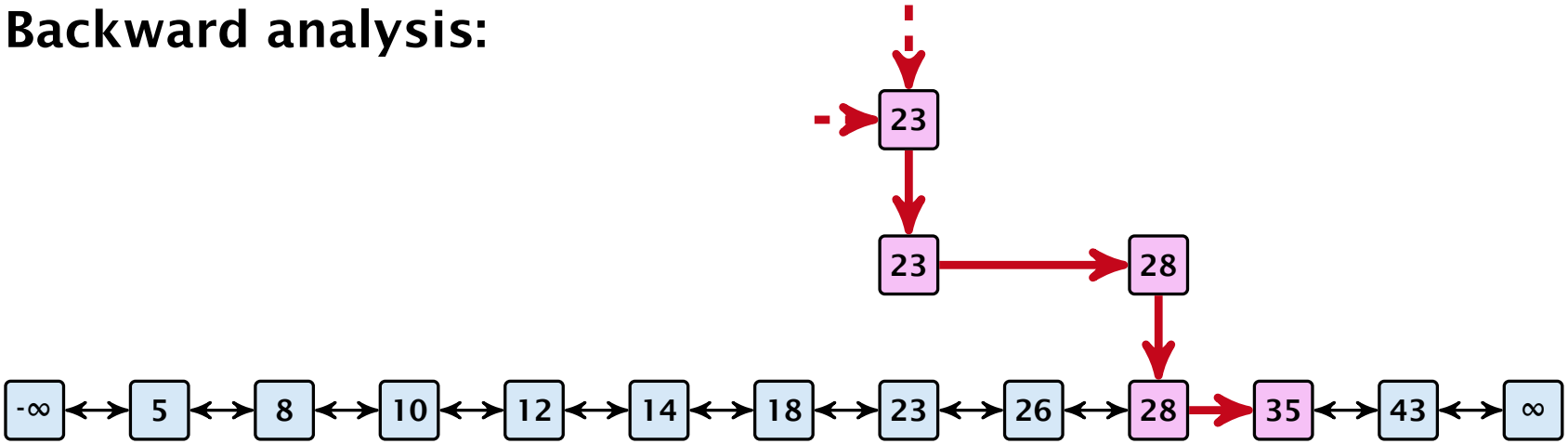

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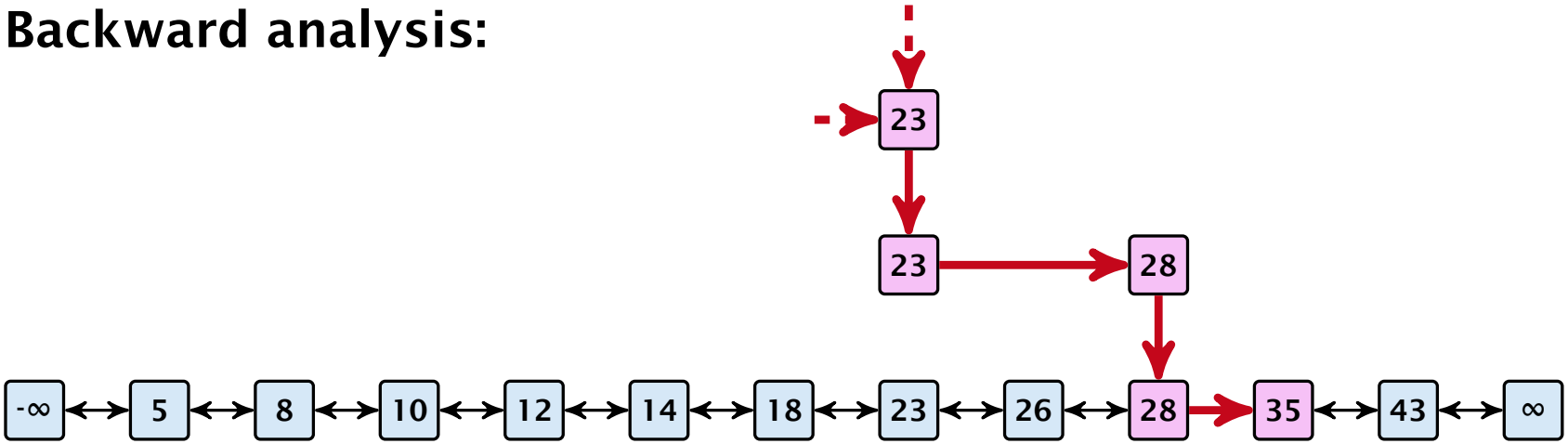

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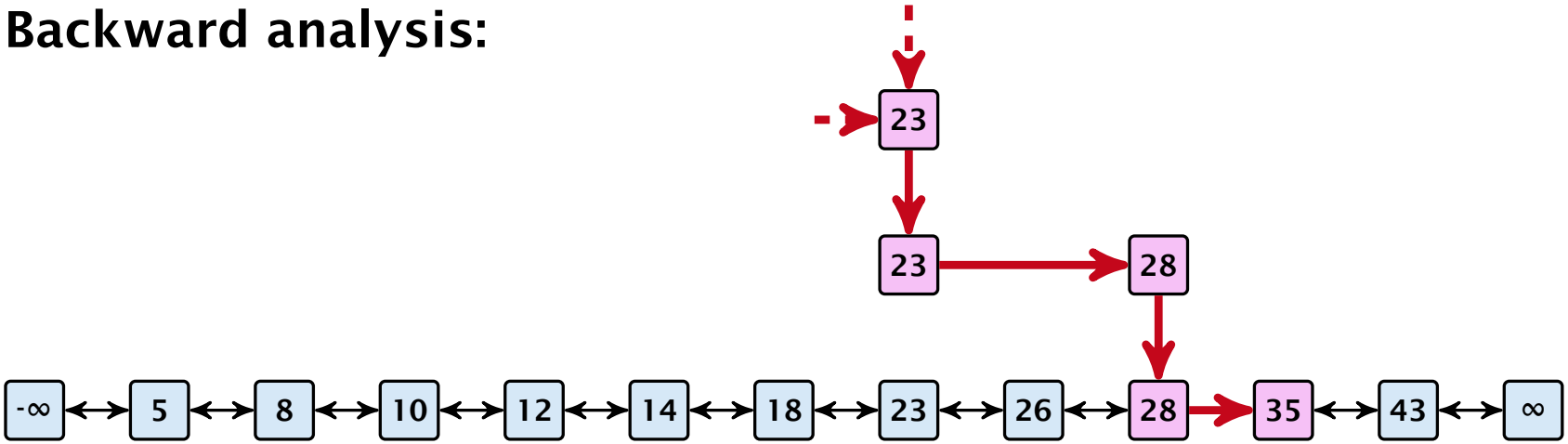

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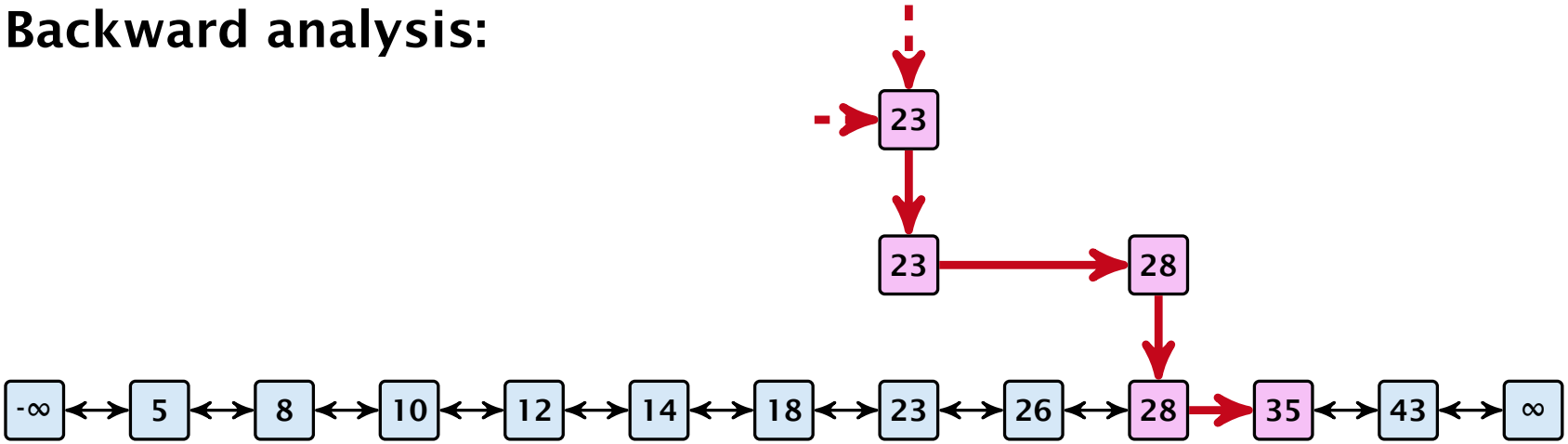

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We show that w.h.p:

- A "long" search path must also go very high.
- There are no elements in high lists.

From this it follows that w.h.p. there are no long paths.

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In particular, this means that during the construction in the backward analysis we see at most k heads (i.e., coin flips that tell you to go up) in z trials.

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This means, the search requires at most z steps, w. h. p.

7.7 Hashing

Dictionary:

- $S.\,\mathbf{insert}(x)$: Insert an element x.
- $S.\,\mathbf{delete}(x)$: Delete the element pointed to by x.
- $S.\,\mathbf{search}(k)$: Return a pointer to an element e with $\mathrm{key}[e] = k$ in S if it exists; otherwise return null.

So far we have implemented the search for a key by carefully choosing split-elements.

Then the memory location of an object x with key k is determined by successively comparing k to split-elements.

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Definitions:

- Universe U of keys, e.g., $U \subseteq \mathbb{N}_0$. U very large.
- Set $S \subseteq U$ of keys, $|S| = m \leq |U|$.
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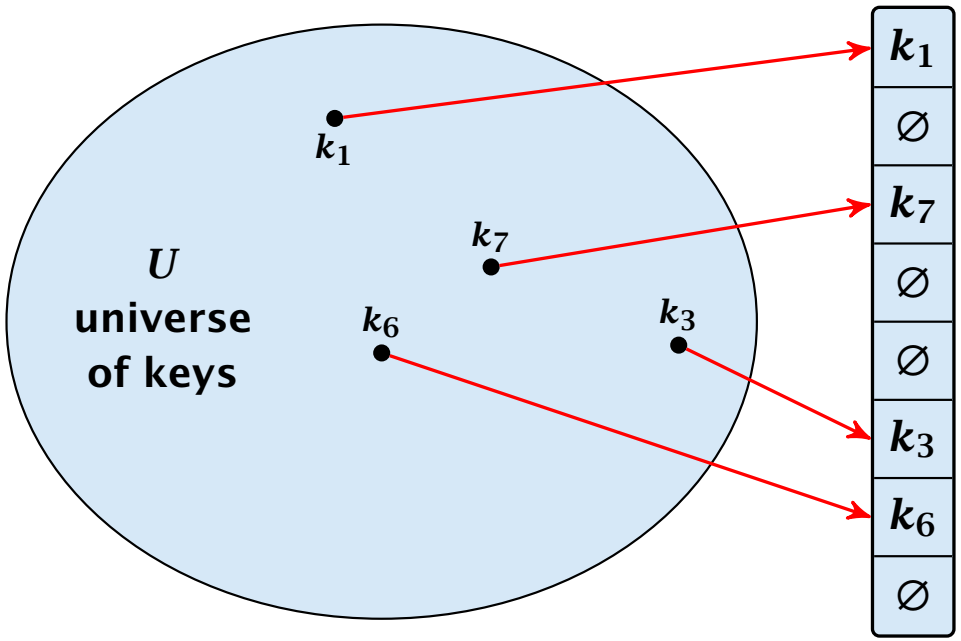
Direct Addressing

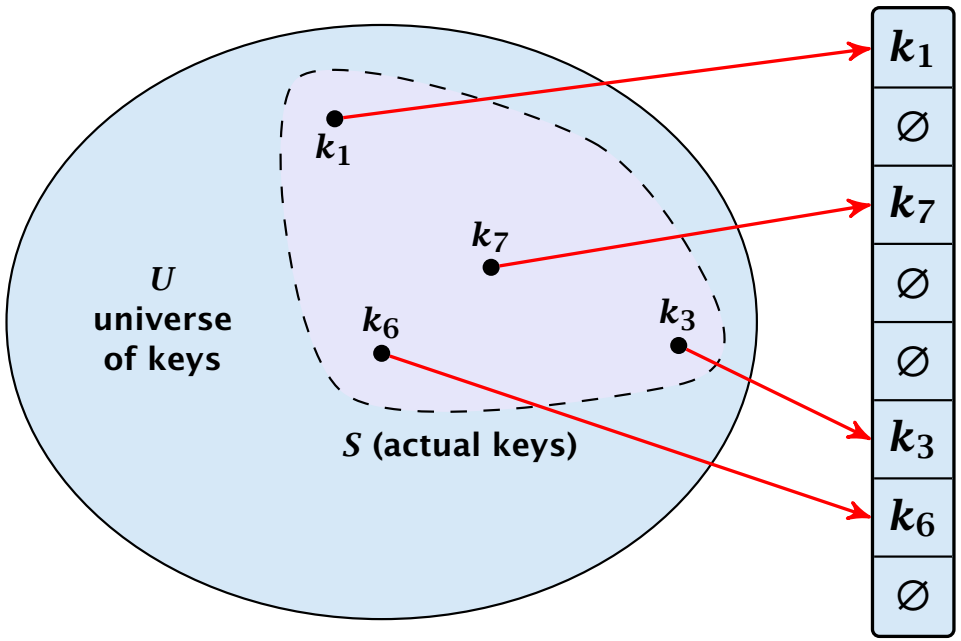
Ideally the hash function maps all keys to different memory locations.



This special case is known as Direct Addressing. It is usually very unrealistic as the universe of keys typically is quite large, and in particular larger than the available memory.

Perfect Hashing

Suppose that we know the set S of actual keys (no insert/no delete). Then we may want to design a simple hash-function that maps all these keys to different memory locations.



Such a hash function h is called a perfect hash function for set S.

Collisions

If we do not know the keys in advance, the best we can hope for is that the hash function distributes keys evenly across the table.

Problem: Collisions

Usually the universe U is much larger than the table-size n.

Hence, there may be two elements k_1, k_2 from the set S that map to the same memory location (i.e., $h(k_1) = h(k_2)$). This is called a collision.

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Typically, collisions do not appear once the size of the set S of actual keys gets close to n, but already when $|S| \geq \omega(\sqrt{n})$.

Lemma 22

The probability of having a collision when hashing m elements into a table of size n under uniform hashing is at least

$$1 - e^{-\frac{m(m-1)}{2n}} \approx 1 - e^{-\frac{m^2}{2n}} .$$

Uniform hashing:

Choose a hash function uniformly at random from all functions $f : U \to [0, \ldots, n-1]$.

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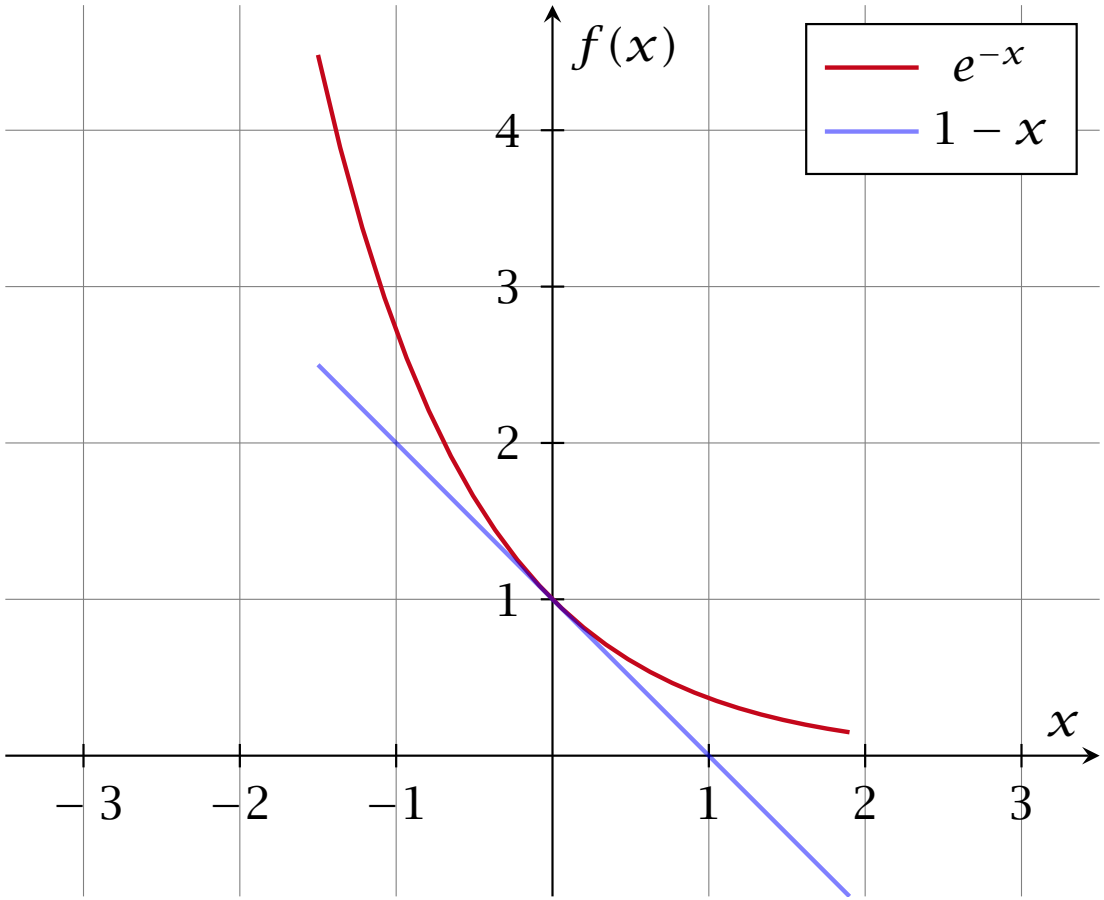
Here the first equality follows since the ℓ-th element that is hashed has a probability of $\frac{n-\ell+1}{n}$ to not generate a collision under the condition that the previous elements did not induce collisions. □

Collisions



The inequality $1 - x \leq e^{-x}$ is derived by stopping the Taylor-expansion of e^{-x} after the second term.

Resolving Collisions

The methods for dealing with collisions can be classified into the two main types

- **open addressing**, aka. closed hashing
- **hashing with chaining**, aka. closed addressing, open hashing.

There are applications e.g. computer chess where you do not resolve collisions at all.

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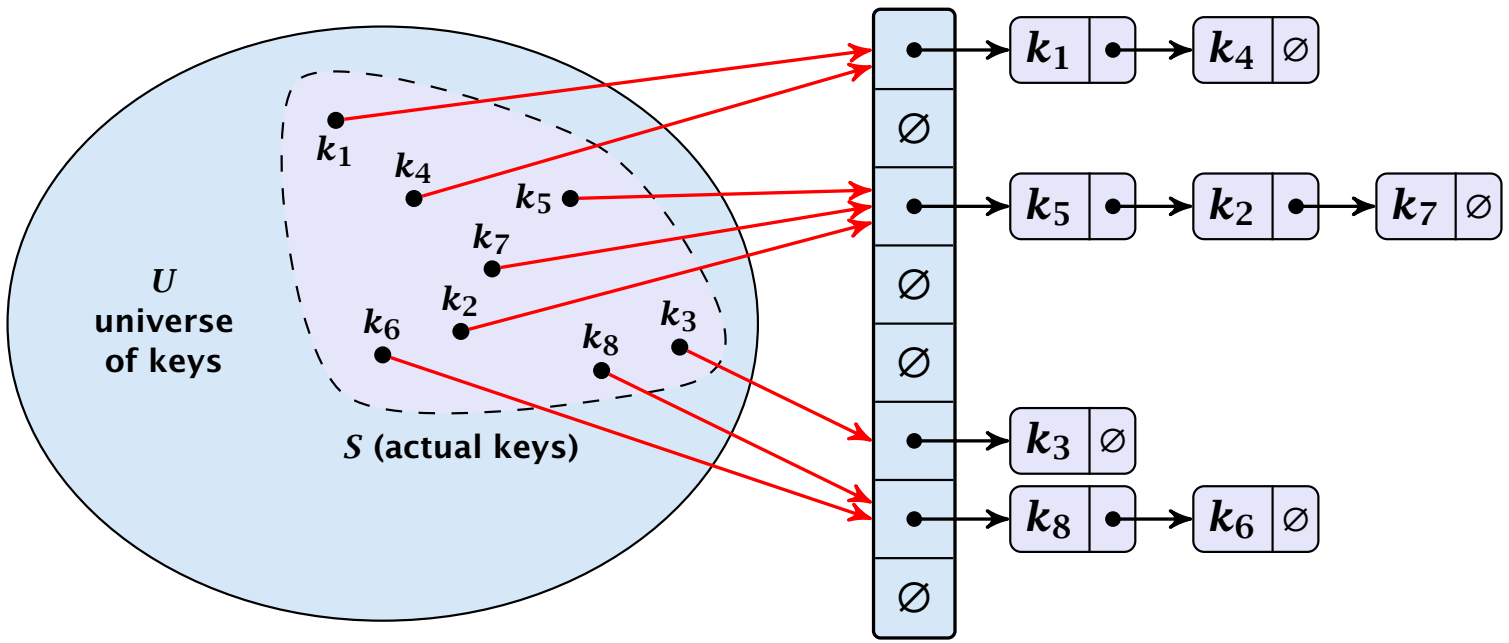
Hashing with Chaining

Arrange elements that map to the same position in a linear list.

- Access: compute $h(x)$ and search list for $\text{key}[x]$.
- Insert: insert at the front of the list.



Hashing with Chaining

Let A denote a strategy for resolving collisions. We use the following notation:

- A^+ denotes the average time for a **successful** search when using A;
- A^- denotes the average time for an **unsuccessful** search when using A;
- We parameterize the complexity results in terms of $\alpha := \frac{m}{n}$, the so-called fill factor of the hash-table.

We assume uniform hashing for the following analysis.

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$$A^- = 1 + \alpha \ .$$

Hashing with Chaining

For a successful search observe that we do **not** choose a list at random, but we consider a random key k in the hash-table and ask for the search-time for k.

This is 1 plus the number of elements that lie before k in k's list.

Let k_ℓ denote the ℓ-th key inserted into the table.

Let for two keys k_i and k_j, X_{ij} denote the indicator variable for the event that k_i and k_j hash to the same position. Clearly, $\Pr[X_{ij} = 1] = 1/n$ for uniform hashing.

The expected successful search cost is

$$\mathrm{E}\left[\frac{1}{m}\sum_{i=1}^{m}\left(1+\sum_{j=i+1}^{m} X_{ij}\right)\right]$$

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Hence, the expected cost for a successful search is $A^+ \leq 1+\frac{\alpha}{2}$.

Hashing with Chaining

Disadvantages:

- pointers increase memory requirements
- pointers may lead to bad cache efficiency

Advantages:

- no à priori limit on the number of elements
- deletion can be implemented efficiently
- by using balanced trees instead of linked list one can also obtain worst-case guarantees.

Open Addressing

All objects are stored in the table itself.

Define a function $h(k,j)$ that determines the table-position to be examined in the j-th step. The values $h(k,0),\ldots,h(k,n-1)$ must form a permutation of $0,\ldots,n-1$.

Search(k): Try position $h(k,0)$; if it is empty your search fails; otw. continue with $h(k,1)$, $h(k,2)$, ...

Insert(x): Search until you find an empty slot; insert your element there. If your search reaches $h(k,n-1)$, and this slot is non-empty then your table is full.

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Choices for $h(k,j)$:

- Linear probing:
  $h(k,i) = h(k) + i \mod n$
  (sometimes: $h(k,i) = h(k) + ci \mod n$).
- Quadratic probing:
  $h(k,i) = h(k) + c_1 i + c_2 i^2 \mod n$.
- Double hashing:
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For quadratic probing and double hashing one has to ensure that the search covers all positions in the table (i.e., for double hashing $h_2(k)$ must be relatively prime to n (teilerfremd); for quadratic probing c_1 and c_2 have to be chosen carefully).

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Linear Probing

- Advantage: Cache-efficiency. The new probe position is very likely to be in the cache.
- Disadvantage: Primary clustering. Long sequences of occupied table-positions get longer as they have a larger probability to be hit. Furthermore, they can merge forming larger sequences.

Lemma 23

Let L be the method of linear probing for resolving collisions:

$$L^+ \approx \frac{1}{2}\left(1+\frac{1}{1-\alpha}\right)$$

$$L^- \approx \frac{1}{2}\left(1+\frac{1}{(1-\alpha)^2}\right)$$

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- Disadvantage: Primary clustering. Long sequences of occupied table-positions get longer as they have a larger probability to be hit. Furthermore, they can merge forming larger sequences.

Lemma 23

Let L be the method of linear probing for resolving collisions:

$$L^+ \approx \frac{1}{2}\left(1 + \frac{1}{1-\alpha}\right)$$

$$L^- \approx \frac{1}{2}\left(1 + \frac{1}{(1-\alpha)^2}\right)$$

Quadratic Probing

- Not as cache-efficient as Linear Probing.
- Secondary clustering: caused by the fact that all keys mapped to the same position have the same probe sequence.

Lemma 24

Let Q be the method of quadratic probing for resolving collisions:

$$Q^+ \approx 1 + \ln\left(\frac{1}{1-\alpha}\right) - \frac{\alpha}{2}$$

$$Q^- \approx \frac{1}{1-\alpha} + \ln\left(\frac{1}{1-\alpha}\right) - \alpha$$

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Double Hashing

- Any probe into the hash-table usually creates a cache-miss.

Lemma 25

Let A be the method of double hashing for resolving collisions:

$$D^+ \approx \frac{1}{\alpha} \ln\left(\frac{1}{1-\alpha}\right)$$

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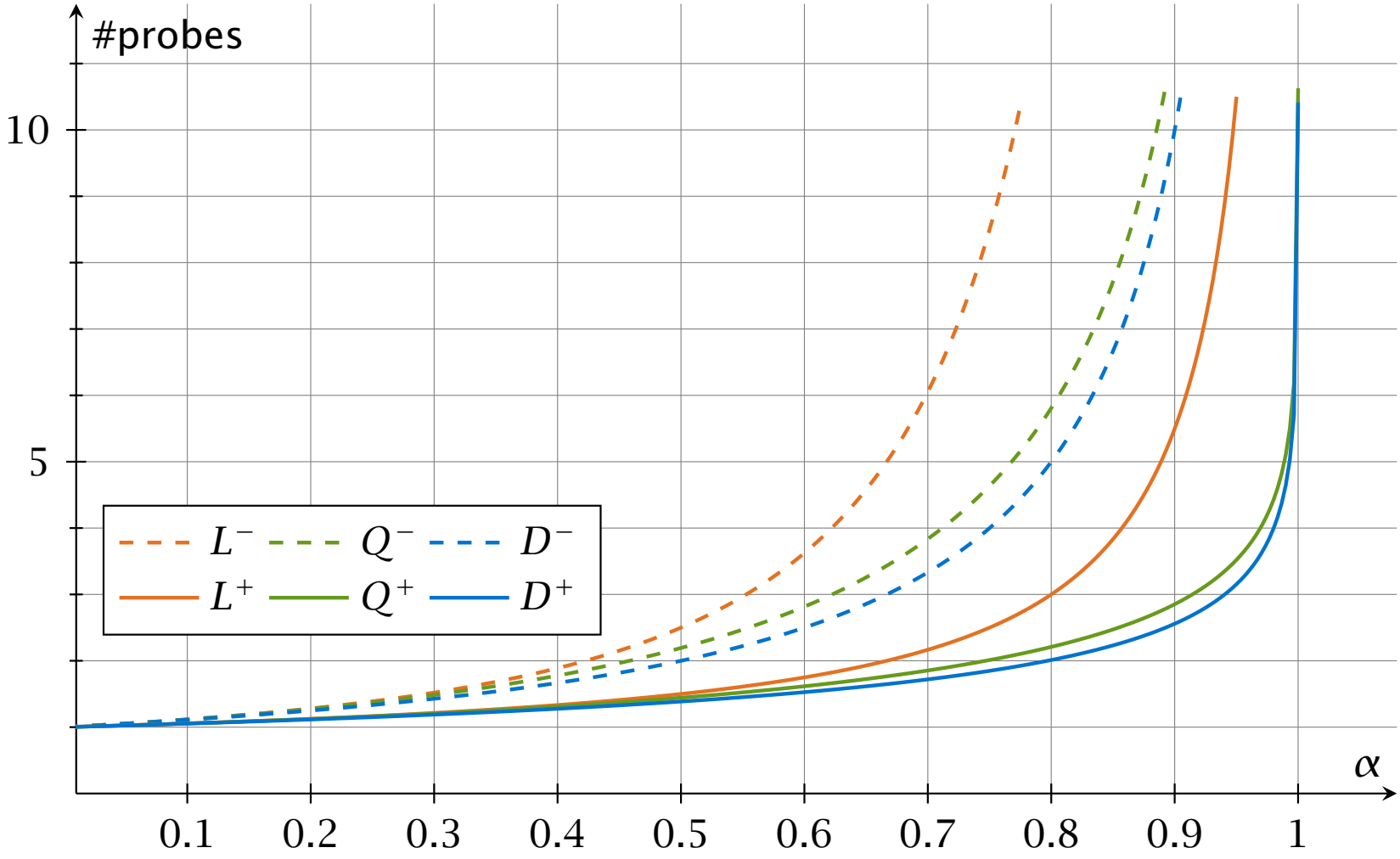
Open Addressing

Some values:

α	*Linear Probing*		*Quadratic Probing*		*Double Hashing*	
	L^+	L^-	Q^+	Q^-	D^+	D^-
0.5	1.5	2.5	1.44	2.19	1.39	2
0.9	5.5	50.5	2.85	11.40	2.55	10
0.95	10.5	200.5	3.52	22.05	3.15	20

Open Addressing



Analysis of Idealized Open Address Hashing

We analyze the time for a search in a very idealized Open Addressing scheme.

- The probe sequence $h(k,0), h(k,1), h(k,2), \ldots$ is equally likely to be any permutation of $\langle 0, 1, \ldots, n-1 \rangle$.

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Let X denote a random variable describing the number of probes in an unsuccessful search.

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$$\Pr[A_1 \cap A_2 \cap \cdots \cap A_{i-1}]$$

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$$= \Pr[A_1] \cdot \Pr[A_2 \mid A_1] \cdot \Pr[A_3 \mid A_1 \cap A_2] \cdot$$
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$$\Pr[X \geq i] = \frac{m}{n} \cdot \frac{m-1}{n-1} \cdot \frac{m-2}{n-2} \cdot \ldots \cdot \frac{m-i+2}{n-i+2}$$

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Analysis of Idealized Open Address Hashing

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Analysis of Idealized Open Address Hashing

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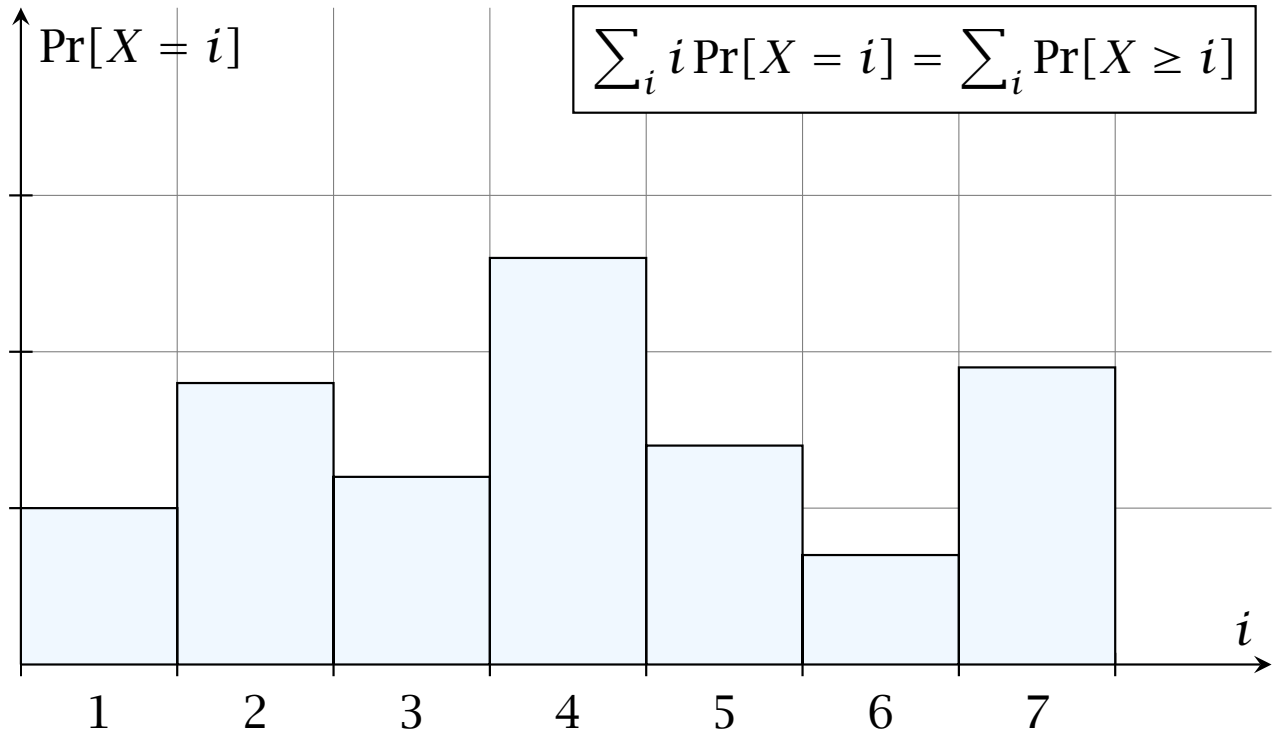
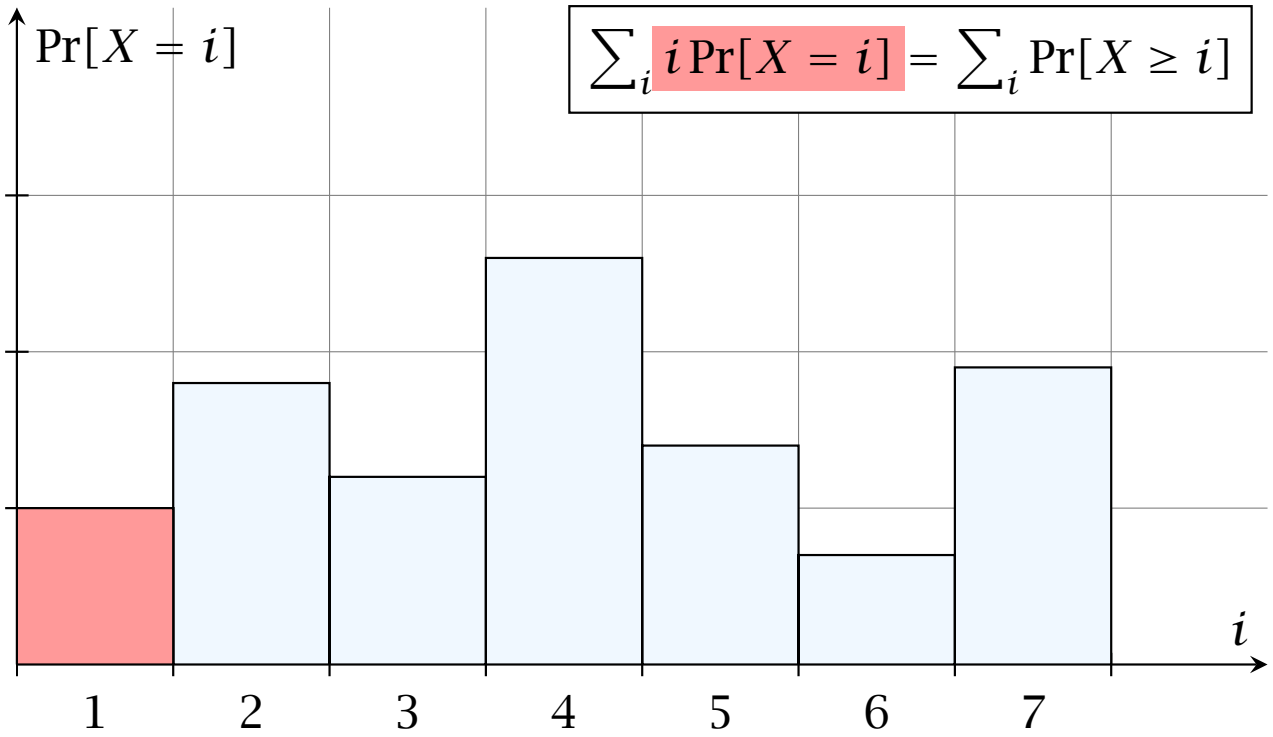
$$\frac{1}{1-\alpha} = 1 + \alpha + \alpha^2 + \alpha^3 + \ldots$$

Analysis of Idealized Open Address Hashing



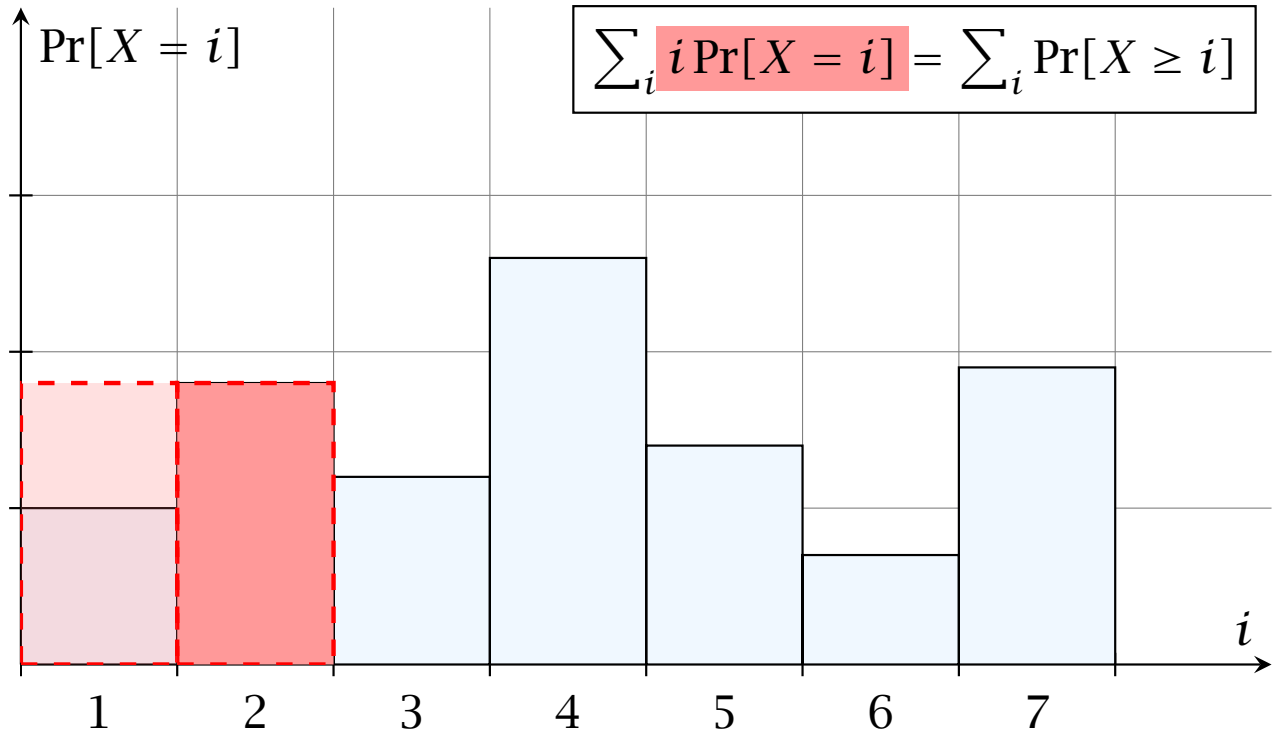
Analysis of Idealized Open Address Hashing

$i = 1$



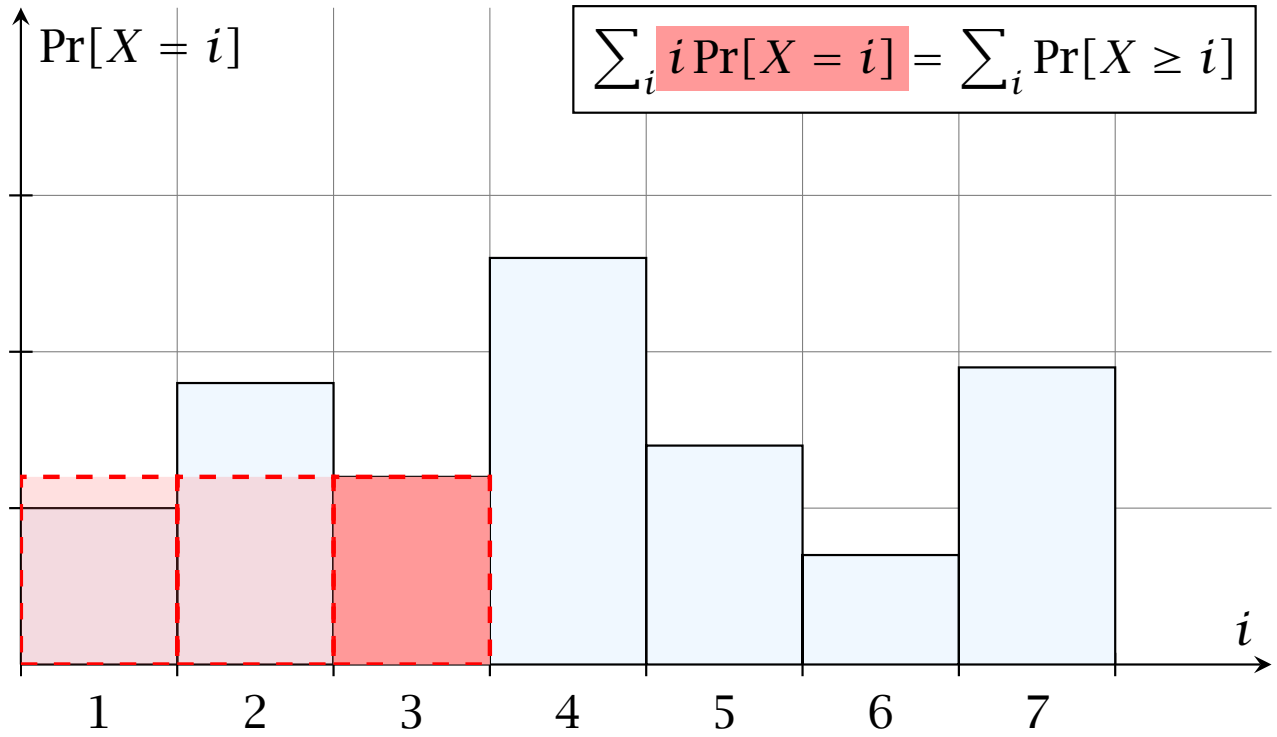
Analysis of Idealized Open Address Hashing

$i = 2$



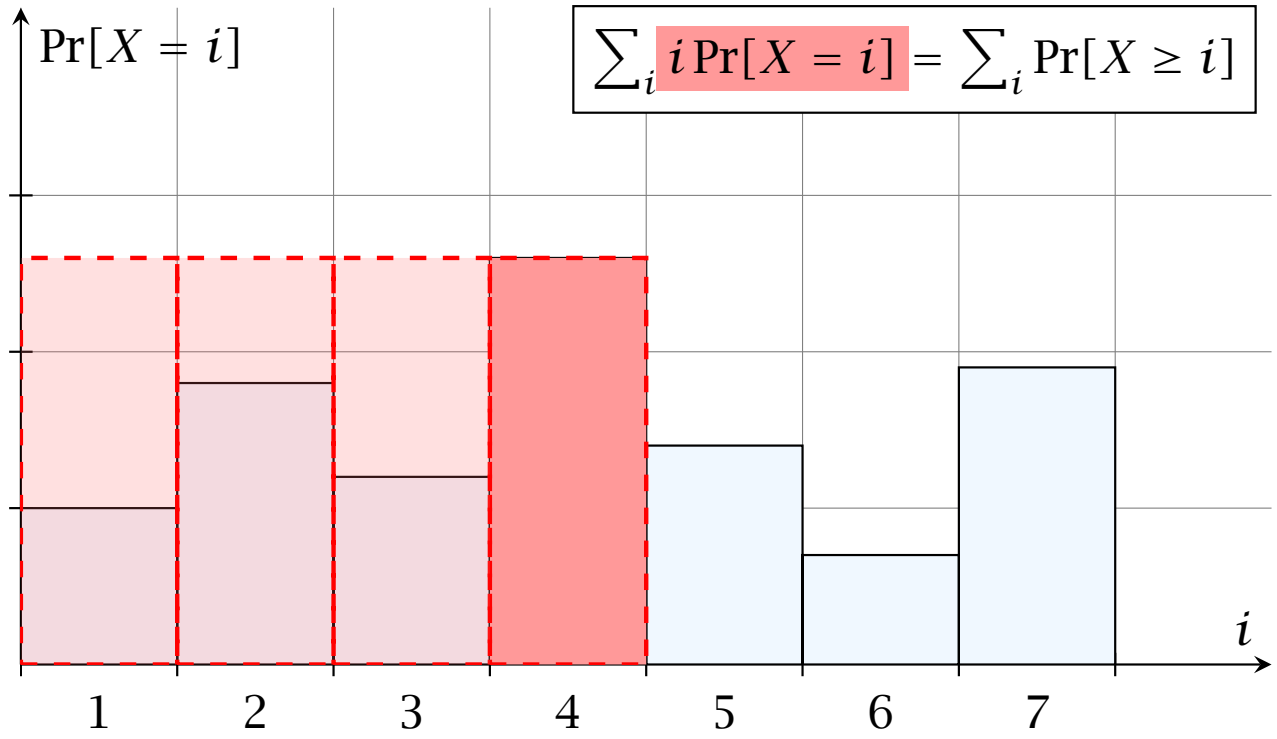
Analysis of Idealized Open Address Hashing

$i = 3$



Analysis of Idealized Open Address Hashing

$i = 4$



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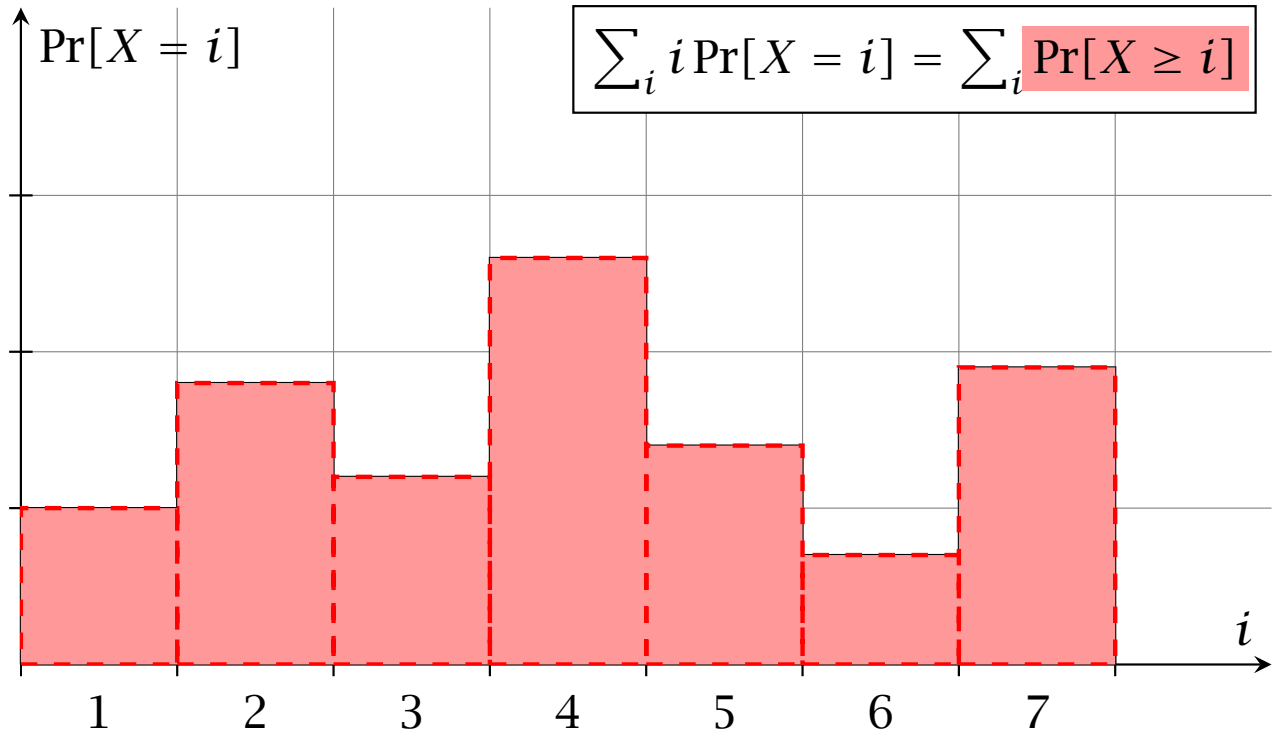

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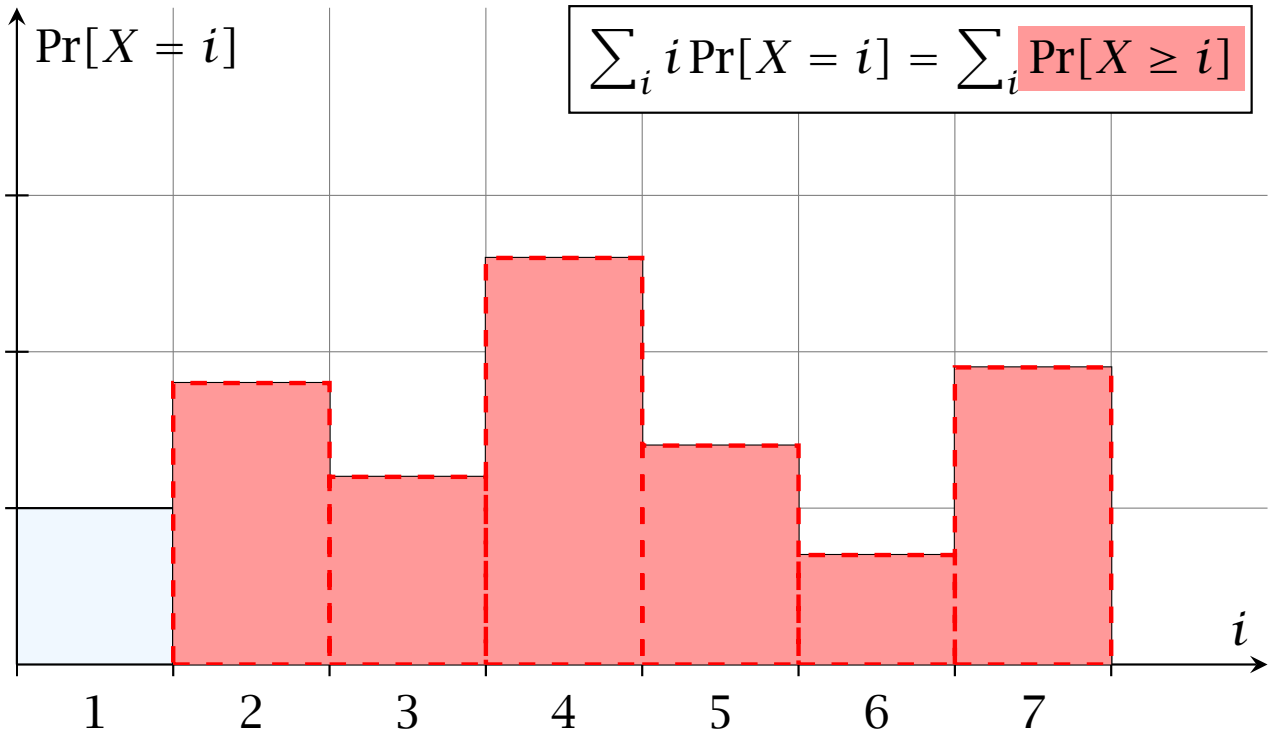

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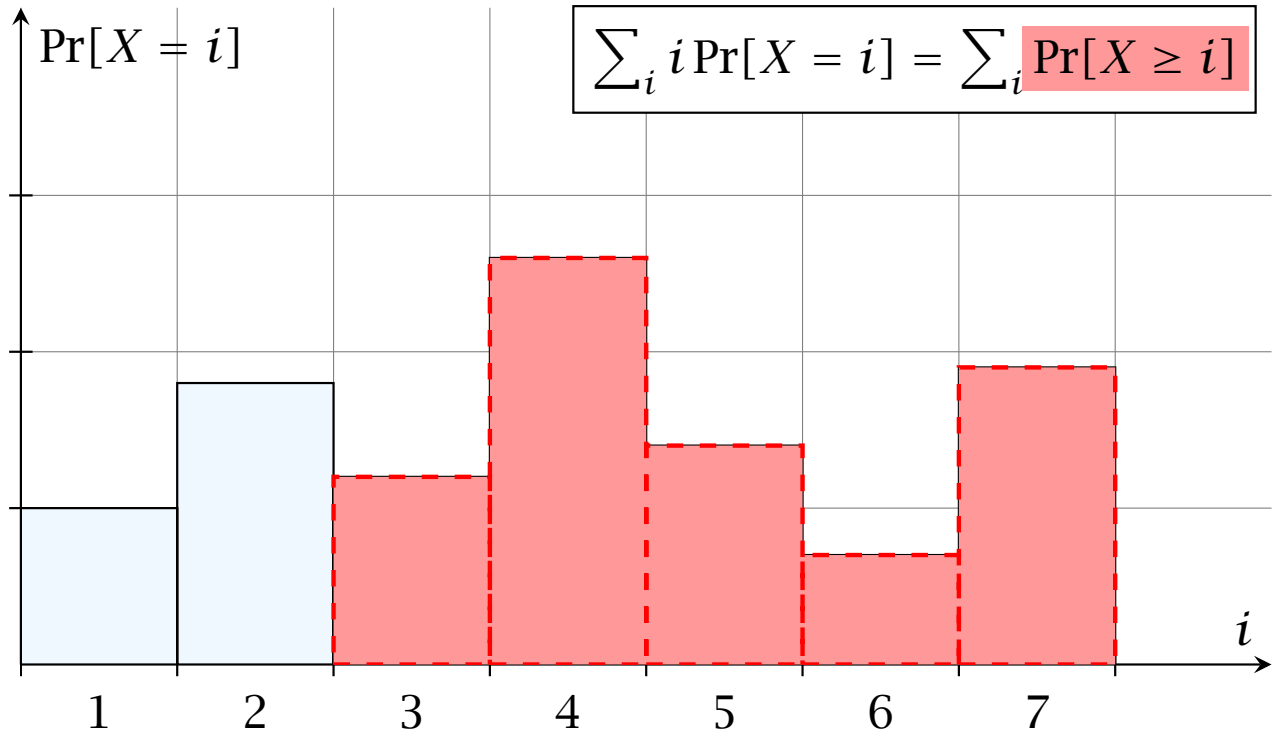

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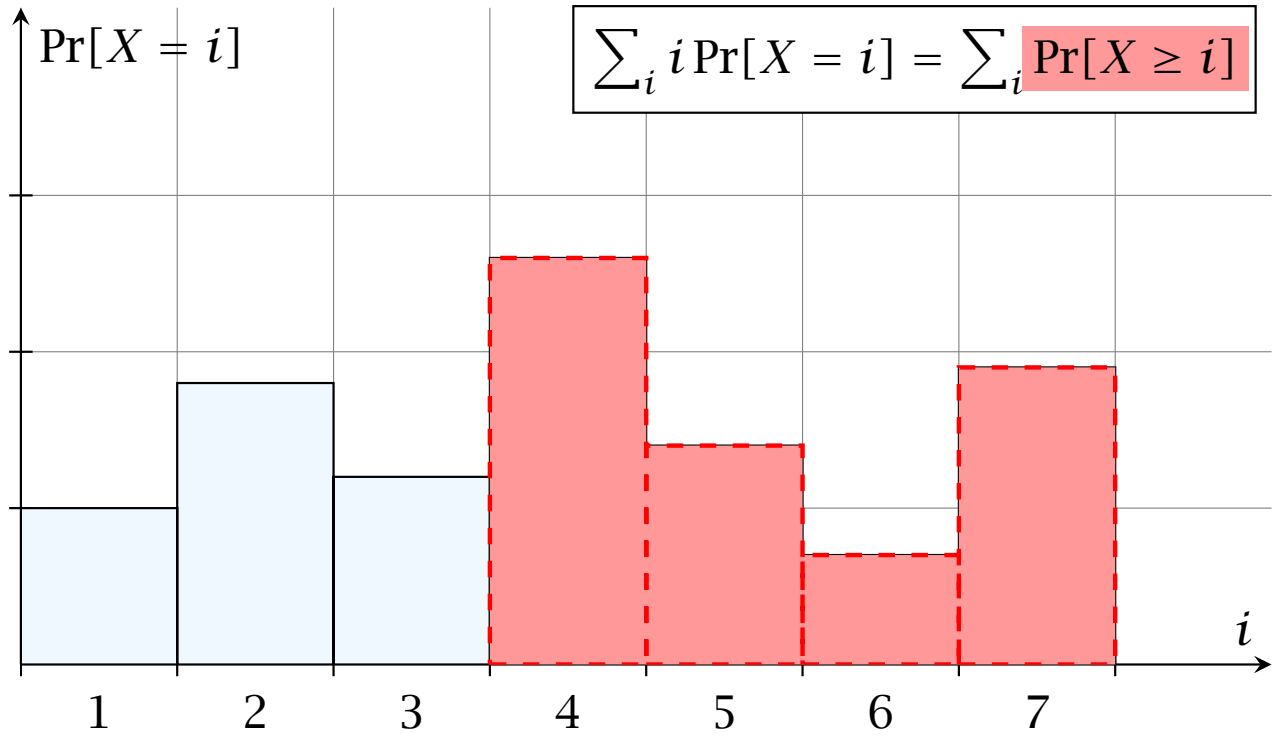

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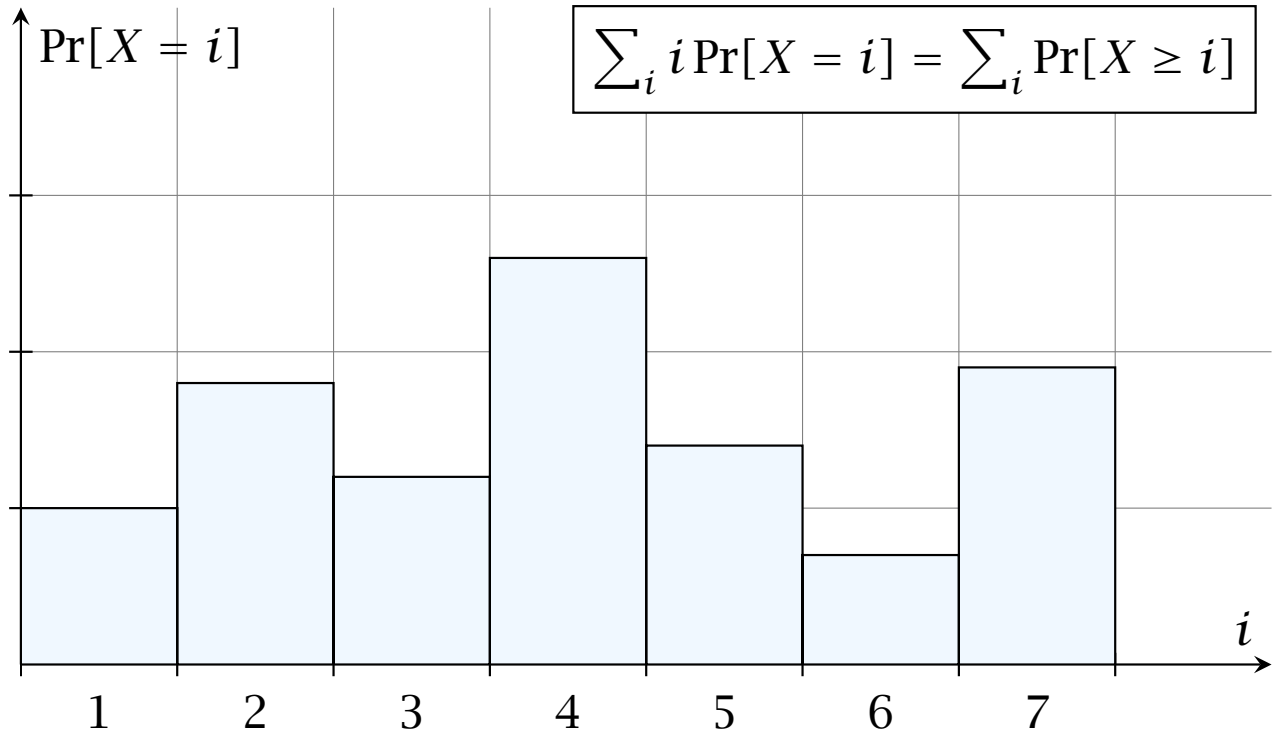

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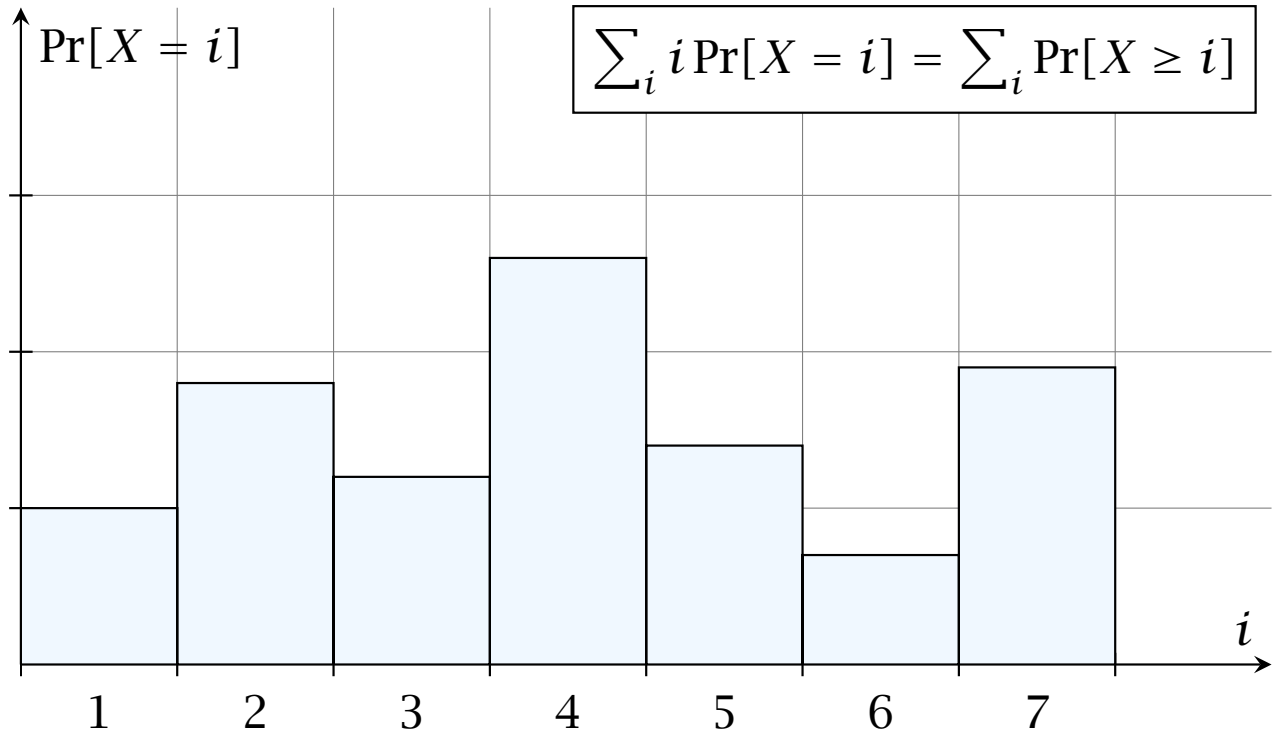

The j-th rectangle appears in both sums j times. (j times in the first due to multiplication with j; and j times in the second for summands $i = 1, 2, \ldots, j$)

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$$\frac{1}{m}\sum_{i=0}^{m-1}\frac{n}{n-i} = \frac{n}{m}\sum_{i=0}^{m-1}\frac{1}{n-i} = \frac{1}{\alpha}\sum_{k=n-m+1}^{n}\frac{1}{k}$$

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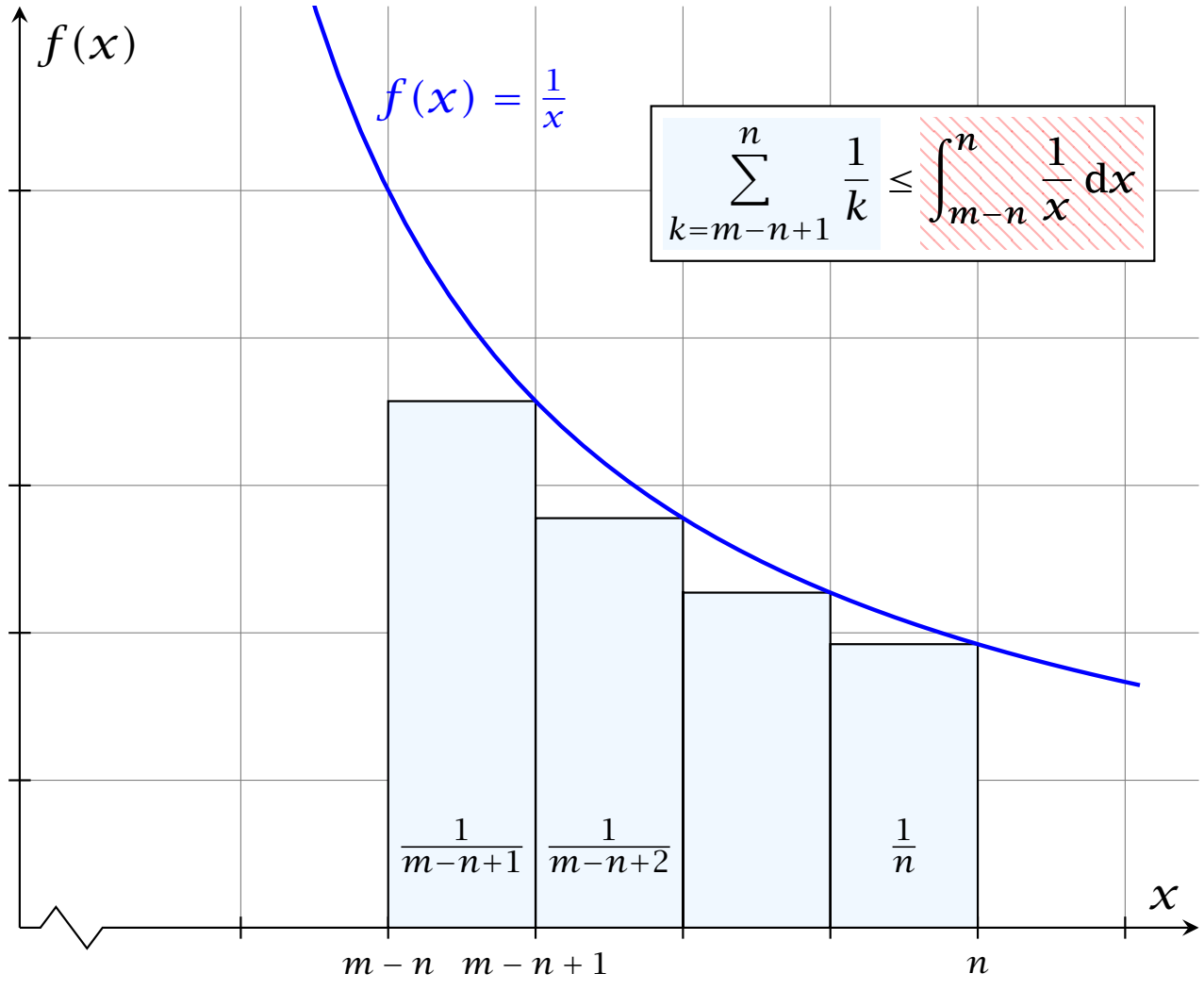

Deletions in Hashtables

How do we delete in a hash-table?

- For hashing with chaining this is not a problem. Simply search for the key, and delete the item in the corresponding list.
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- Simply removing a key might interrupt the probe sequence of other keys which then cannot be found anymore.
- One can delete an element by replacing it with a deleted-marker.
  - During an insertion if a deleted-marker is encountered an element can be inserted there.
  - During a search a deleted-marker must not be used to terminate the probe sequence.
- The table could fill up with deleted-markers leading to bad performance.
- If a table contains many deleted-markers (linear fraction of the keys) one can rehash the whole table and amortize the cost for this rehash against the cost for the deletions.

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Algorithm 16 delete(p)

```
1: T[p] ← null
2: p ← succ(p)
3: while T[p] ≠ null do
4:         y ← T[p]
5:         T[p] ← null
6:         p ← succ(p)
7:         insert(y)
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p is the index into the table-cell that contains the object to be deleted.

Pointers into the hash-table become invalid.

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Universal Hashing

Regardless, of the choice of hash-function there is always an input (a set of keys) that has a very poor worst-case behaviour.

Therefore, so far we assumed that the hash-function is random so that regardless of the input the average case behaviour is good.

However, the assumption of uniform hashing that h is chosen randomly from all functions $f : U \to [0, \ldots, n-1]$ is clearly unrealistic as there are $n^{|U|}$ such functions. Even writing down such a function would take $|U| \log n$ bits.

Universal hashing tries to define a set $\mathcal{H}$ of functions that is much smaller but still leads to good average case behaviour when selecting a hash-function uniformly at random from $\mathcal{H}$.

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Universal Hashing

Definition 26

A class $\mathcal{H}$ of hash-functions from the universe U into the set $\{0,\ldots,n-1\}$ is called universal if for all $u_1, u_2 \in U$ with $u_1 \neq u_2$

$$\Pr[h(u_1) = h(u_2)] \leq \frac{1}{n} \ ,$$

where the probability is w. r. t. the choice of a random hash-function from set $\mathcal{H}$.

Note that this means that the probability of a collision between two arbitrary elements is at most $\frac{1}{n}$.

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Universal Hashing

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A class $\mathcal{H}$ of hash-functions from the universe U into the set $\{0,\ldots,n-1\}$ is called 2-independent (pairwise independent) if the following two conditions hold

- For any key $u \in U$, and $t \in \{0,\ldots,n-1\}$ $\Pr[h(u) = t] = \frac{1}{n}$, i.e., a key is distributed uniformly within the hash-table.
- For all $u_1, u_2 \in U$ with $u_1 \neq u_2$, and for any two hash-positions t_1, t_2:

$$\Pr[h(u_1) = t_1 \wedge h(u_2) = t_2] \leq \frac{1}{n^2} \ .$$

This requirement clearly implies a universal hash-function.

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A class $\mathcal{H}$ of hash-functions from the universe U into the set $\{0,\ldots,n-1\}$ is called k-independent if for any choice of $\ell \leq k$ distinct keys $u_1,\ldots,u_\ell \in U$, and for any set of ℓ not necessarily distinct hash-positions $t_1,\ldots,t_\ell$:

$$\Pr[h(u_1) = t_1 \wedge \cdots \wedge h(u_\ell) = t_\ell] \leq \frac{1}{n^\ell} \ ,$$

where the probability is w. r. t. the choice of a random hash-function from set $\mathcal{H}$.

Universal Hashing

Definition 29

A class $\mathcal{H}$ of hash-functions from the universe U into the set $\{0,\ldots,n-1\}$ is called (μ,k)-independent if for any choice of $\ell \leq k$ distinct keys $u_1,\ldots,u_\ell \in U$, and for any set of ℓ not necessarily distinct hash-positions $t_1,\ldots,t_\ell$:

$$\Pr[h(u_1) = t_1 \wedge \cdots \wedge h(u_\ell) = t_\ell] \leq \frac{\mu}{n^\ell} \ ,$$

where the probability is w. r. t. the choice of a random hash-function from set $\mathcal{H}$.

Universal Hashing

Let $U := \{0, \ldots, p-1\}$ for a prime p. Let $\mathbb{Z}_p := \{0, \ldots, p-1\}$, and let $\mathbb{Z}_p^* := \{1, \ldots, p-1\}$ denote the set of invertible elements in $\mathbb{Z}_p$.

Define

$$h_{a,b}(x) := (ax + b \bmod p) \bmod n$$

Lemma 30

The class

$$\mathcal{H} = \{h_{a,b} \mid a \in \mathbb{Z}_p^*, b \in \mathbb{Z}_p\}$$

is a universal class of hash-functions from U to $\{0, \ldots, n-1\}$.

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If $x \neq y$ then $(x - y) \not\equiv 0 \pmod{p}$.

Multiplying with $a \not\equiv 0 \pmod{p}$ gives

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- The hash-function does not generate collisions before the $(\mathrm{mod}\ n)$-operation. Furthermore, every choice (a, b) is mapped to a different pair (t_x, t_y) with $t_x := ax + b$ and $t_y := ay + b$.

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$$\begin{aligned} a &\equiv (t_x - t_y)(x - y)^{-1} && (\mathrm{mod}\ p) \\ b &\equiv t_y - ay && (\mathrm{mod}\ p) \end{aligned}$$

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There is a one-to-one correspondence between hash-functions (pairs (a,b), $a \neq 0$) and pairs (t_x, t_y), $t_x \neq t_y$.

Therefore, we can view the first step (before the $\bmod n$-operation) as choosing a pair (t_x, t_y), $t_x \neq t_y$ uniformly at random.

What happens when we do the $\bmod n$ operation?

Fix a value t_x. There are $p-1$ possible values for choosing t_y.

From the range $0, \ldots, p-1$ the values $t_x, t_x + n, t_x + 2n, \ldots$ map to t_x after the modulo-operation. These are at most $\lceil p/n \rceil$ values.

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As $t_y \neq t_x$ there are

possibilities for choosing t_y such that the final hash-value creates a collision.

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It is also possible to show that $\mathcal{H}$ is an (almost) pairwise independent class of hash-functions.

$$\Pr_{t_x \neq t_y \in \mathbb{Z}_p^2} \left[\begin{array}{c} t_x \bmod n = h_1 \\ \wedge \\ t_y \bmod n = h_2 \end{array} \right]$$

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It is also possible to show that $\mathcal{H}$ is an (almost) pairwise independent class of hash-functions.

$$\frac{\left\lfloor \frac{p}{n} \right\rfloor^2}{p(p-1)} \leq \Pr_{t_x \neq t_y \in \mathbb{Z}_p^2} \left[\begin{array}{c} t_x \bmod n = h_1 \\ \wedge \\ t_y \bmod n = h_2 \end{array} \right] \leq \frac{\left\lceil \frac{p}{n} \right\rceil^2}{p(p-1)}$$

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Note that the middle is the probability that $h(x) = h_1$ and $h(y) = h_2$. The total number of choices for (t_x, t_y) is $p(p-1)$. The number of choices for t_x (t_y) such that $t_x \bmod n = h_1$ ($t_y \bmod n = h_2$) lies between $\lfloor \frac{p}{n} \rfloor$ and $\lceil \frac{p}{n} \rceil$.

Universal Hashing

Definition 31

Let $d \in \mathbb{N}$; $q \geq (d+1)n$ be a prime; and let $\bar{a} \in \{0,\ldots,q-1\}^{d+1}$. Define for $x \in \{0,\ldots,q-1\}$

$$h_{\bar{a}}(x) := \Big(\sum_{i=0}^{d} a_i x^i \bmod q \Big) \bmod n \ .$$

Let $\mathcal{H}_n^d := \{h_{\bar{a}} \mid \bar{a} \in \{0,\ldots,q-1\}^{d+1}\}$. The class $\mathcal{H}_n^d$ is $(e, d+1)$-independent.

Note that in the previous case we had $d = 1$ and chose $a_d \neq 0$.

Universal Hashing

For the coefficients $\bar{a} \in \{0, \ldots, q-1\}^{d+1}$ let $f_{\bar{a}}$ denote the polynomial

$$f_{\bar{a}}(x) = \left(\sum_{i=0}^{d} a_i x^i \right) \bmod q$$

The polynomial is defined by $d+1$ distinct points.

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Universal Hashing

Fix $\ell \leq d+1$; let $x_1, \ldots, x_\ell \in \{0, \ldots, q-1\}$ be keys, and let $t_1, \ldots, t_\ell$ denote the corresponding hash-function values.

Let $A^\ell = \{h_{\bar{a}} \in \mathcal{H} \mid h_{\bar{a}}(x_i) = t_i \text{ for all } i \in \{1, \ldots, \ell\}\}$
Then

$$h_{\bar{a}} \in A^\ell \Leftrightarrow h_{\bar{a}} = f_{\bar{a}} \bmod n \text{ and}$$

$$f_{\bar{a}}(x_i) \in \underbrace{\{t_i + \alpha \cdot n \mid \alpha \in \{0, \ldots, \lceil \tfrac{q}{n} \rceil - 1\}\}}_{=:B_i}$$

In order to obtain the cardinality of A^ℓ we choose our polynomial by fixing $d+1$ points.

We first fix the values for inputs $x_1, \ldots, x_\ell$.
We have

$$|B_1| \cdot \ldots \cdot |B_\ell|$$

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$$h_{\bar{a}} \in A^\ell \Leftrightarrow h_{\bar{a}} = f_{\bar{a}} \bmod n \text{ and}$$

$$f_{\bar{a}}(x_i) \in \underbrace{\{t_i + \alpha \cdot n \mid \alpha \in \{0, \ldots, \lceil \tfrac{q}{n} \rceil - 1\}\}}_{=: B_i}$$

In order to obtain the cardinality of A^ℓ we choose our polynomial by fixing $d+1$ points.

We first fix the values for inputs $x_1, \ldots, x_\ell$.
We have

$$|B_1| \cdot \ldots \cdot |B_\ell|$$

possibilities to do this (so that $h_{\bar{a}}(x_i) = t_i$).

Universal Hashing

Now, we choose $d - \ell + 1$ other inputs and choose their value arbitrarily. We have $q^{d-\ell+1}$ possibilities to do this.

Therefore we have

$$|B_1| \cdot \ldots \cdot |B_\ell| \cdot q^{d-\ell+1} \leq \lceil \frac{q}{n} \rceil^\ell \cdot q^{d-\ell+1}$$

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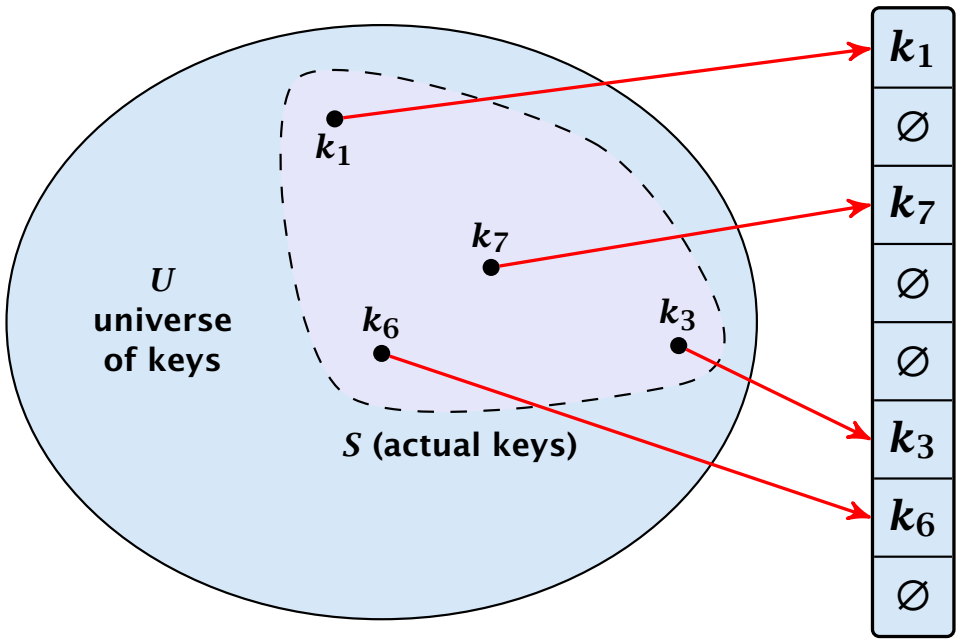
This shows that the $\mathcal{H}$ is $(e, d+1)$-universal.

The last step followed from $q \geq (d+1)n$, and $\ell \leq d+1$.

Perfect Hashing

Suppose that we **know** the set S of actual keys (no insert/no delete). Then we may want to design a **simple** hash-function that maps all these keys to different memory locations.



Perfect Hashing

Let $m = |S|$. We could simply choose the hash-table size very large so that we don't get any collisions.

Using a universal hash-function the expected number of collisions is

$$\mathrm{E}[\#\text{Collisions}] = \binom{m}{2} \cdot \frac{1}{n} \ .$$

If we choose $n = m^2$ the expected number of collisions is strictly less than $\frac{1}{2}$.

Can we get an upper bound on the probability of having collisions?

The probability of having 1 or more collisions can be at most $\frac{1}{2}$ as otherwise the expectation would be larger than $\frac{1}{2}$.

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We can find such a hash-function by a few trials.

However, a hash-table size of $n = m^2$ is very very high.

We construct a two-level scheme. We first use a hash-function that maps elements from S to m buckets.

Let m_j denote the number of items that are hashed to the j-th bucket. For each bucket we choose a second hash-function that maps the elements of the bucket into a table of size m_j^2. The second function can be chosen such that all elements are mapped to different locations.

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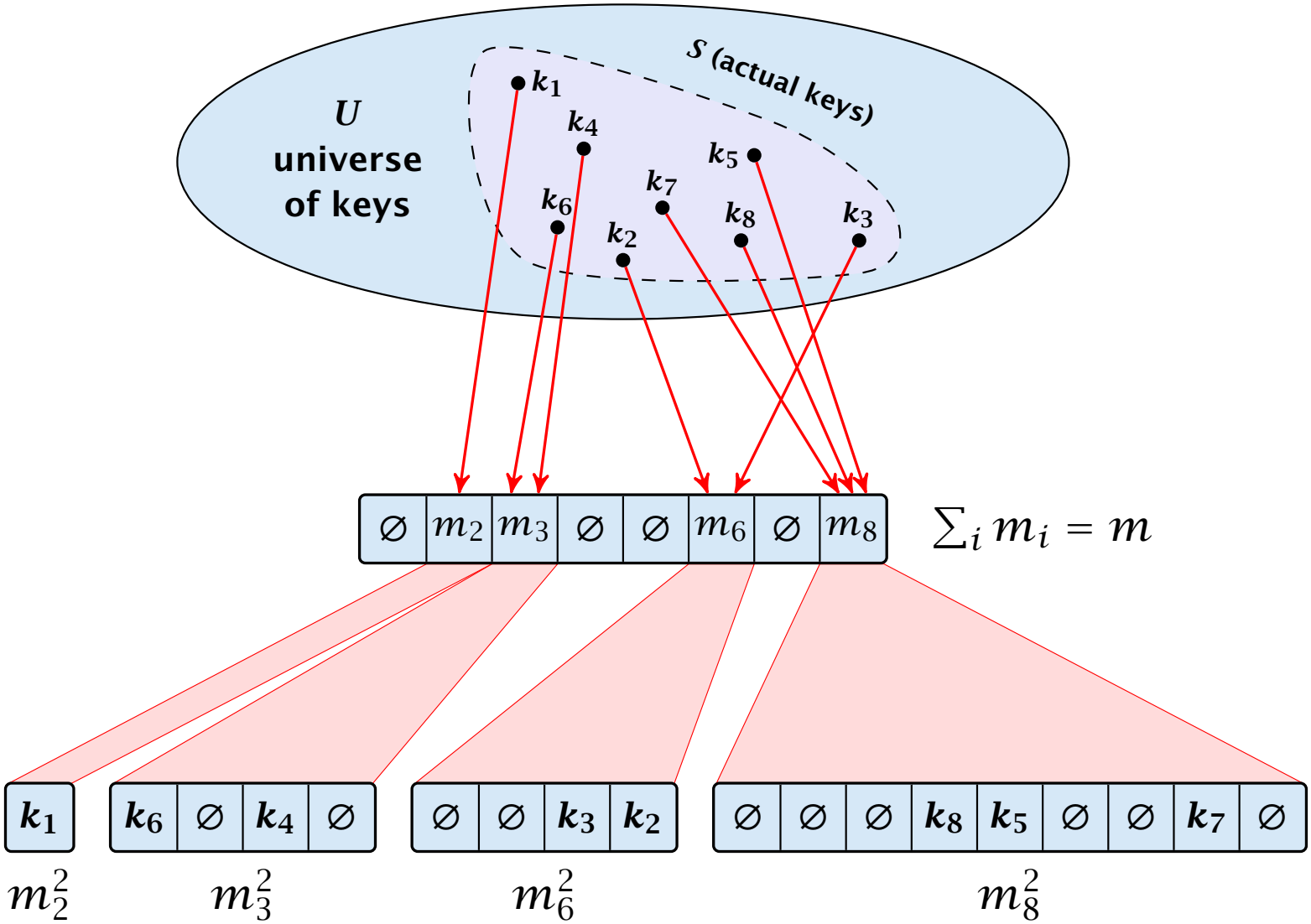

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$$= 2\binom{m}{2}\frac{1}{m} + m = 2m - 1 \ .$$

Perfect Hashing

We need only $\mathcal{O}(m)$ time to construct a hash-function h with $\sum_j m_j^2 = \mathcal{O}(4m)$, because with probability at least $1/2$ a random function from a universal family will have this property.

Then we construct a hash-table h_j for every bucket. This takes expected time $\mathcal{O}(m_j)$ for every bucket. A random function h_j is collision-free with probability at least $1/2$. We need $\mathcal{O}(m_j)$ to test this.

We only need that the hash-functions are chosen from a universal family!!!

Cuckoo Hashing

Goal:
Try to generate a hash-table with constant worst-case search time in a dynamic scenario.

- Two hash-tables $T_1[0,\dots,n-1]$ and $T_2[0,\dots,n-1]$ with hash-functions h_1 and h_2.
- An object x is either stored at location $T_1[h_1(x)]$ or $T_2[h_2(x)]$.
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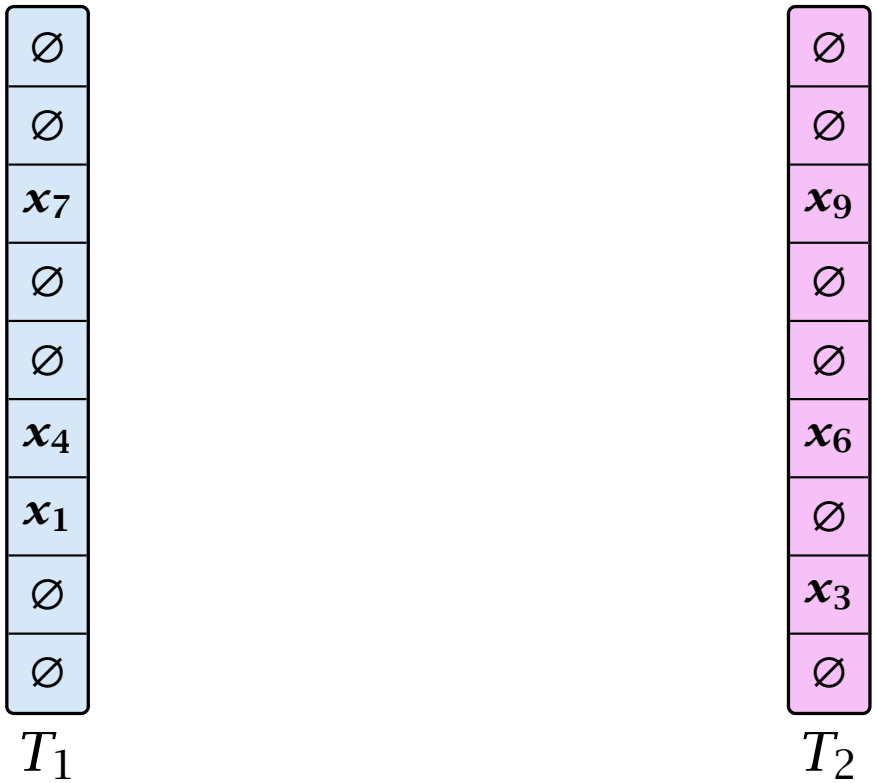

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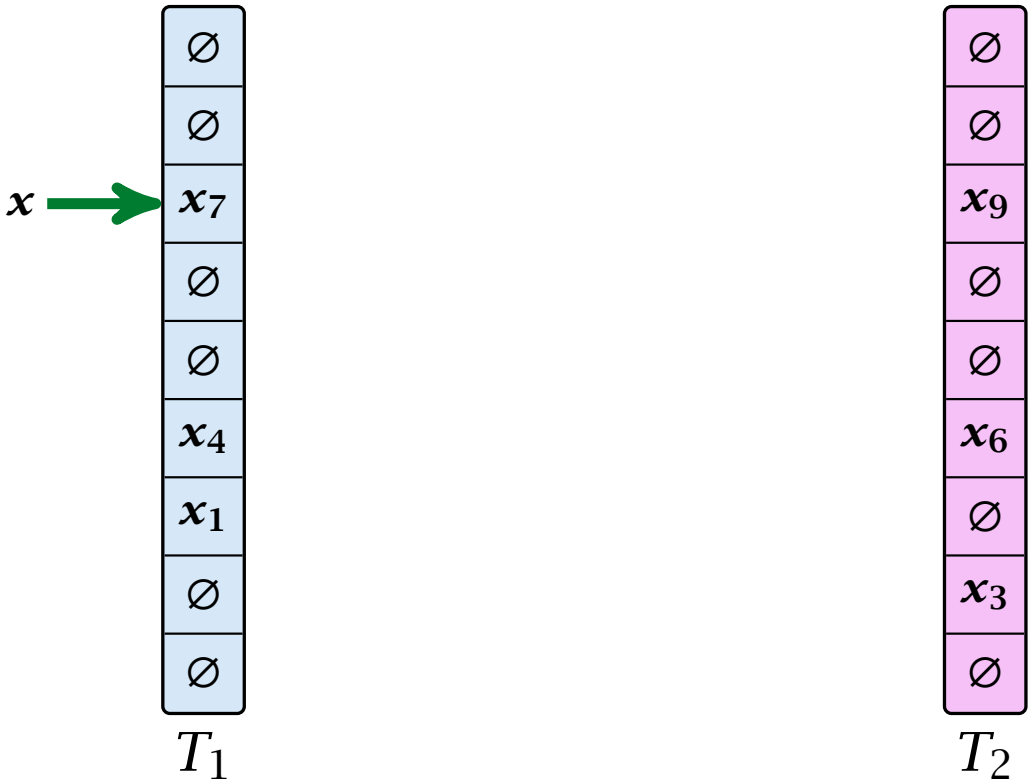

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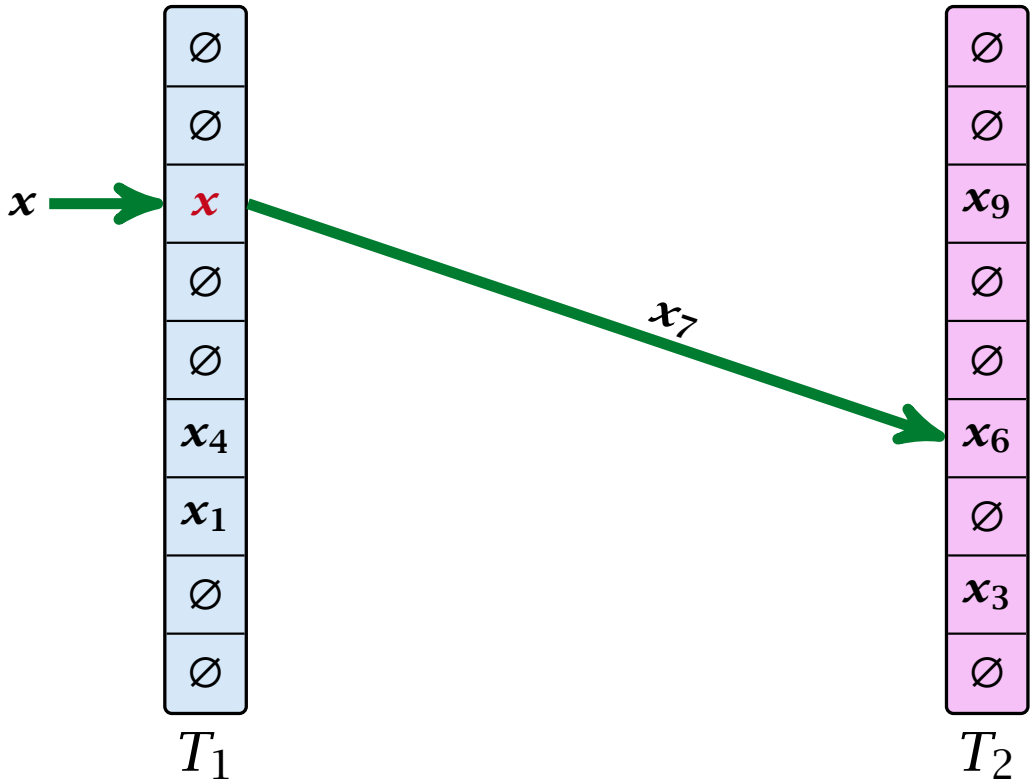

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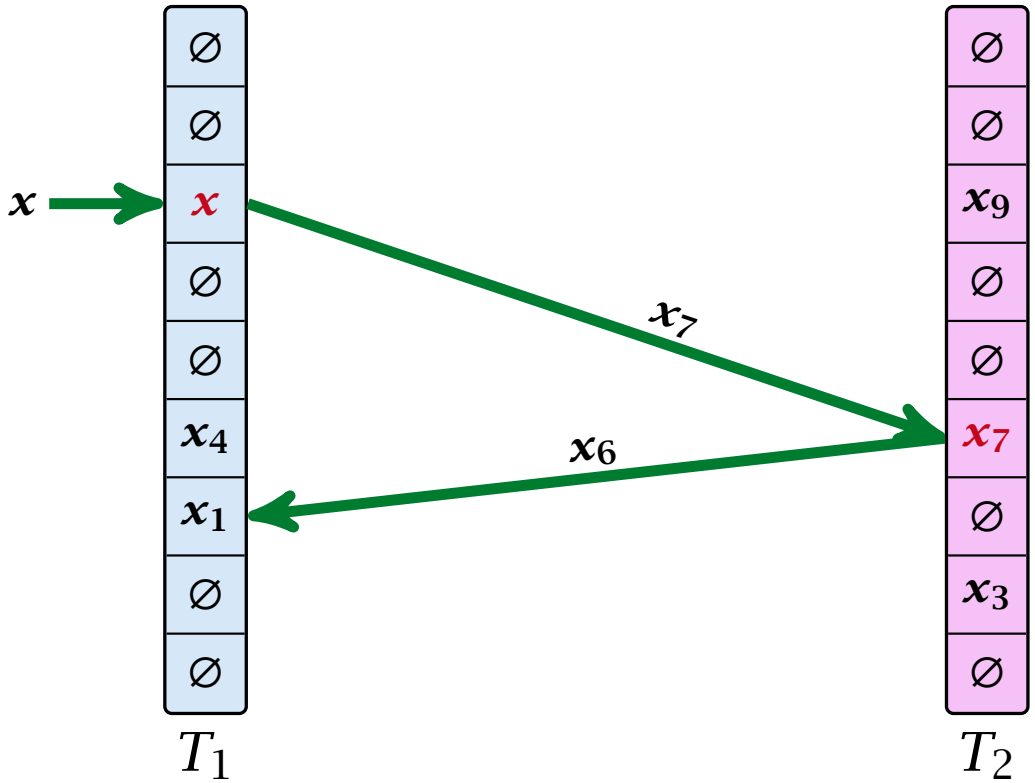

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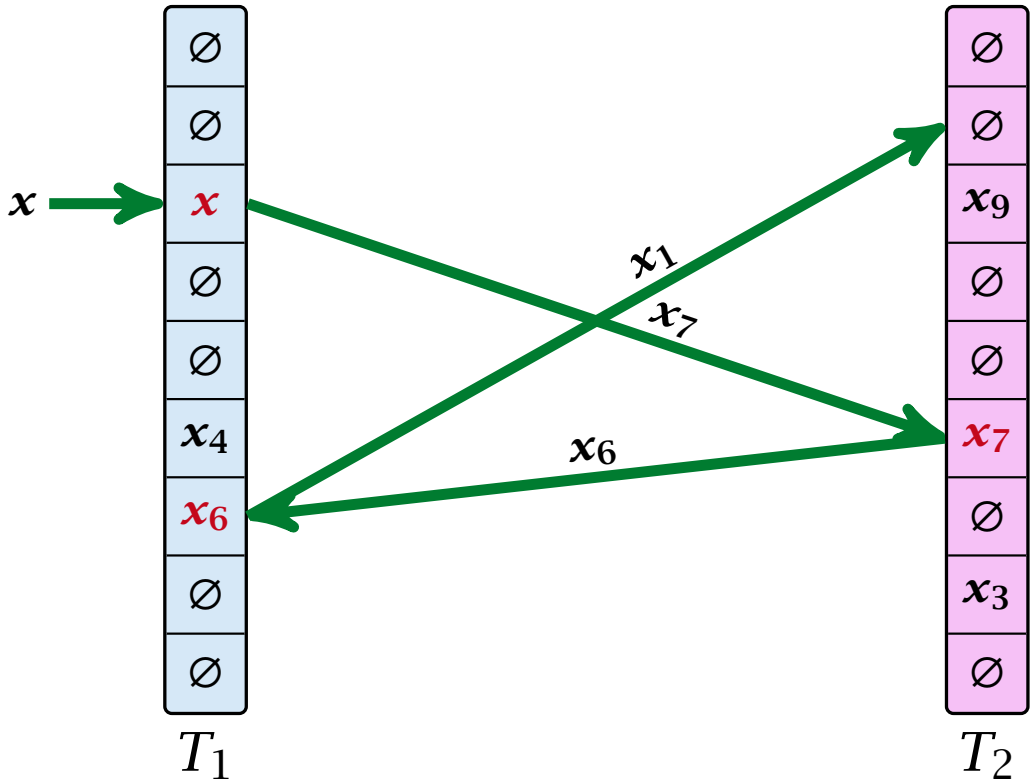

Cuckoo Hashing

Algorithm 17 Cuckoo-Insert(x)

```
 1: if T_1[h_1(x)] = x ∨ T_2[h_2(x)] = x then return
 2: steps ← 1
 3: while steps ≤ maxsteps do
 4:         exchange x and T_1[h_1(x)]
 5:         if x = null then return
 6:         exchange x and T_2[h_2(x)]
 7:         if x = null then return
 8:         steps ← steps +1
 9: rehash() // change hash-functions; rehash everything
10: Cuckoo-Insert(x)
```

Cuckoo Hashing

- We call one iteration through the while-loop a step of the algorithm.
- We call a sequence of iterations through the while-loop without the termination condition becoming true a phase of the algorithm.
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What is the expected time for an insert-operation?

We first analyze the probability that we end-up in an infinite loop (that is then terminated after maxsteps steps).

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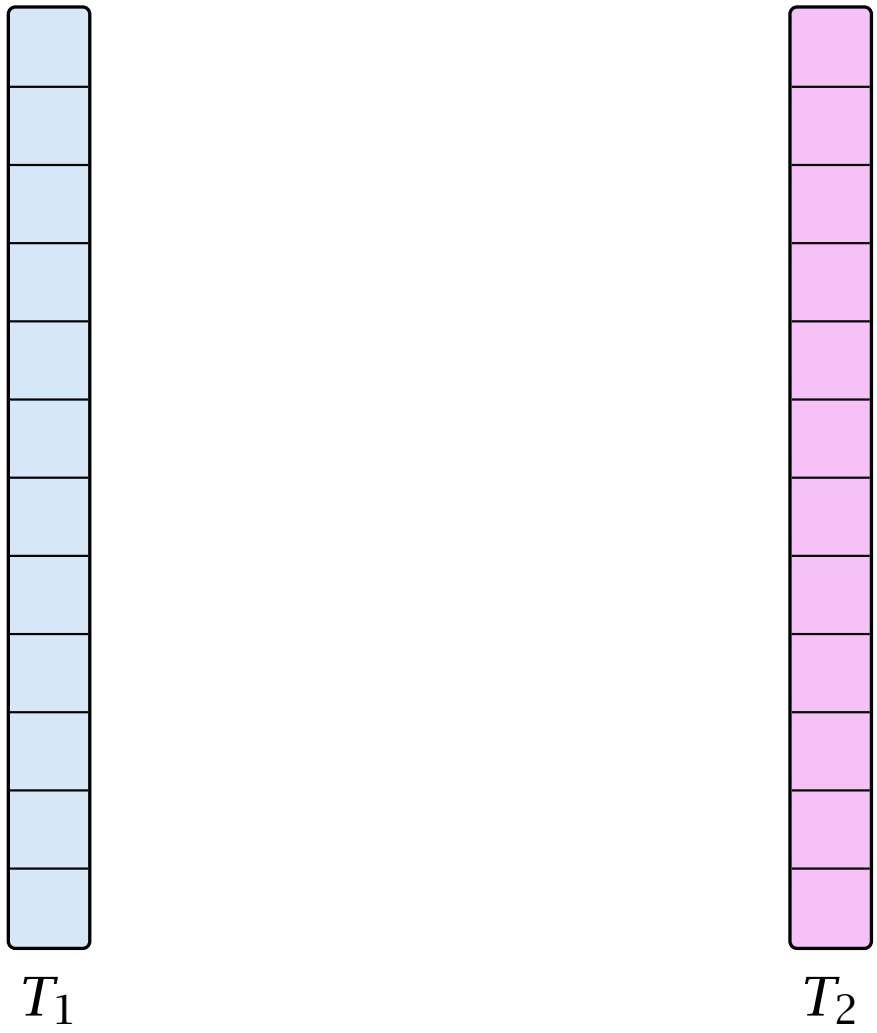

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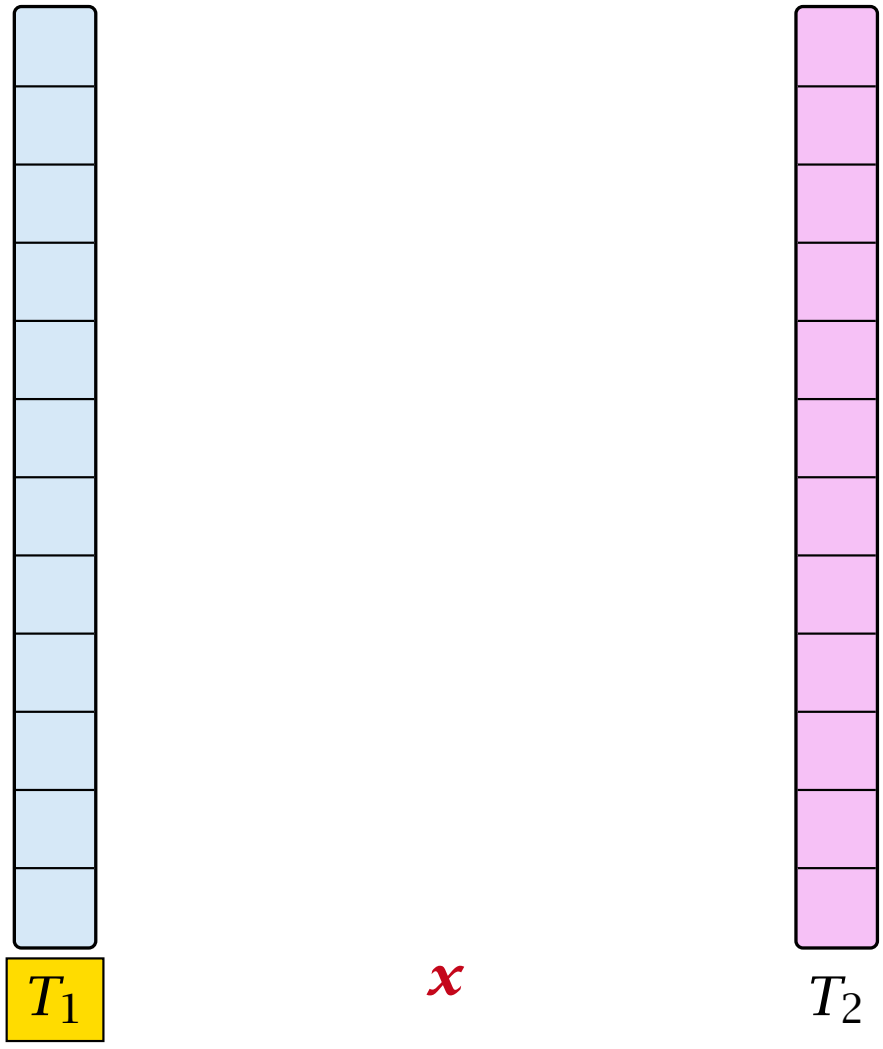

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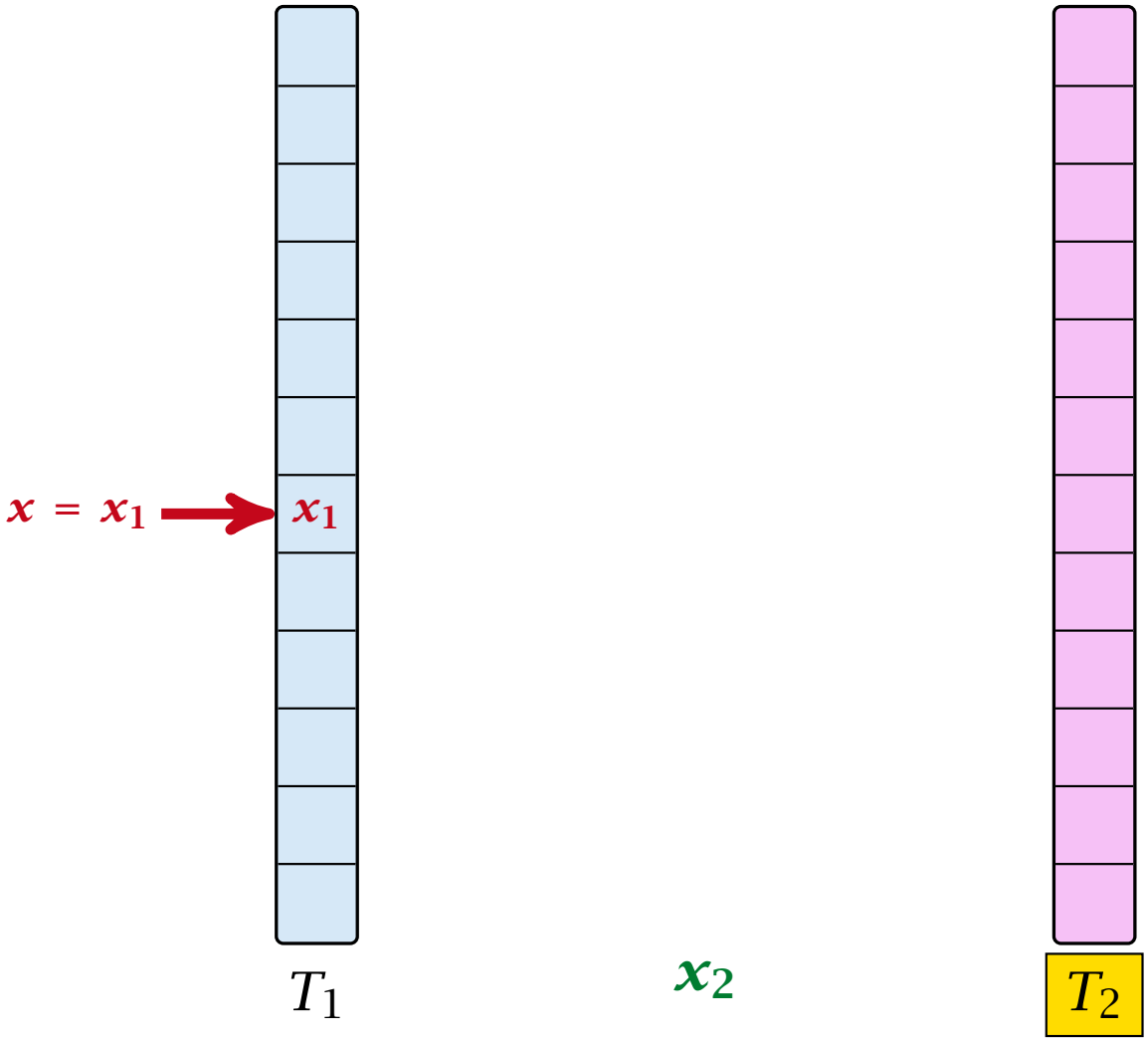

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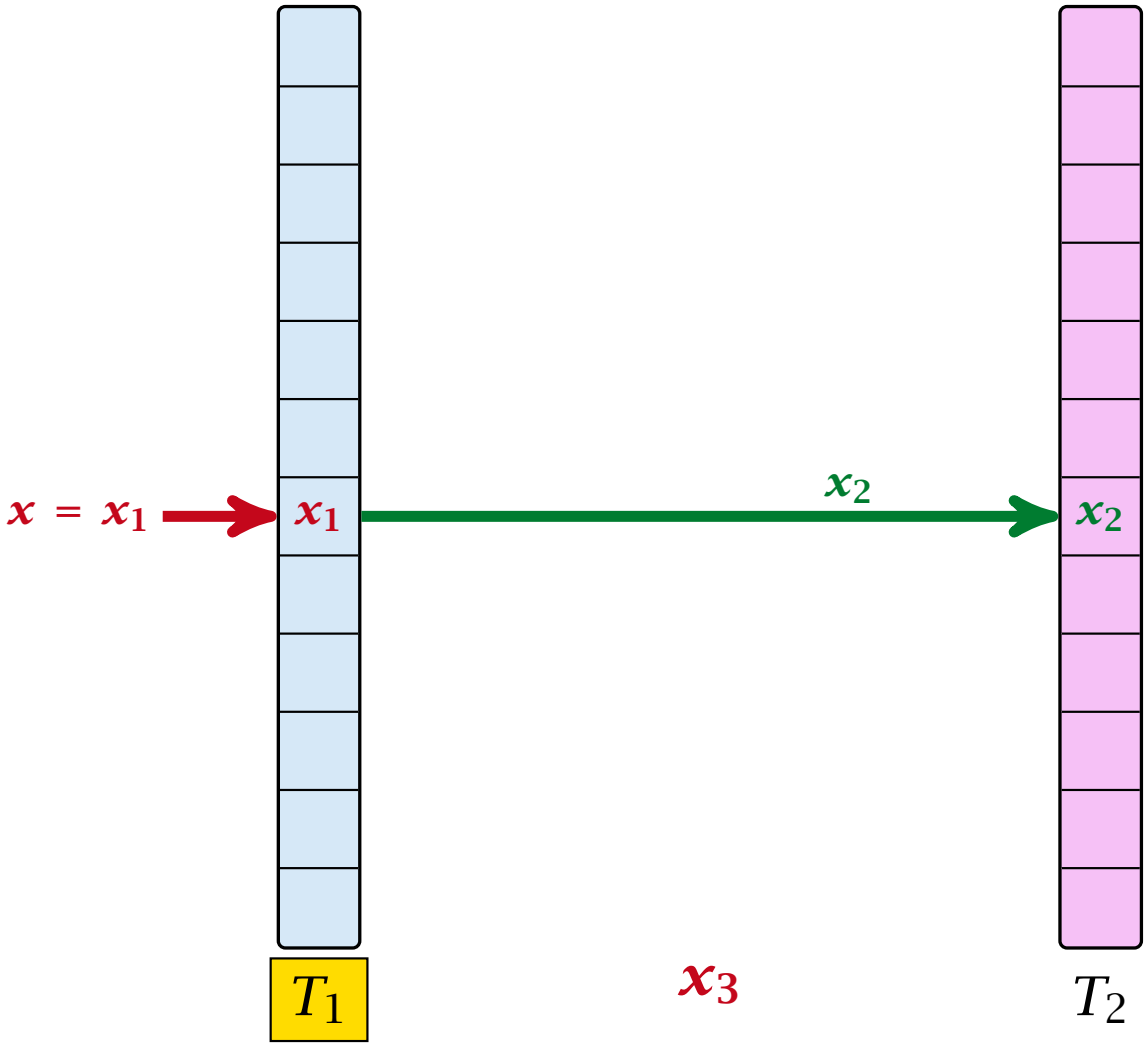

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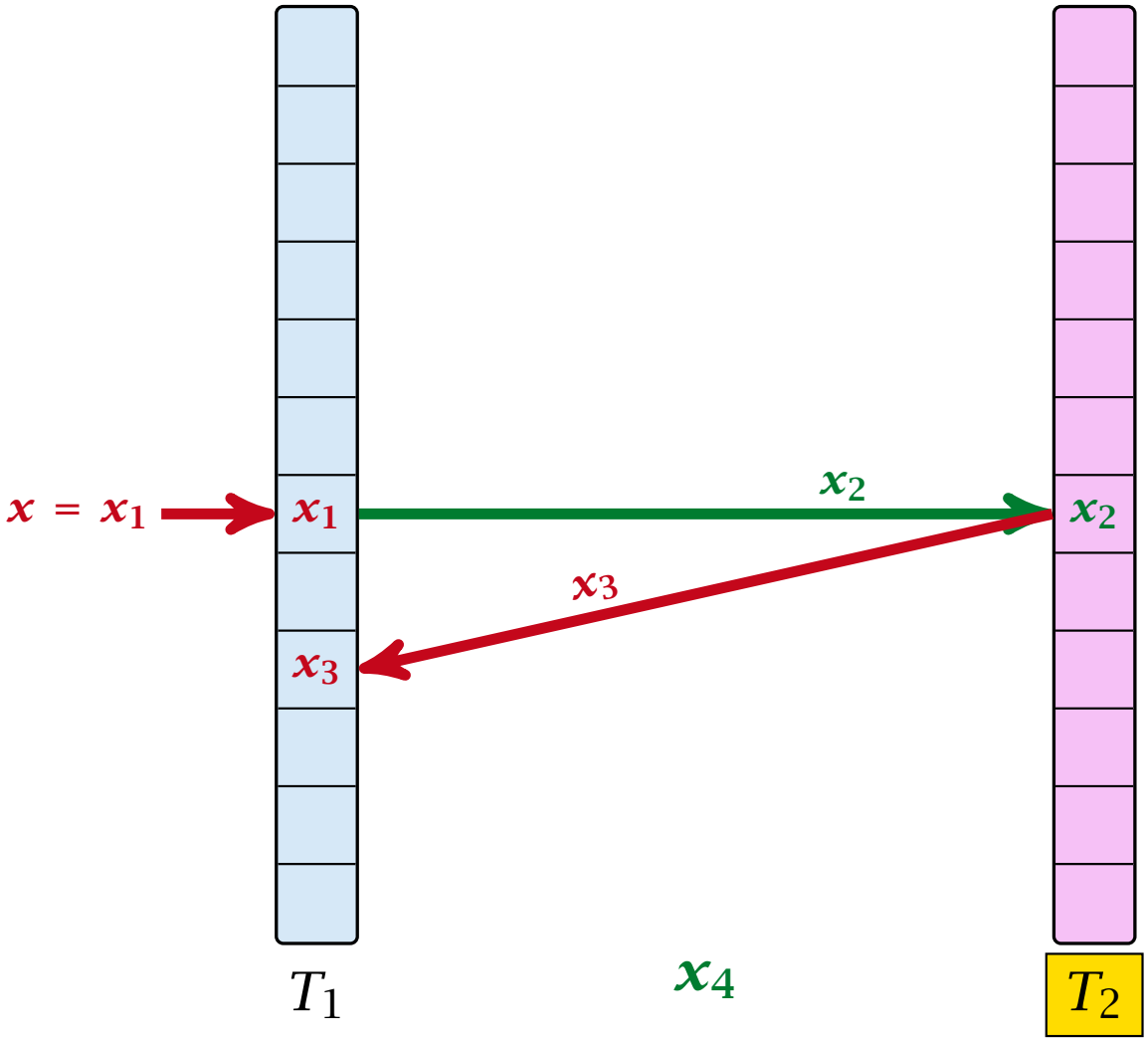

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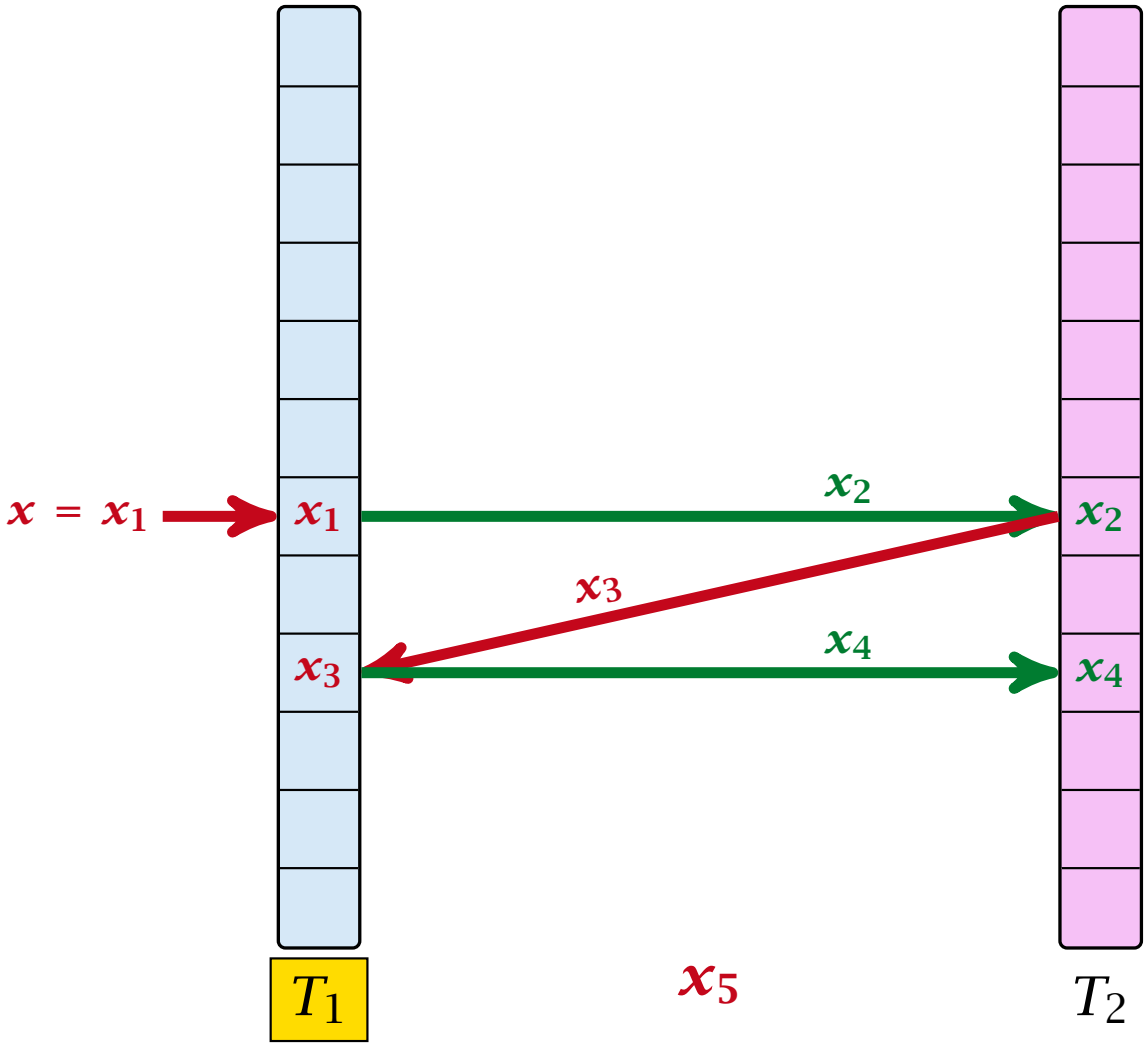

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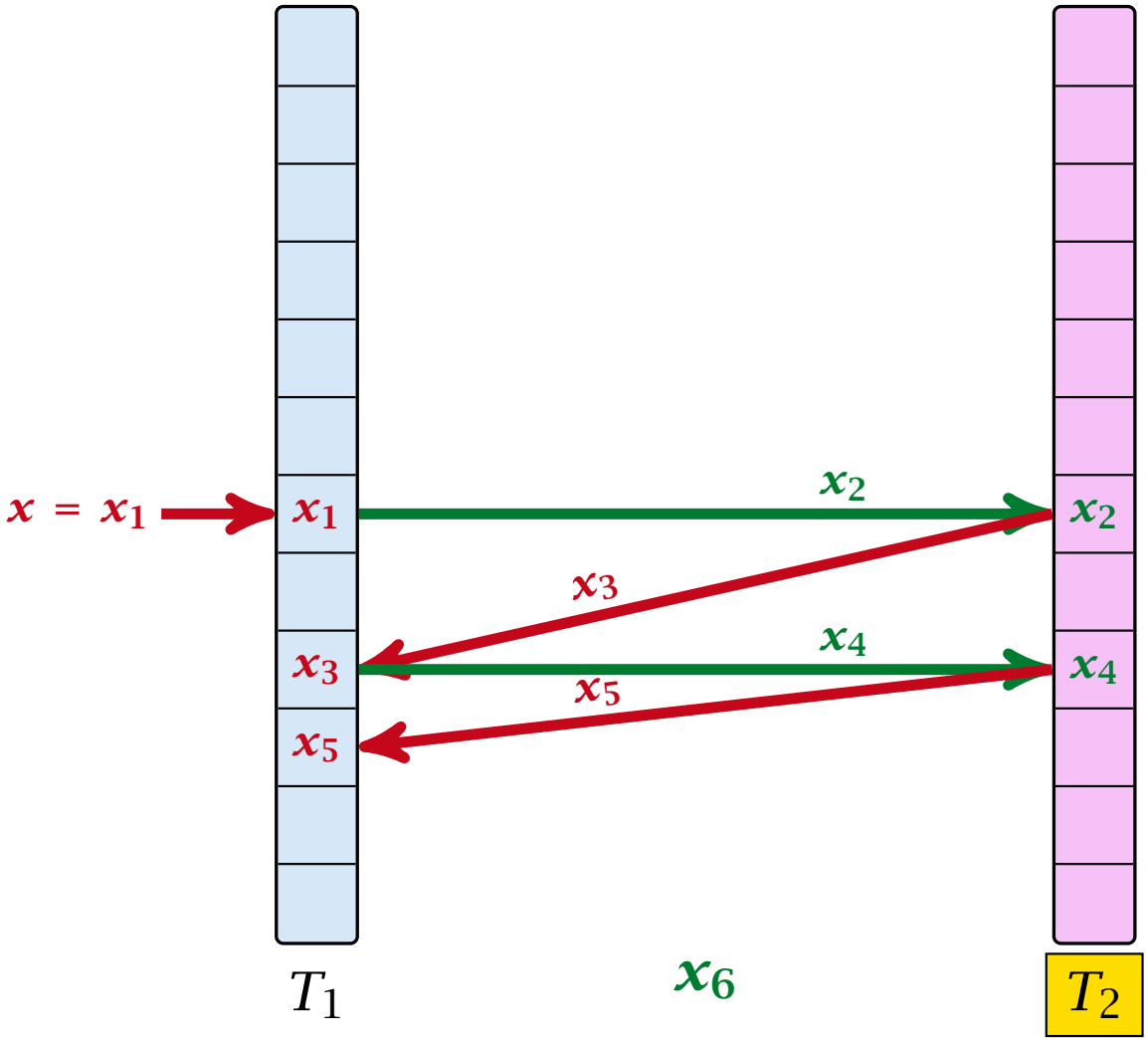

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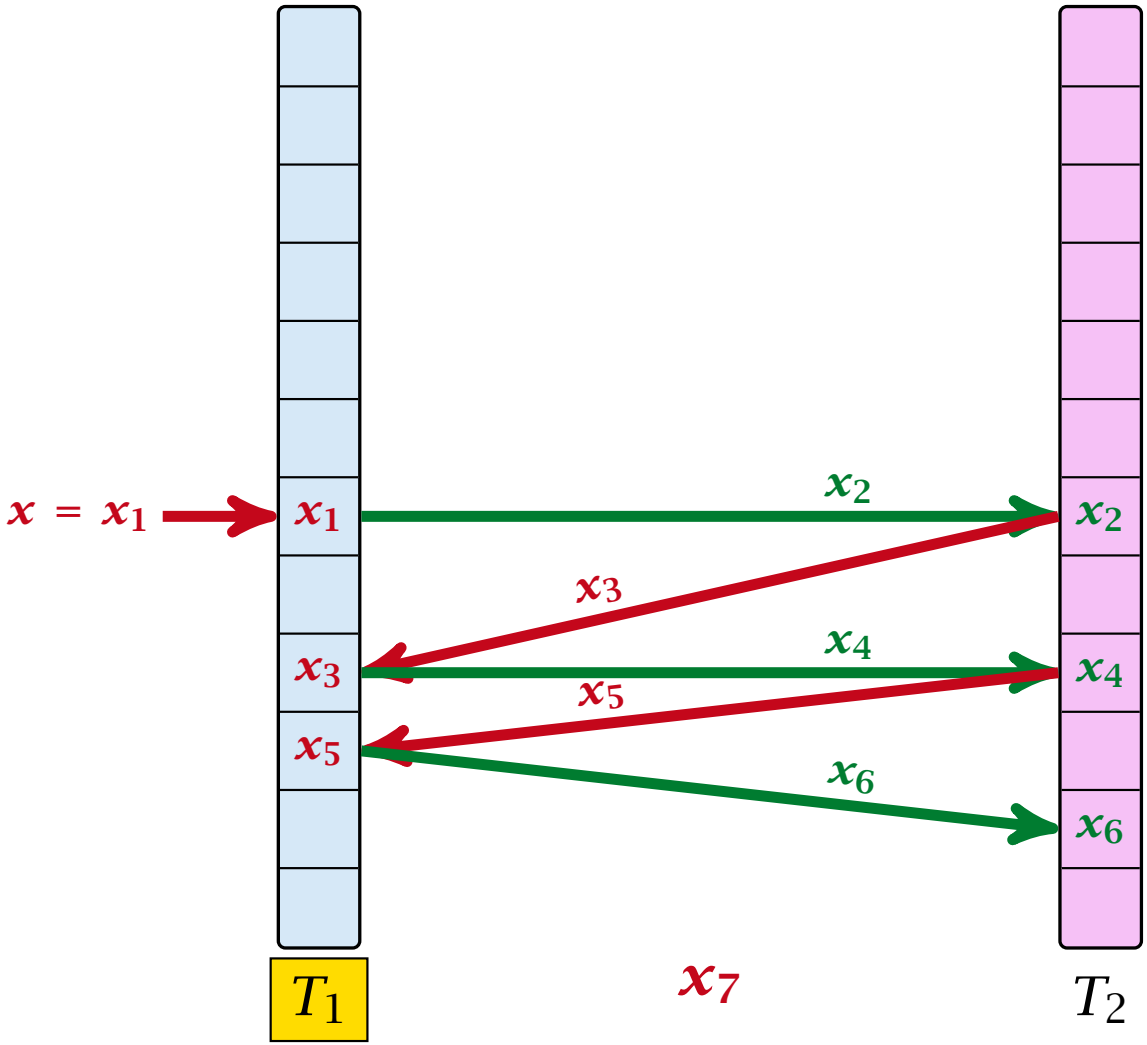

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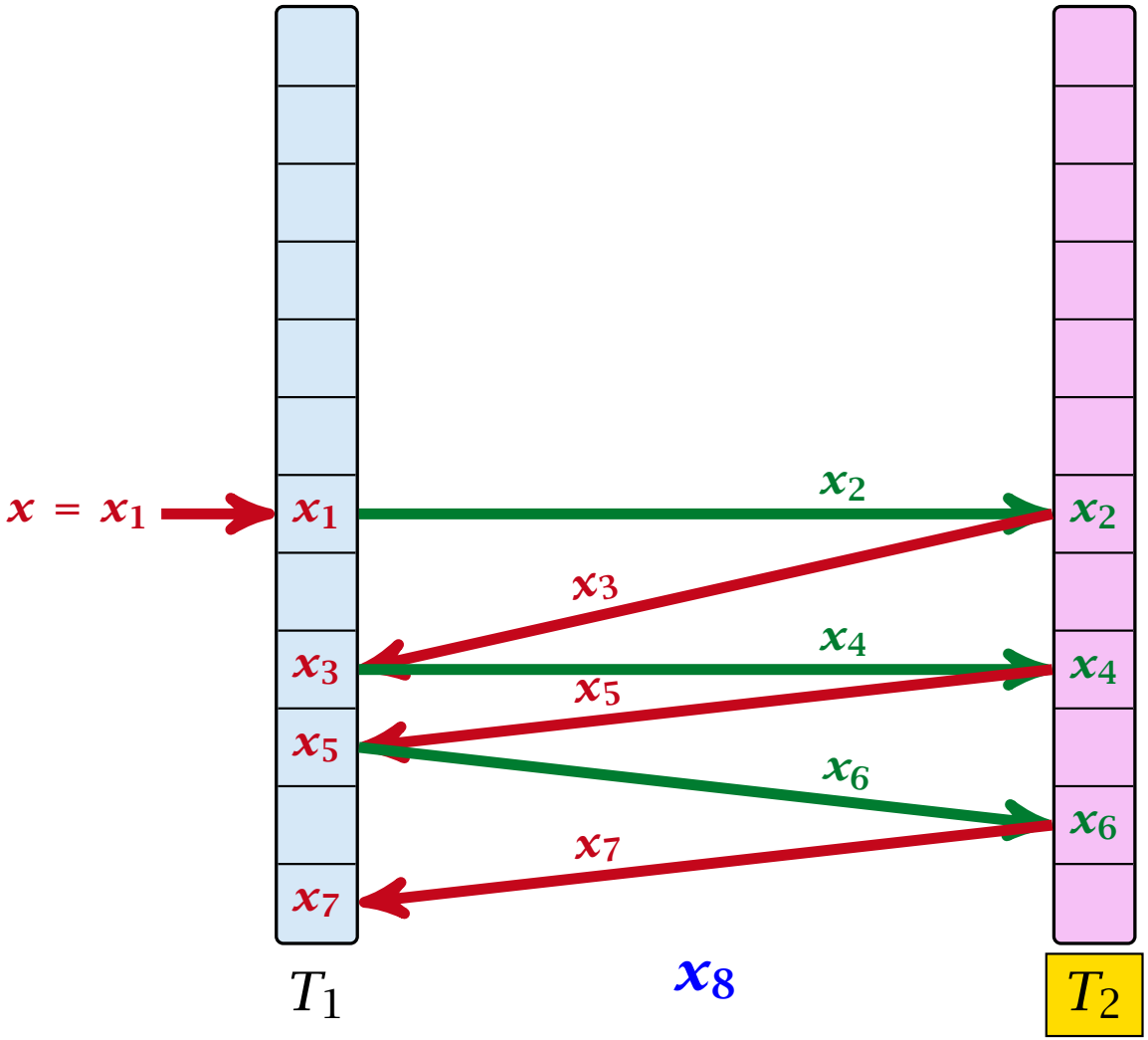

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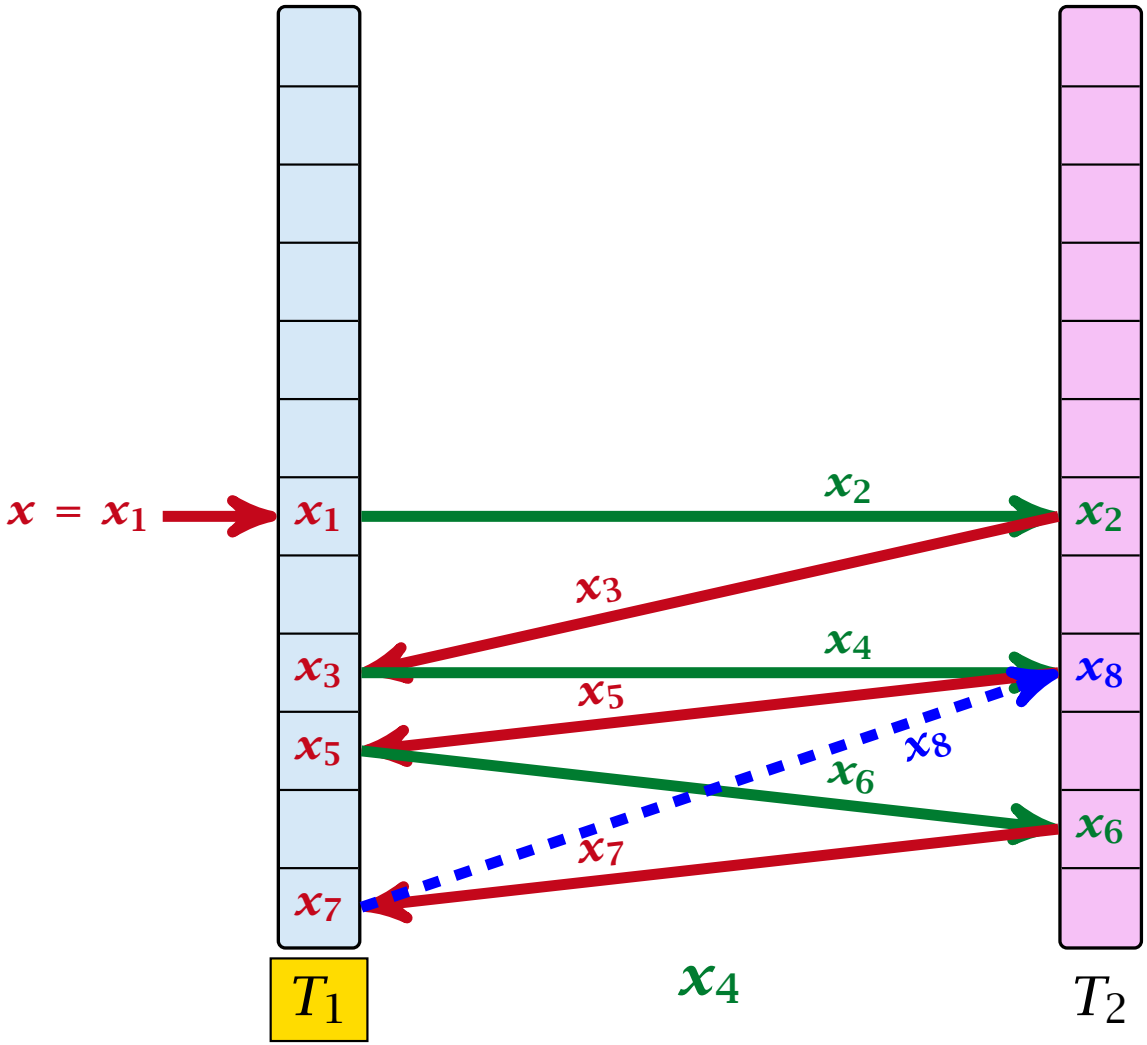

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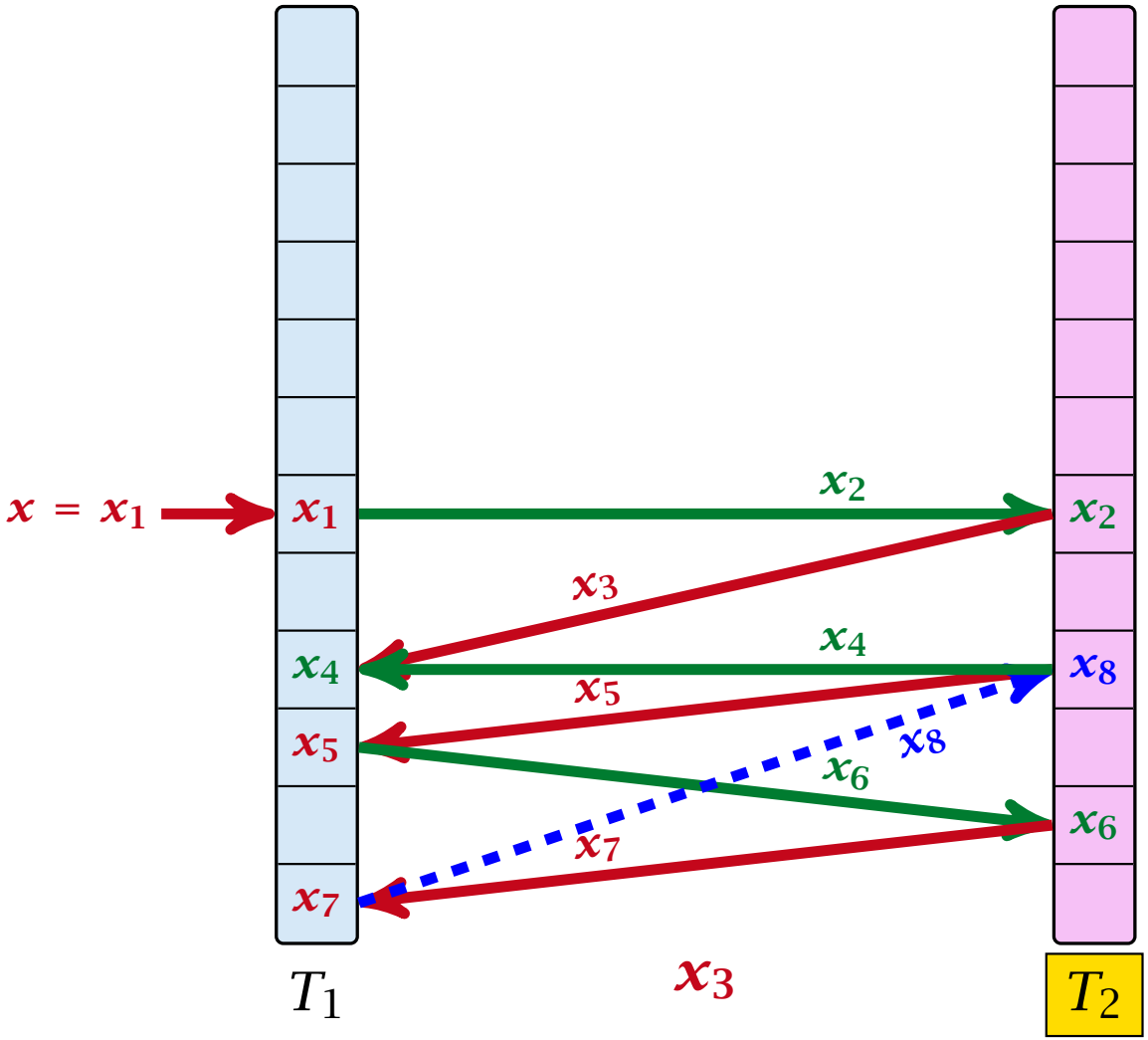

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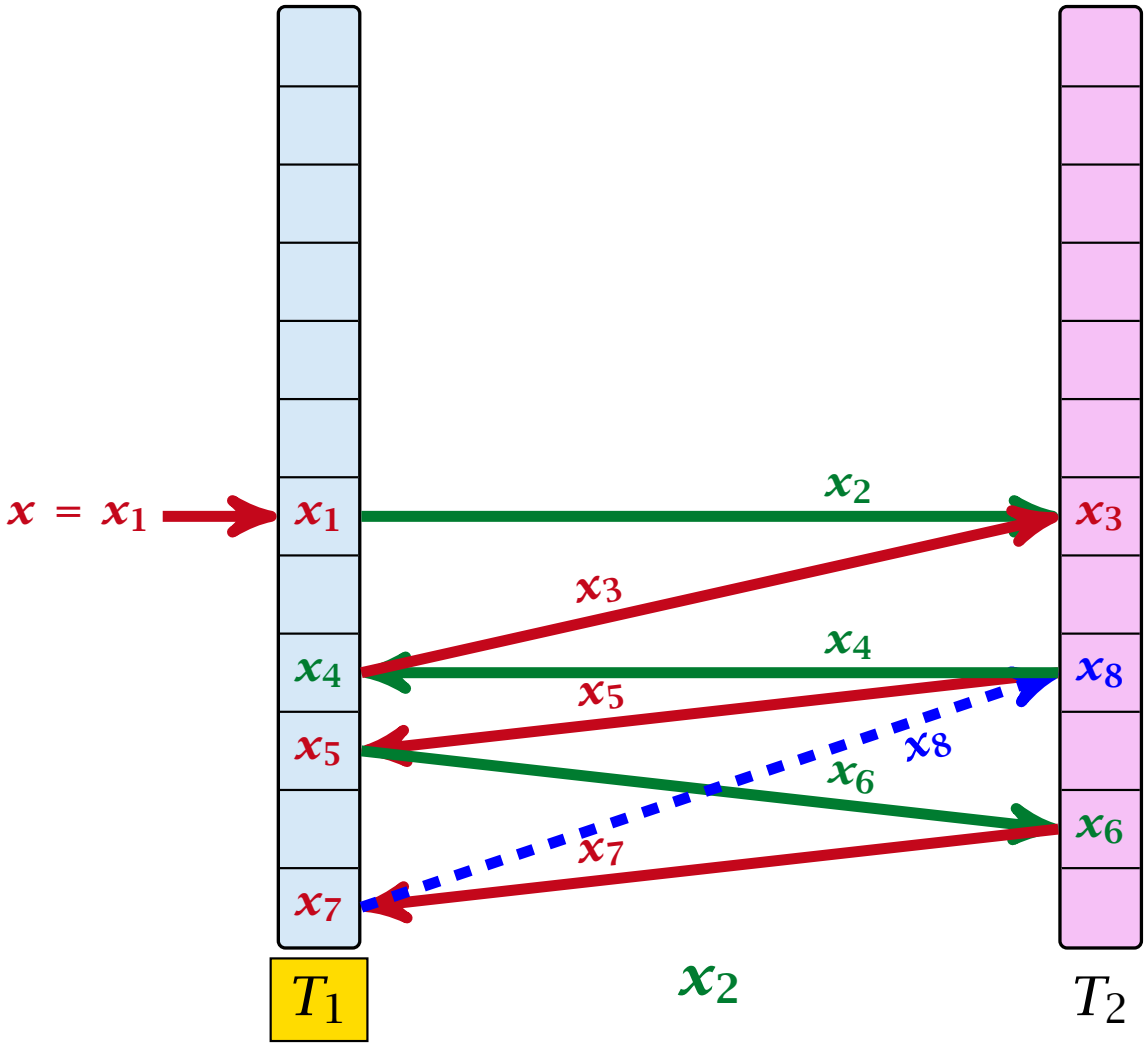
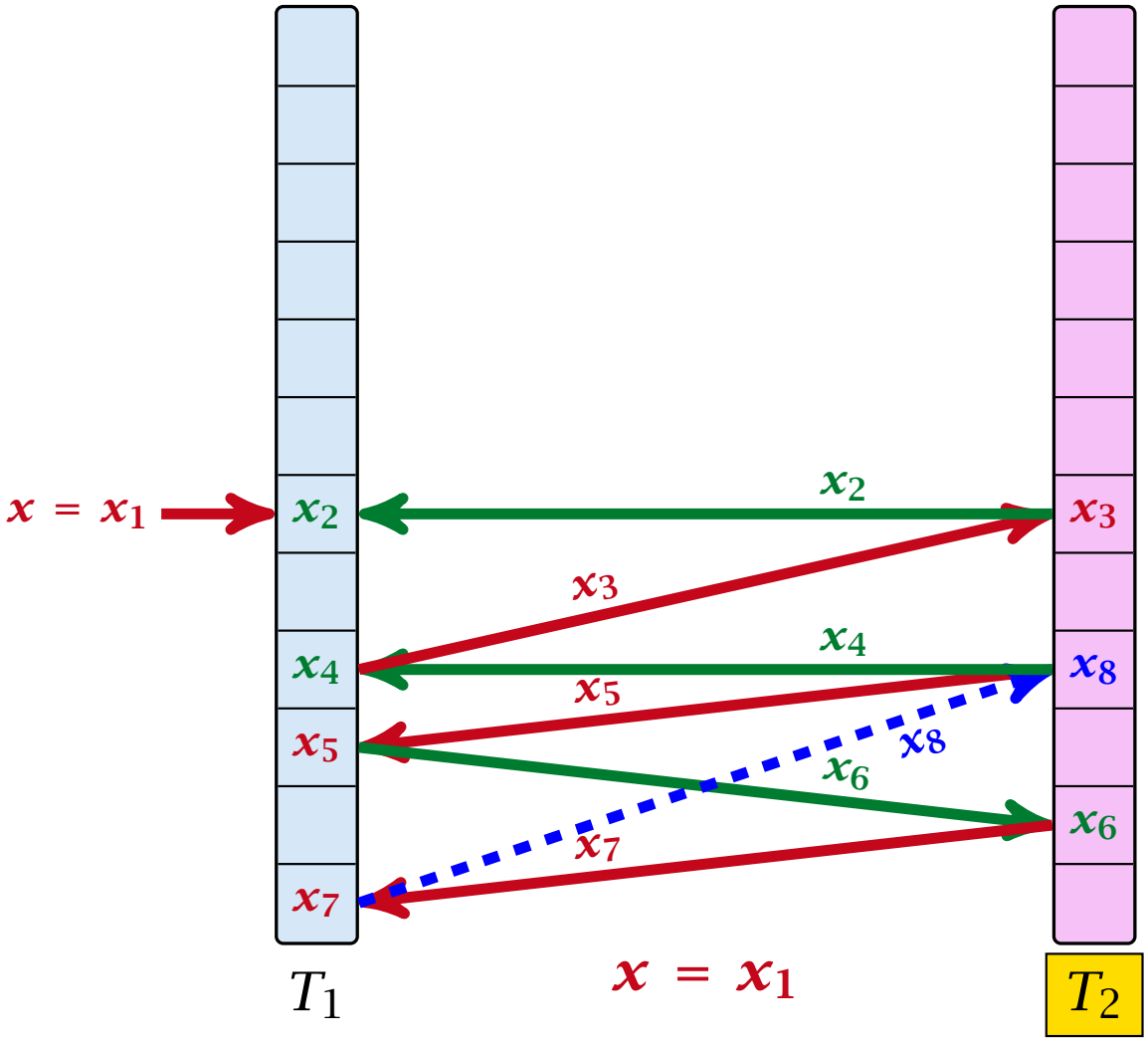


Cuckoo Hashing: Insert



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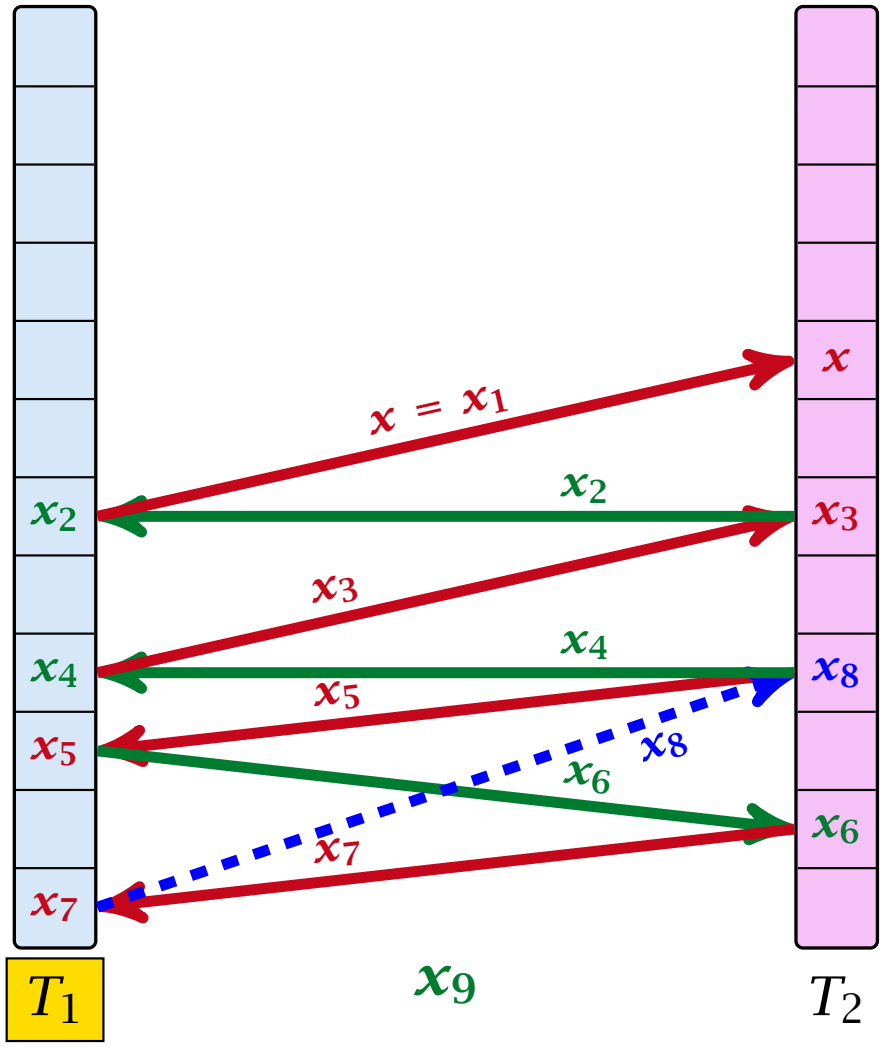

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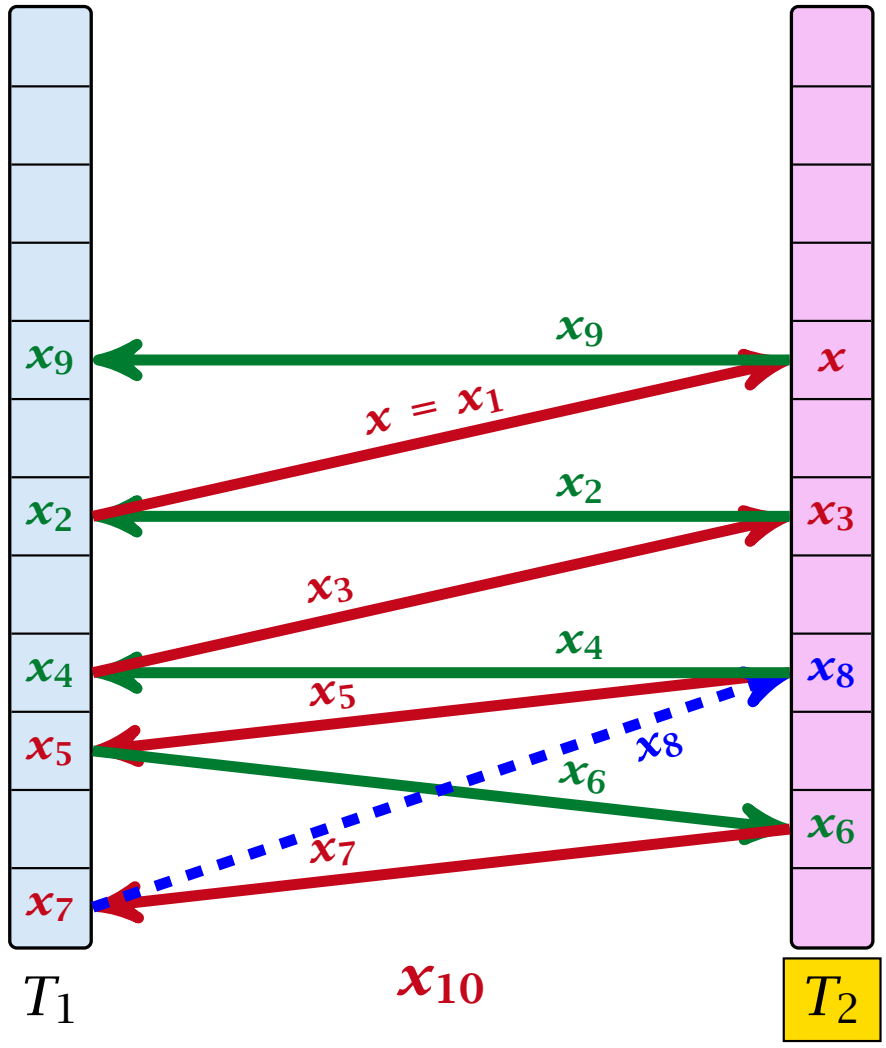

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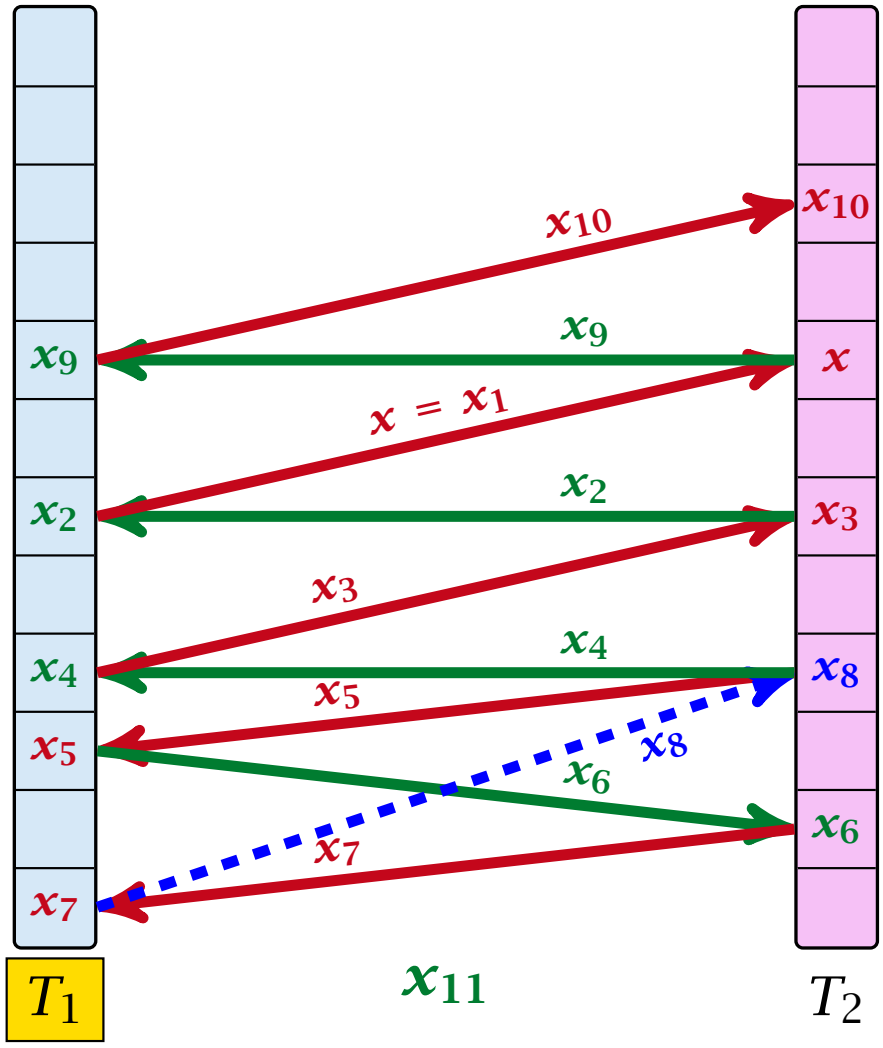

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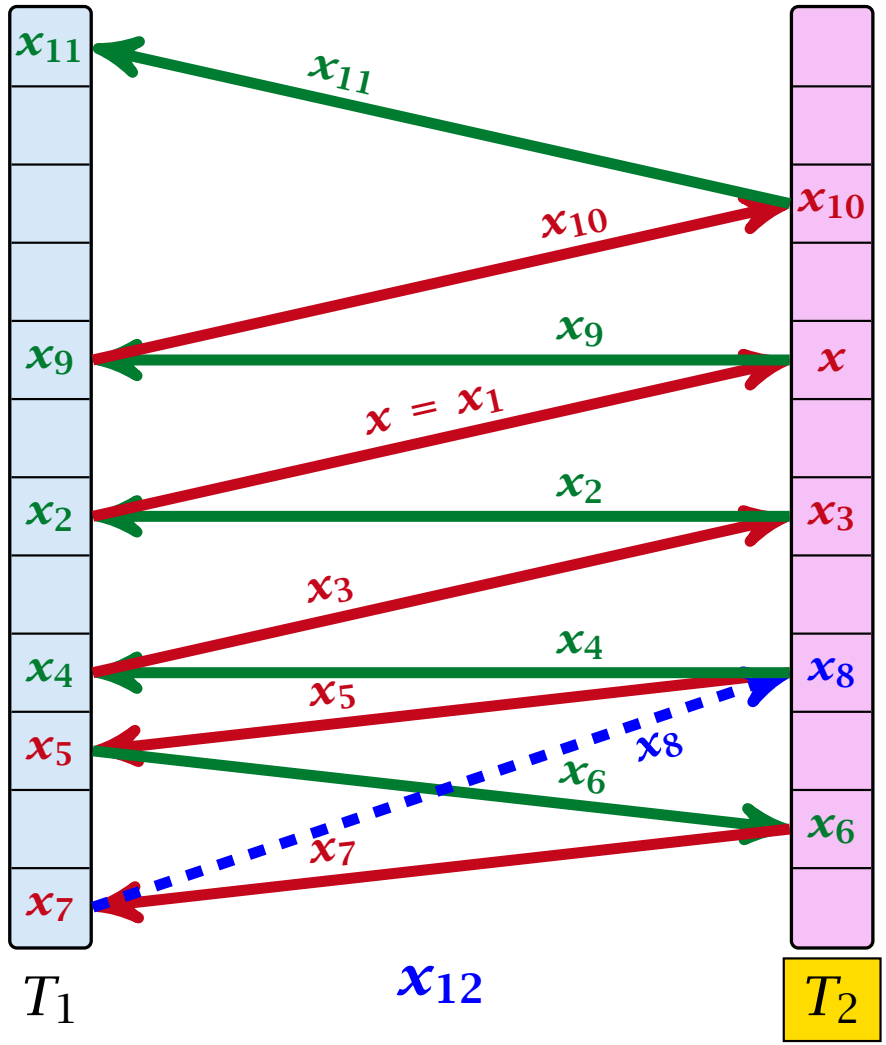

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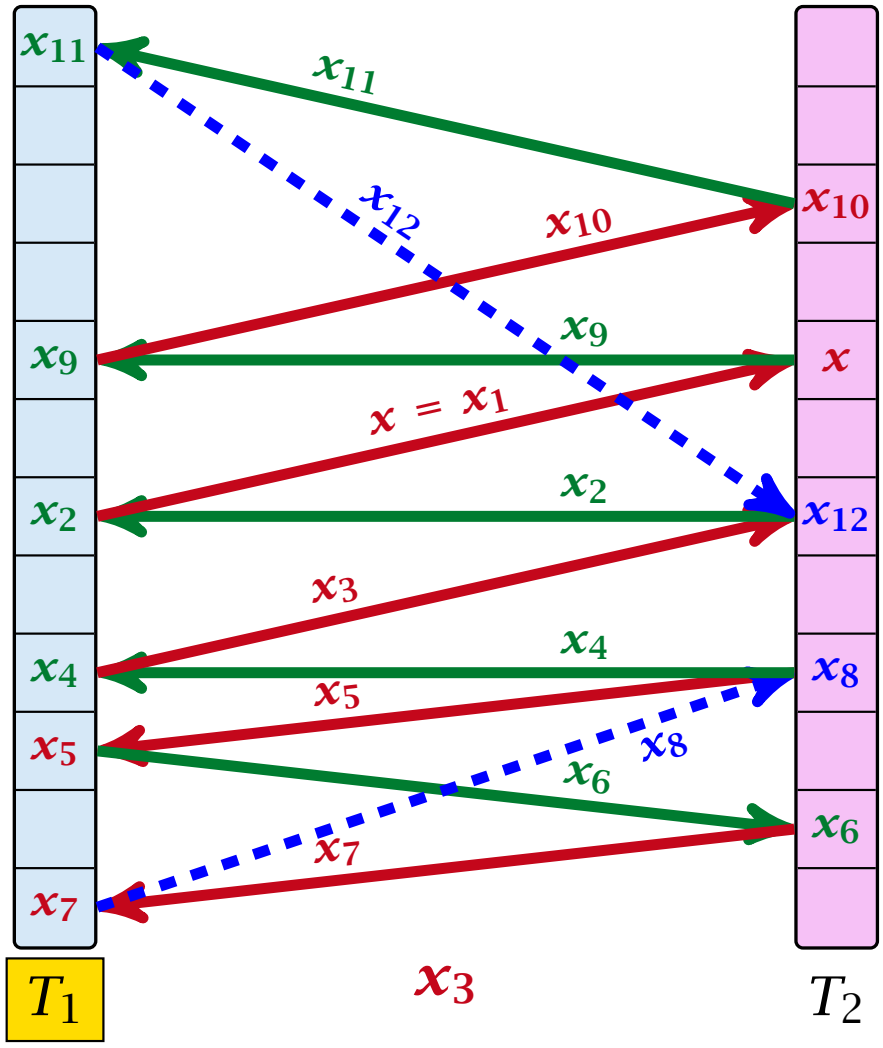
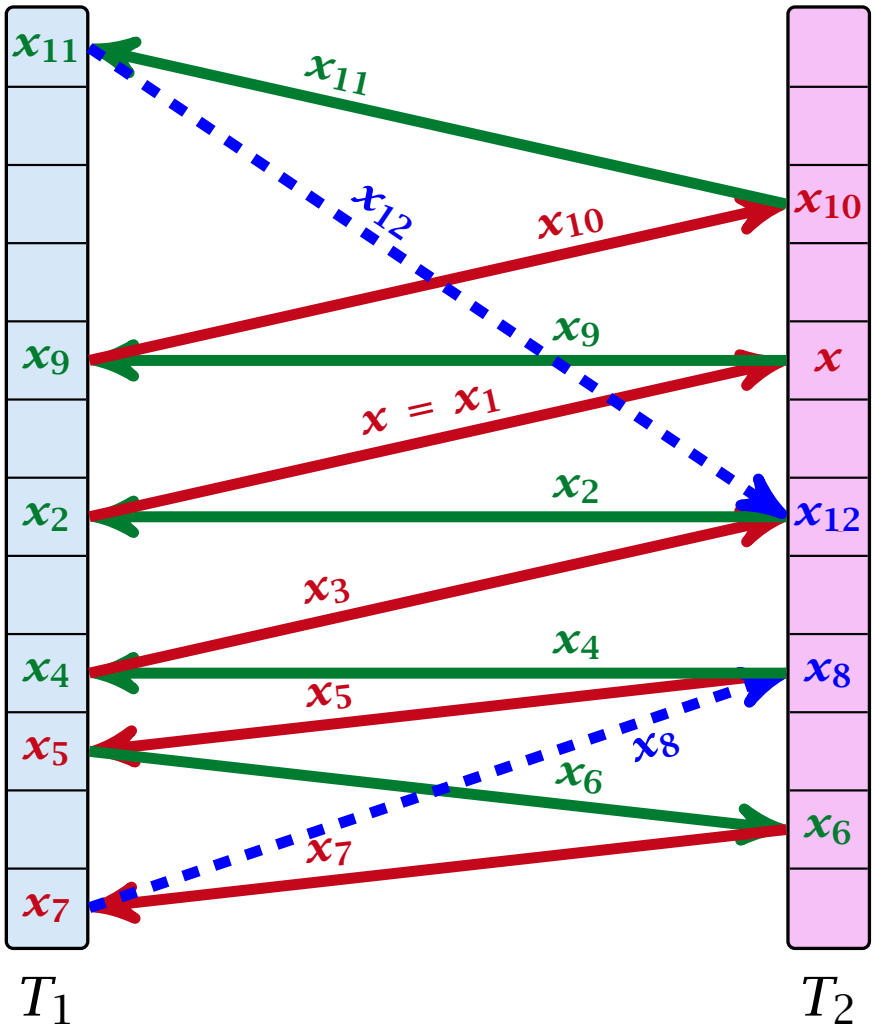


Cuckoo Hashing: Insert



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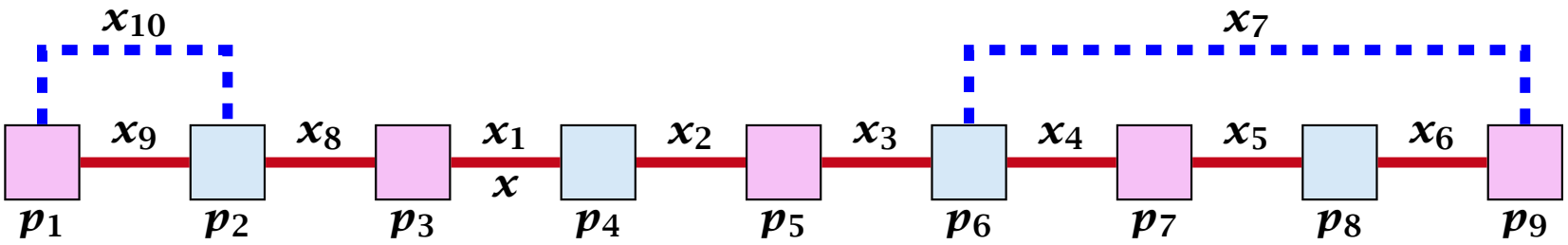
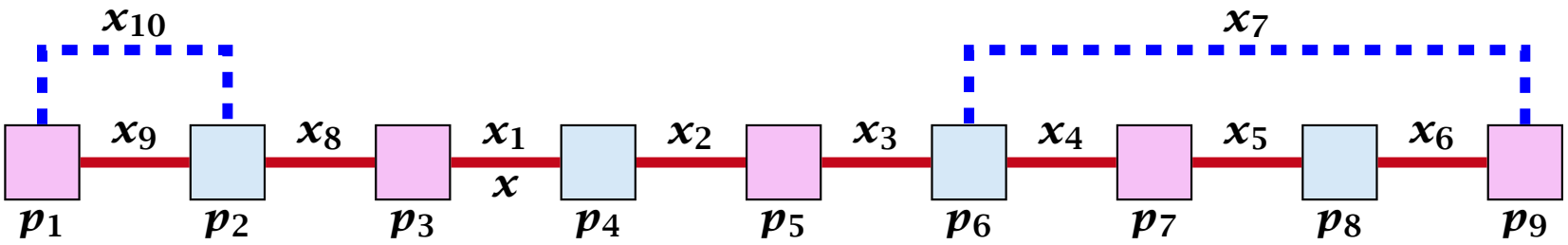


A cycle-structure of size s is defined by

- s different cells (alternating btw. cells from A and B).
- s distinct keys $x_1, \ldots, x_s$ linking the cells.
- The leftmost cell is "linked forward" to some cell on the right.
- The rightmost cell is "linked backward" to a cell on the left.
- One link represents key x; this is where the hash table starts.

Cuckoo Hashing



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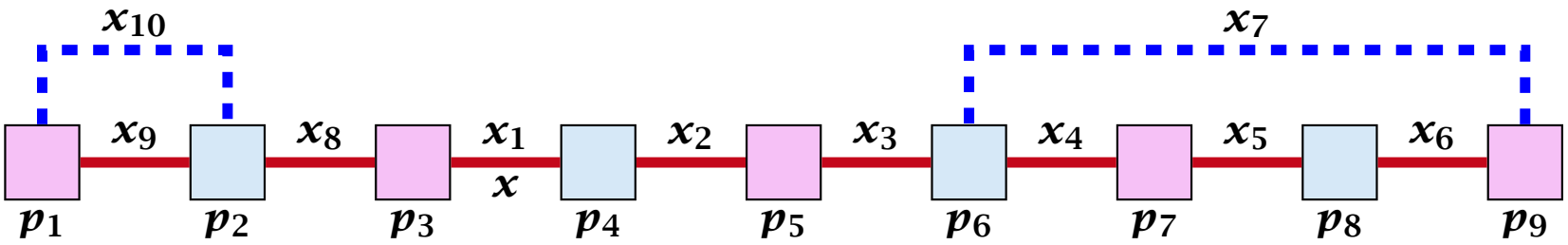

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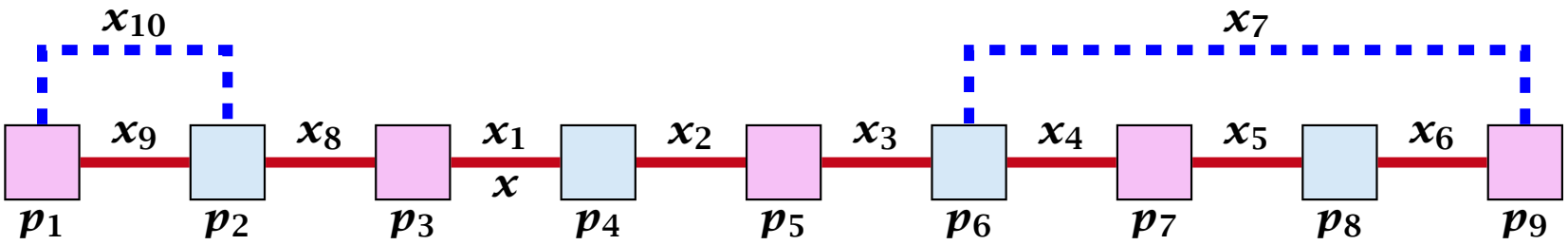

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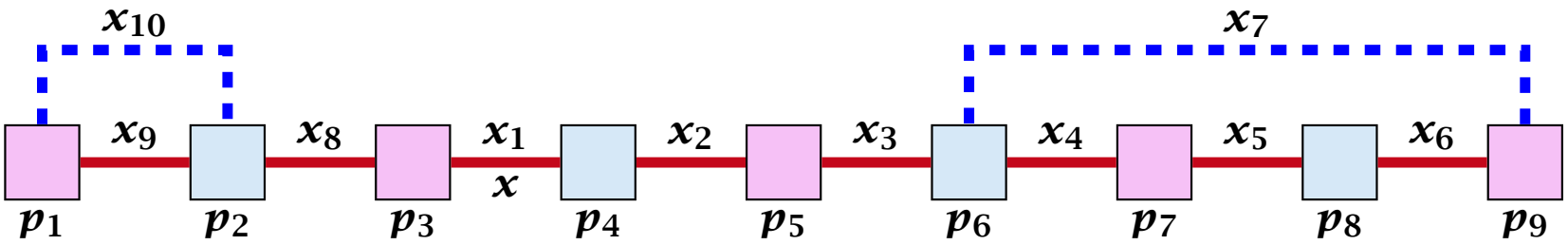

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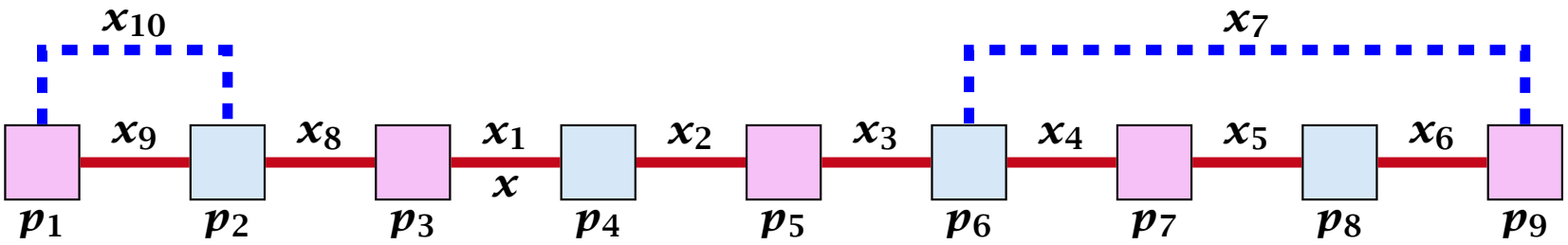

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A cycle-structure is active if for every key x_ℓ (linking a cell p_i from T_1 and a cell p_j from T_2) we have

$$h_1(x_\ell) = p_i \qquad \text{and} \qquad h_2(x_\ell) = p_j$$

Observation:
If during a phase the insert-procedure runs into a cycle there must exist an active cycle structure of size $s \geq 3$.

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Cuckoo Hashing

What is the probability that all keys in a cycle-structure of size s correctly map into their T_1-cell?

This probability is at most $\frac{\mu}{n^s}$ since h_1 is a (μ, s)-independent hash-function.

What is the probability that all keys in the cycle-structure of size s correctly map into their T_2-cell?

This probability is at most $\frac{\mu}{n^s}$ since h_2 is a (μ, s)-independent hash-function.

These events are independent.

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The probability that a given cycle-structure of size s is active is at most $\frac{\mu^2}{n^{2s}}$.

What is the probability that there exists an active cycle structure of size s?

Cuckoo Hashing

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What is the probability that there exists an active cycle structure of size s?

Cuckoo Hashing

The number of cycle-structures of size s is at most

$$s^3 \cdot n^{s-1} \cdot m^{s-1} \ .$$

- There are at most s^3 possibilities where to attach the forward and backward links.
- There are at most s possibilities to choose where to place key x_1.
- There are n^{s-1} possibilities to choose the keys apart from x_1.
- There are m^{s-1} possibilities to choose the cells.

Cuckoo Hashing

The number of cycle-structures of size s is at most

$$s^3 \cdot n^{s-1} \cdot m^{s-1} \ .$$

- There are at most s^2 possibilities where to attach the forward and backward links.
- There are at most s possibilities to choose where to place key x.
- There are m^{s-1} possibilities to choose the keys apart from x.
- There are n^{s-1} possibilities to choose the cells.

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- There are n^{s-1} possibilities to choose the cells.

Cuckoo Hashing

The probability that there exists an active cycle-structure is therefore at most

$$\sum_{s=3}^{\infty} s^3 \cdot n^{s-1} \cdot m^{s-1} \cdot \frac{\mu^2}{n^{2s}}$$

Cuckoo Hashing

The probability that there exists an active cycle-structure is therefore at most

$$\sum_{s=3}^{\infty} s^3 \cdot n^{s-1} \cdot m^{s-1} \cdot \frac{\mu^2}{n^{2s}} = \frac{\mu^2}{nm} \sum_{s=3}^{\infty} s^3 \left(\frac{m}{n}\right)^s$$

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Here we used the fact that $(1+\epsilon)m \leq n$.

Cuckoo Hashing

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Here we used the fact that $(1+\epsilon)m \leq n$.

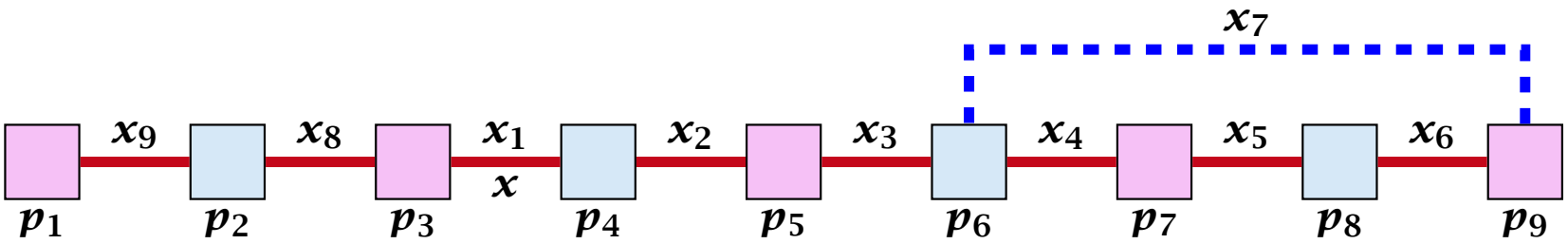
Hence,

$$\Pr[\text{cycle}] = \mathcal{O}\left(\frac{1}{m^2}\right) \ .$$

Cuckoo Hashing

Now, we analyze the probability that a phase is not successful without running into a closed cycle.

Cuckoo Hashing



Sequence of visited keys:

$x = x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_3, x_2, x_1 = x, x_8, x_9, \ldots$

Cuckoo Hashing

Consider the sequence of not necessarily distinct keys starting with x in the order that they are visited during the phase.

Lemma 32

If the sequence is of length p then there exists a sub-sequence of at least $\frac{p+2}{3}$ keys starting with x of distinct keys.

Cuckoo Hashing

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Lemma 32

If the sequence is of length p then there exists a sub-sequence of at least $\frac{p+2}{3}$ keys starting with x of distinct keys.

Cuckoo Hashing

Proof.

Let i be the number of keys (including x) that we see before the first repeated key. Let j denote the total number of distinct keys.

The sequence is of the form:

$$x = x_1 \to x_2 \to \cdots \to x_i \to x_r \to x_{r-1} \to \cdots \to x_1 \to x_{i+1} \to \cdots \to x_j$$

As $r \leq i-1$ the length p of the sequence is

$$p = i + r + (j-i) \leq i + j - 1 \ .$$

Either sub-sequence $x_1 \to x_2 \to \cdots \to x_i$ or sub-sequence $x_1 \to x_{i+1} \to \cdots \to x_j$ has at least $\frac{p+2}{3}$ elements. □

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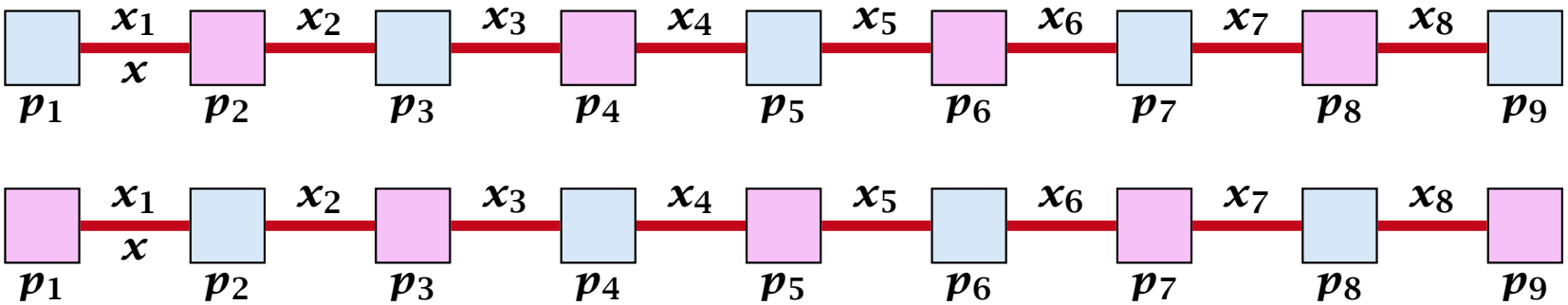
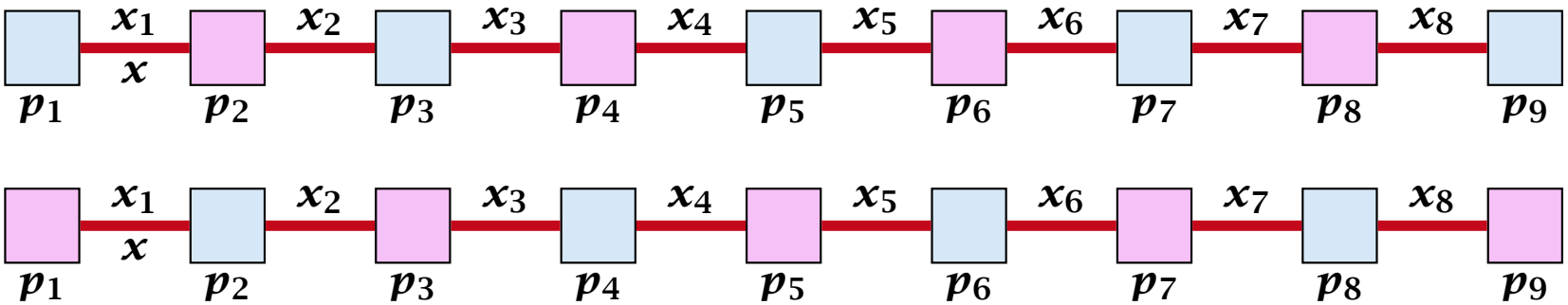
Cuckoo Hashing



A path-structure of size s is defined by

- $s+1$ different cells (alternating between cells from T_1 and T_2)
- s distinct keys
- s edges, each linking two cells
- The leftmost cell is either from T_1 or T_2

Cuckoo Hashing



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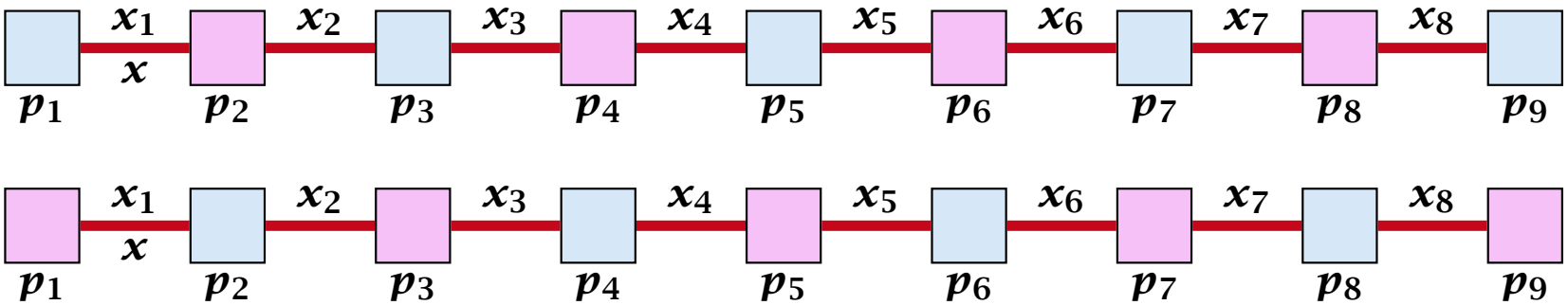

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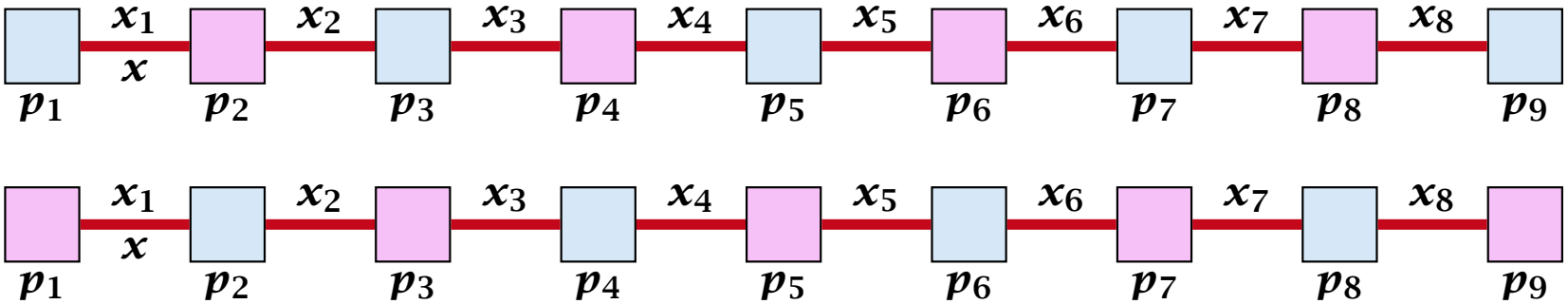

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A path-structure is active if for every key x_ℓ (linking a cell p_i from T_1 and a cell p_j from T_2) we have

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Observation:
If a phase takes at least t steps without running into a cycle there must exist an active path-structure of size $(2t+2)/3$.

Cuckoo Hashing

The probability that a given path-structure of size s is active is at most $\frac{\mu^2}{n^{2s}}$.

Cuckoo Hashing

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The probability that there exists an active path-structure of size s is at most

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$$\leq 2\mu^2 \left(\frac{1}{1+\epsilon}\right)^{(2t+2)/3-1}$$

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$$
\begin{aligned}
& 2 \cdot n^{s+1} \cdot m^{s-1} \cdot \frac{\mu^2}{n^{2s}} \\
& \qquad \leq 2\mu^2 \left(\frac{m}{n}\right)^{s-1} \leq 2\mu^2 \left(\frac{1}{1+\epsilon}\right)^{s-1} \\
& \qquad \leq 2\mu^2 \left(\frac{1}{1+\epsilon}\right)^{(2t+2)/3-1} = 2\mu^2 \left(\frac{1}{1+\epsilon}\right)^{(2t-1)/3} .
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This gives $\text{maxsteps} = \Theta(\log m)$.

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$$\Pr[\text{cycle}] \leq \mathcal{O}\Big(\frac{1}{m^2}\Big)$$

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This means the expected cost for a successful phase is constant (even after accounting for the cost of the incomplete step that finishes the phase).

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A phase that is not successful induces cost for doing a complete rehash (this dominates the cost for the steps in the phase).

The probability that a phase is not successful is $p = \mathcal{O}(1/m^2)$ (probability $\mathcal{O}(1/m^2)$ of running into a cycle and probability $\mathcal{O}(1/m^2)$ of reaching $\mathrm{maxsteps}$ without running into a cycle).

A rehash try requires m insertions and takes expected constant time per insertion. It fails with probability $p := \mathcal{O}(1/m)$.

The expected number of unsuccessful rehashes is $\sum_{i \geq 1} p^i = \frac{1}{1-p} - 1 = \frac{p}{1-p} = \mathcal{O}(p)$.

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The expected cost for all rehashes is

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Note that Z_i is independent of X_j^s, $j \geq i$ (however, it is not independent of X_j^s, $j < i$). Hence,

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What kind of hash-functions do we need?

Since maxsteps is $\Theta(\log m)$ the largest size of a path-structure or cycle-structure contains just $\Theta(\log m)$ different keys.

Therefore, it is sufficient to have $(\mu, \Theta(\log m))$-independent hash-functions.

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How do we make sure that $n \geq (1+\epsilon)m$?

- Let $\alpha := 1/(1+\epsilon)$.
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8 Priority Queues

A Priority Queue S is a dynamic set data structure that supports the following operations:

- $S.\mathbf{build}(x_1, \ldots, x_n)$: Creates a data-structure that contains just the elements $x_1, \ldots, x_n$.
- $S.\mathbf{insert}(x)$: Adds element x to the data-structure.
- $\mathbf{element}\ S.\mathbf{minimum}()$: Returns an element $x \in S$ with minimum key-value $\mathrm{key}[x]$.
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An addressable Priority Queue also supports:

- **handle S.insert(x)**: Adds element x to the data-structure, and returns a handle to the object for future reference.
- **S.delete(h)**: Deletes element specified through handle h.
- **S.decrease-key(h, k)**: Decreases the key of the element specified by handle h to k. Assumes that the key is at least k before the operation.

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Dijkstra's Shortest Path Algorithm

Algorithm 18 Shortest-Path$(G = (V, E, d), s \in V)$

```
 1: Input: weighted graph G = (V,E,d); start vertex s;
 2: Output: key-field of every node contains distance from s;
 3: S.build(); // build empty priority queue
 4: for all v ∈ V \ {s} do
 5:         v.key ← ∞;
 6:         h_v ← S.insert(v);
 7: s.key ← 0; S.insert(s);
 8: while S.is-empty() = false do
 9:         v ← S.delete-min();
10:         for all x ∈ V s.t. (v,x) ∈ E do
11:             if x.key > v.key + d(v,x) then
12:                 S.decrease-key(h_x, v.key + d(v,x));
13:                 x.key ← v.key + d(v,x);
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Prim's Minimum Spanning Tree Algorithm

Algorithm 19 Prim-MST$(G = (V, E, d), s \in V)$

```
 1: Input: weighted graph G = (V,E,d); start vertex s;
 2: Output: pred-fields encode MST;
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13:                 x.key ← d(v,x);
14:                 x.pred ← v;
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Analysis of Dijkstra and Prim

Both algorithms require:

- 1 build() operation
- $|V|$ insert() operations
- $|V|$ delete-min() operations
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How good a running time can we obtain?

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8 Priority Queues

Operation	*Binary Heap*	*BST*	*Binomial Heap*	*Fibonacci Heap*[*]
build	n	$n \log n$	$n \log n$	n
minimum	1	$\log n$	$\log n$	1
is-empty	1	1	1	1
insert	$\log n$	$\log n$	$\log n$	1
delete	$\log n$[**]	$\log n$	$\log n$	$\log n$
delete-min	$\log n$	$\log n$	$\log n$	$\log n$
decrease-key	$\log n$	$\log n$	$\log n$	1
merge	n	$n \log n$	$\log n$	1

Note that most applications use **build**() only to create an empty heap which then costs time 1.

The standard version of binary heaps is not addressable, and hence does not support a delete operation.

Fibonacci heaps only give an amortized guarantee.

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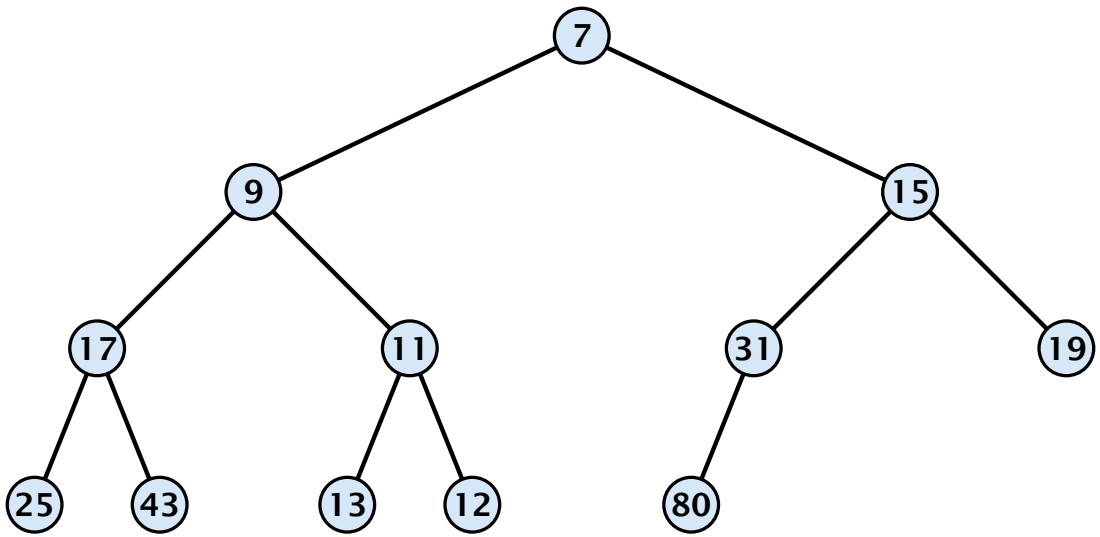
8 Priority Queues

Using Binary Heaps, Prim and Dijkstra run in time $\mathcal{O}((|V| + |E|) \log |V|)$.

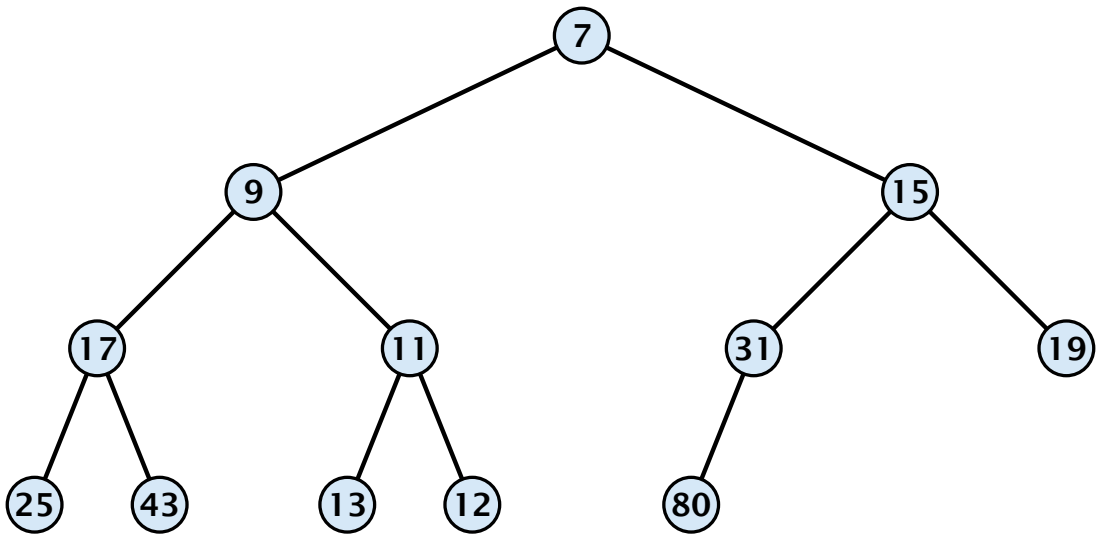
Using Fibonacci Heaps, Prim and Dijkstra run in time $\mathcal{O}(|V| \log |V| + |E|)$.

8.1 Binary Heaps



8.1 Binary Heaps

- Nearly complete binary tree; only the last level is not full, and this one is filled from left to right.



8.1 Binary Heaps

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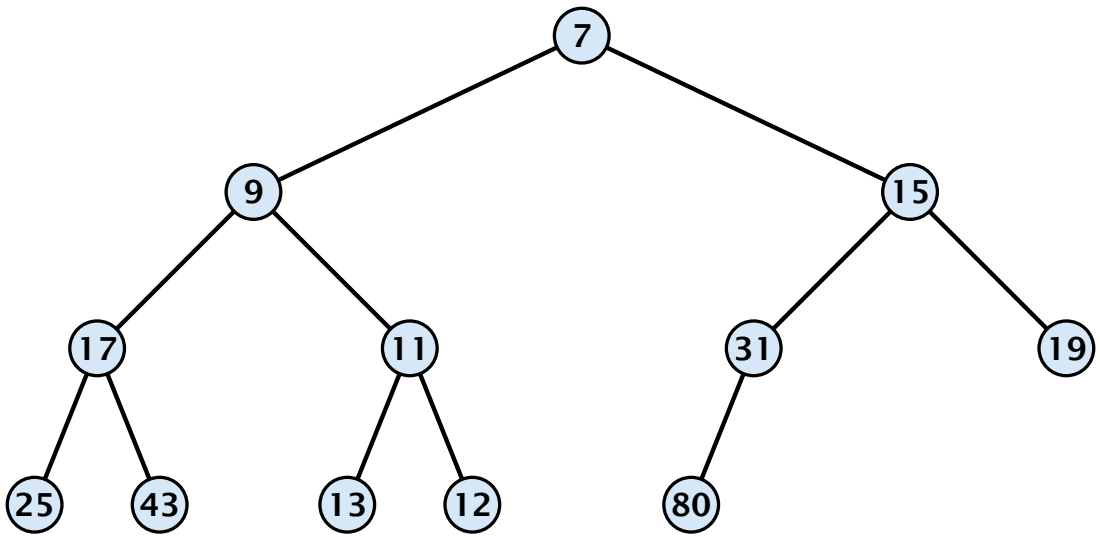
- Heap property: A node's key is not larger than the key of one of its children.



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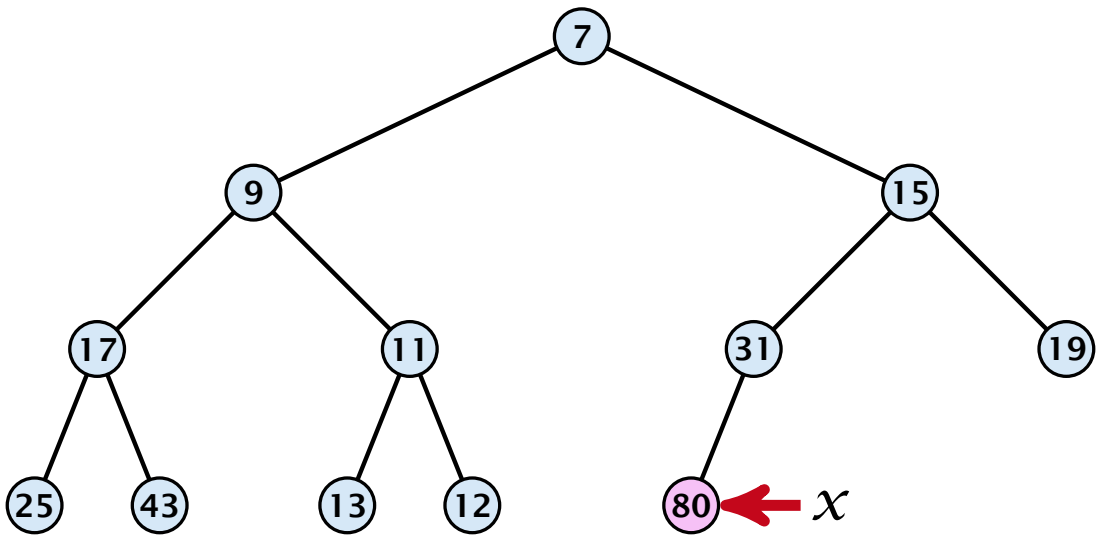
Maintain a pointer to the last element x.

- We can compute the predecessor of x (last element when x is deleted) in time $\mathcal{O}(\log n)$.
  - go up until the last edge used was a right edge.
  - go left, go right until you reach a leaf
  - if you hit the root on the way up, go to the rightmost element



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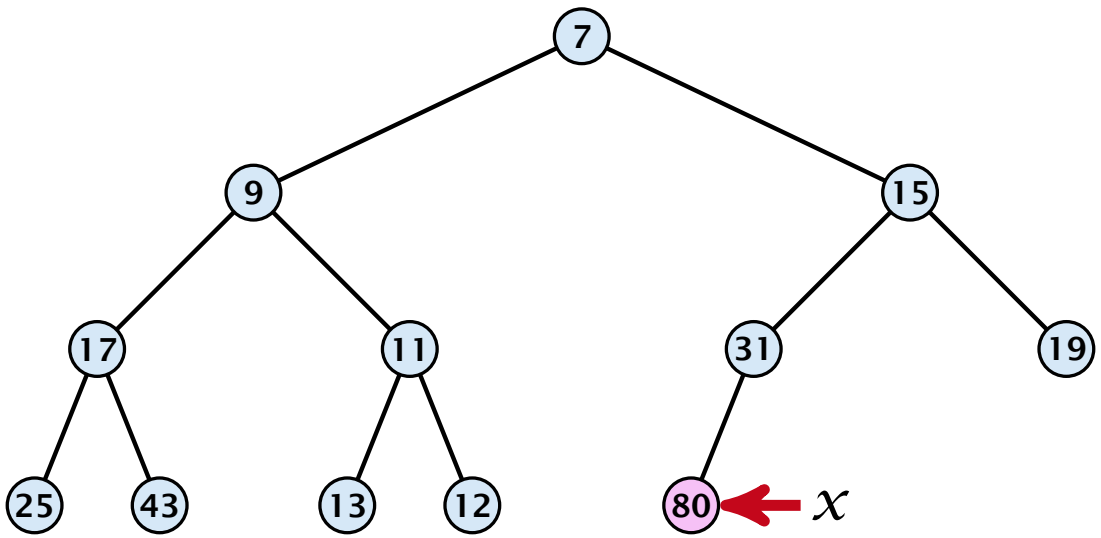

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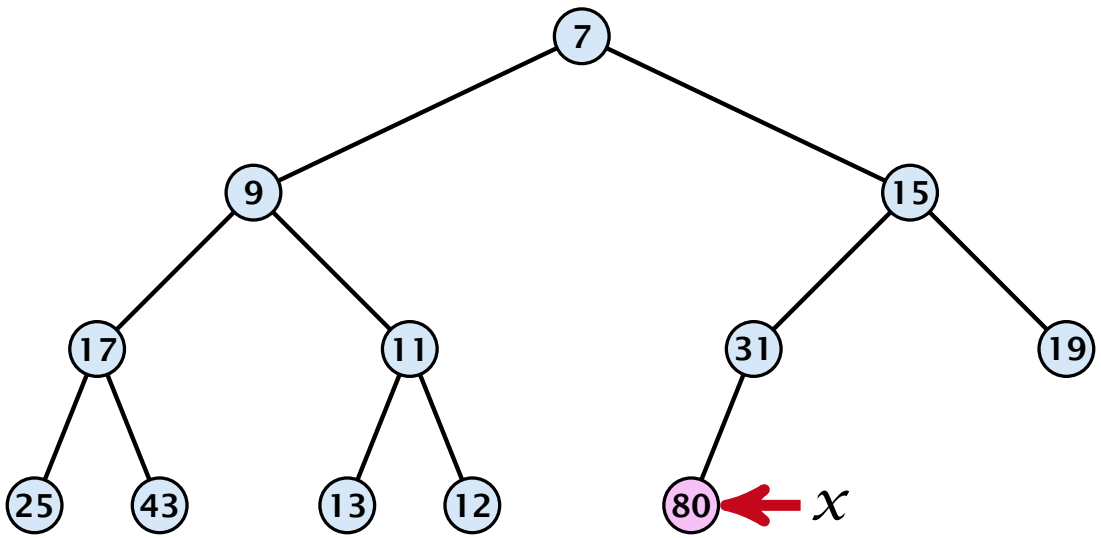

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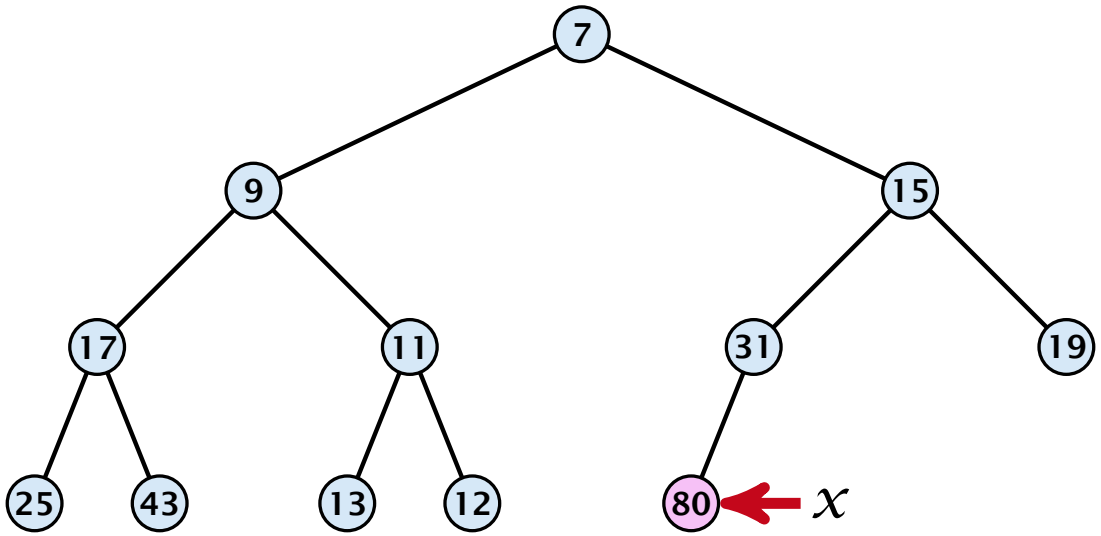

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Maintain a pointer to the last element x.

- We can compute the successor of x (last element when an element is inserted) in time $\Theta(\log n)$.
  - go up until the last edge used was a left edge
  - go right, go left until you reach a node pointer.
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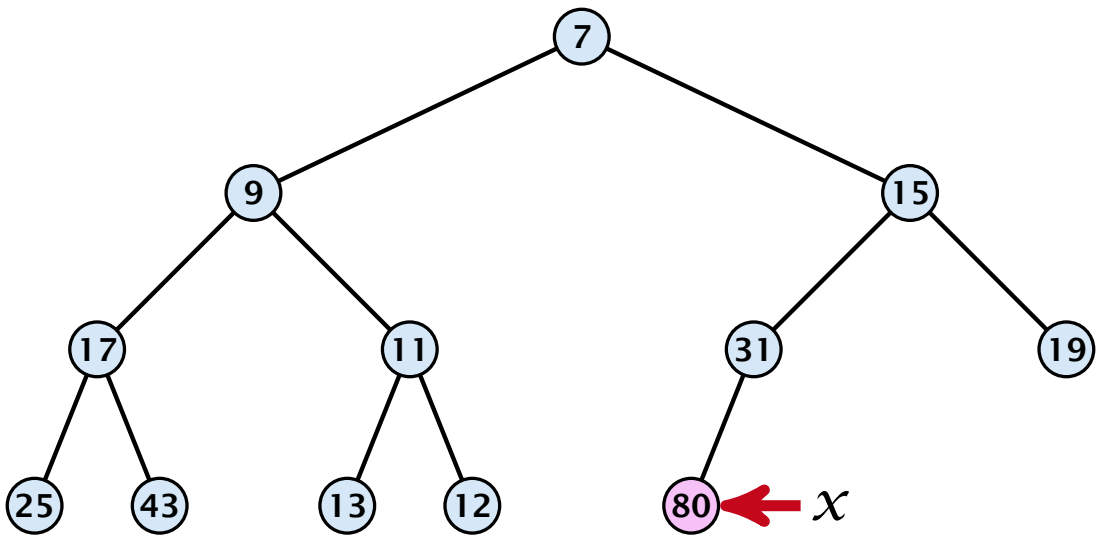

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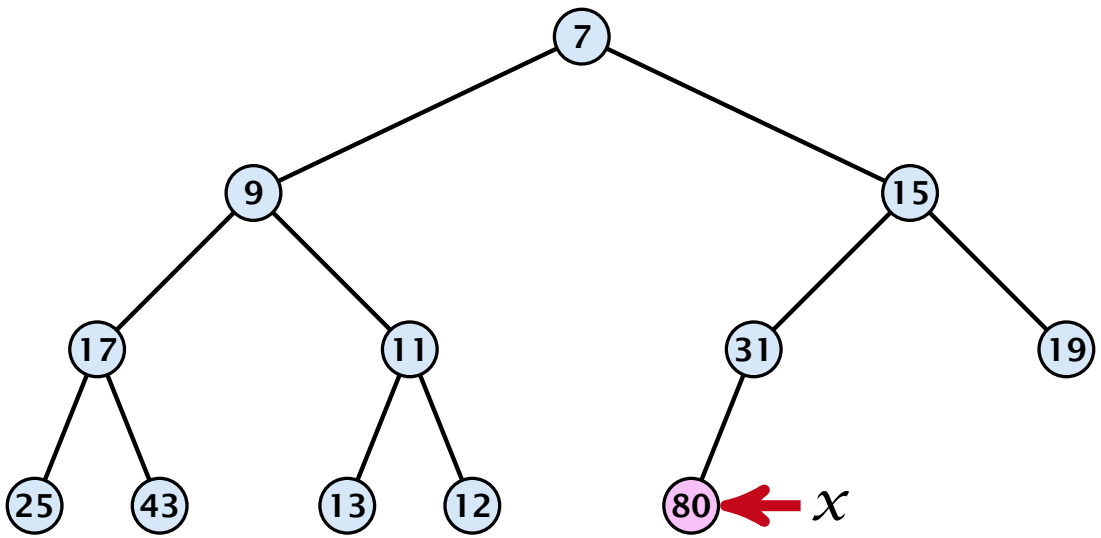

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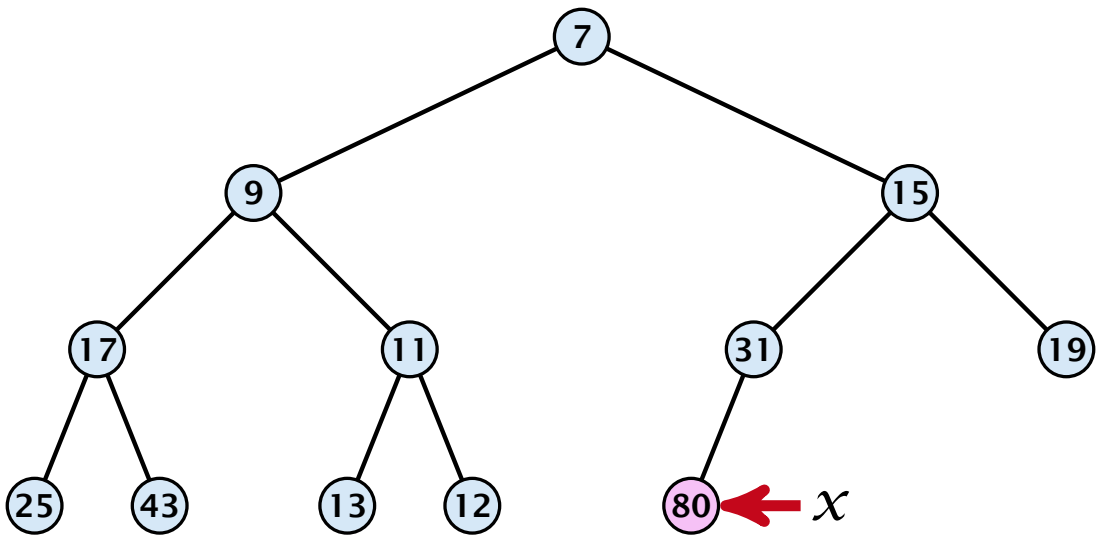

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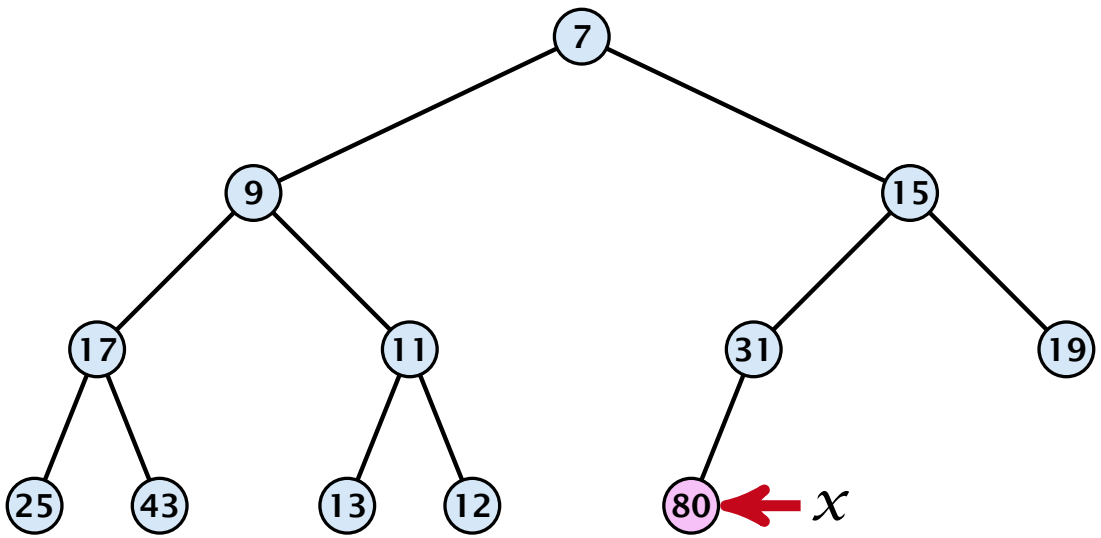

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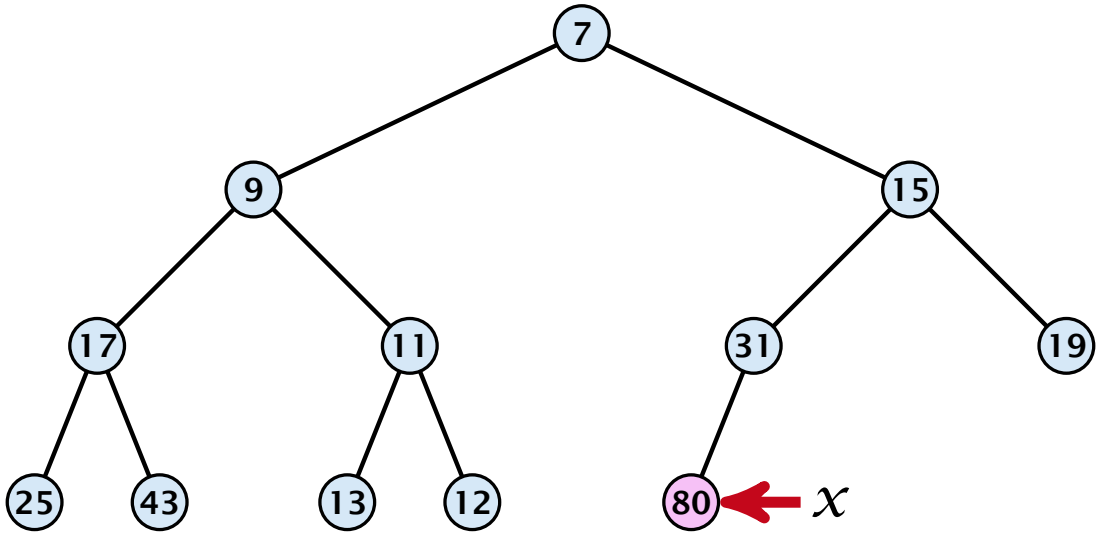

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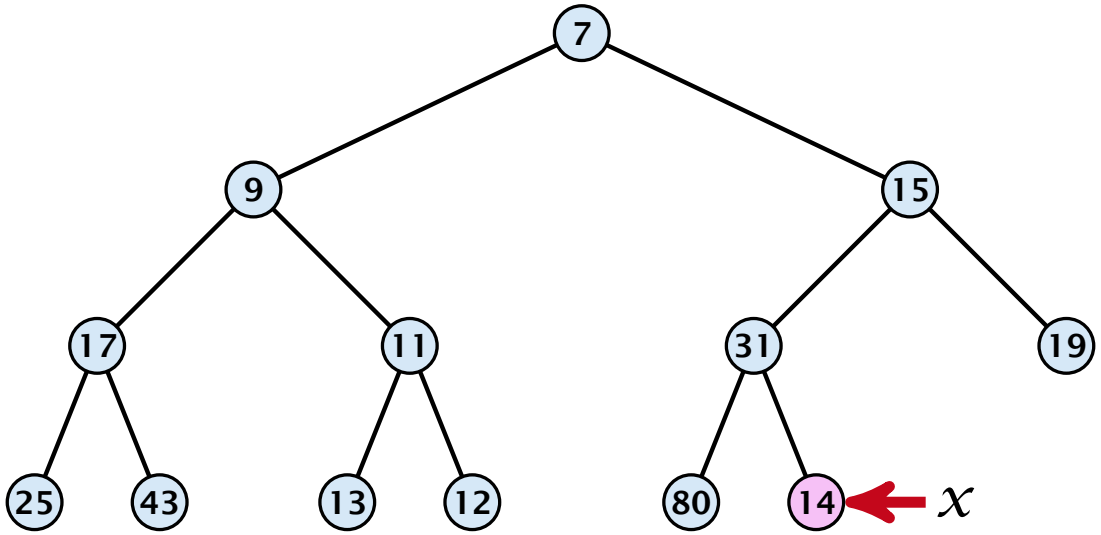

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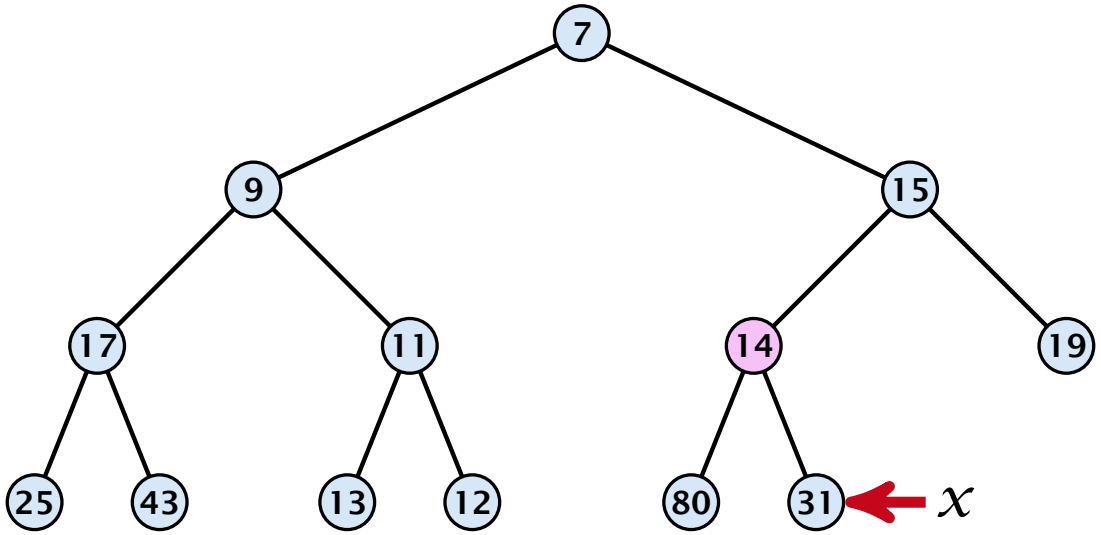

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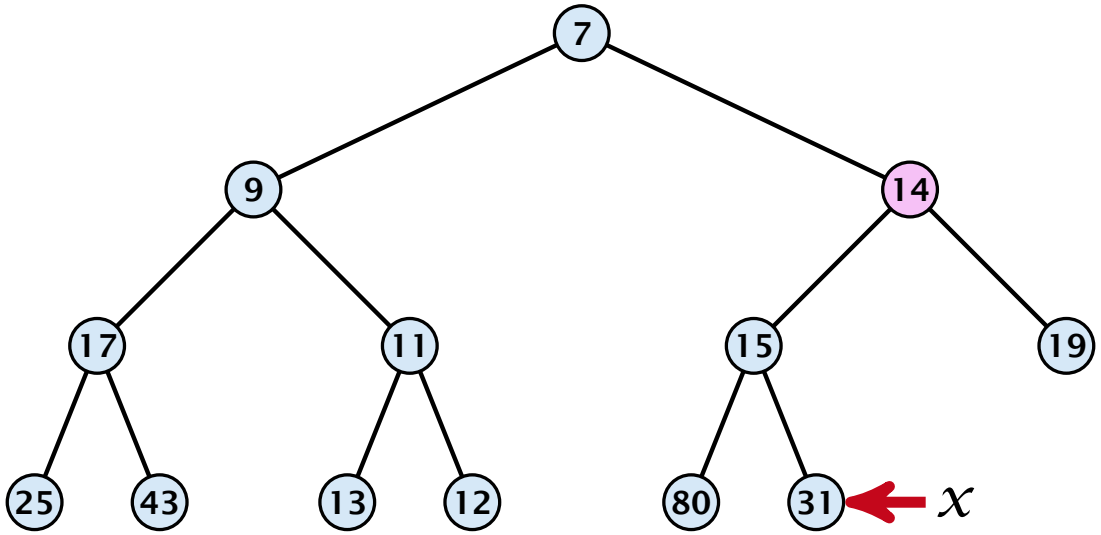

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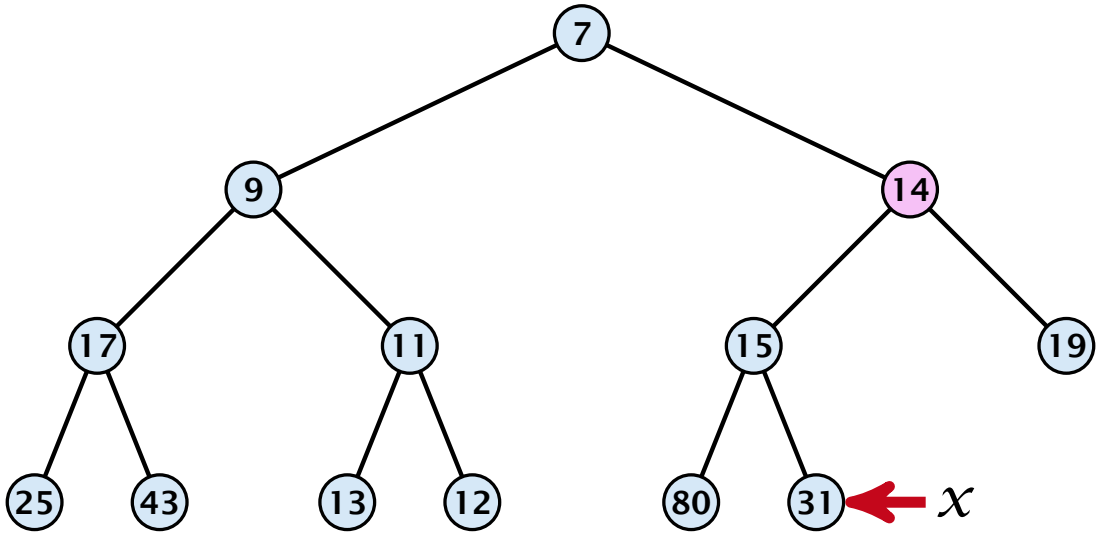

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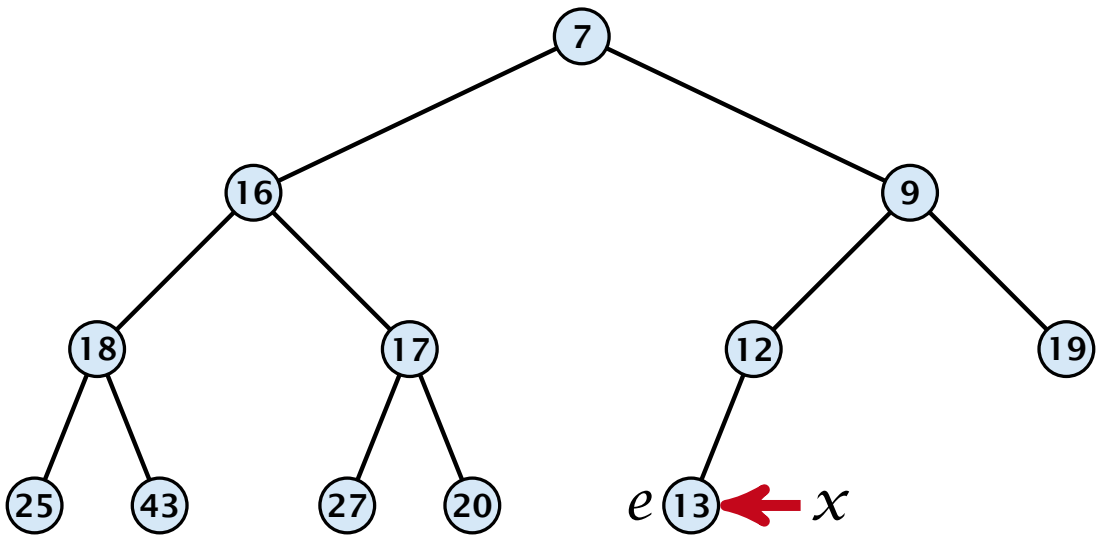

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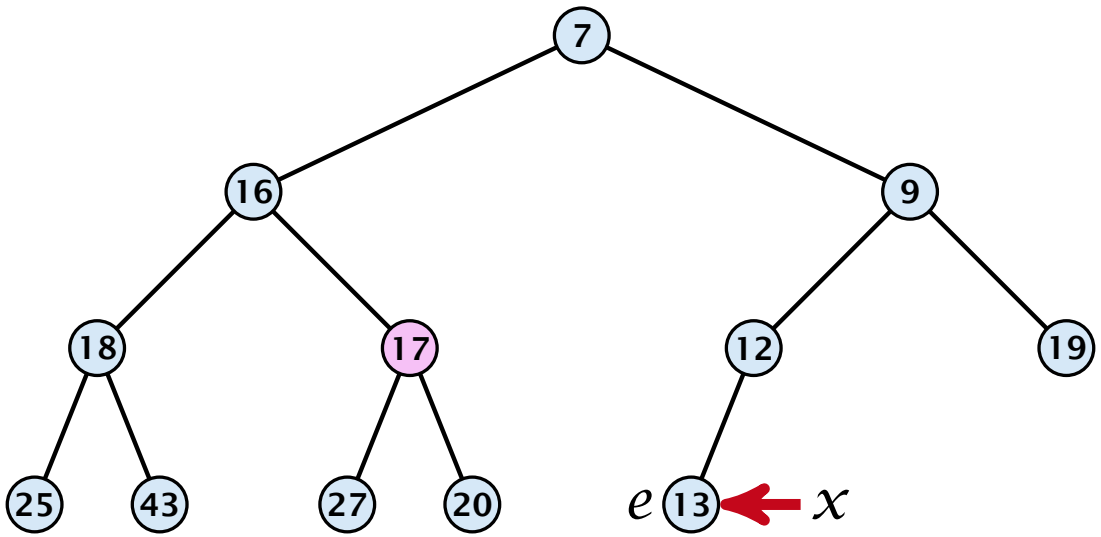

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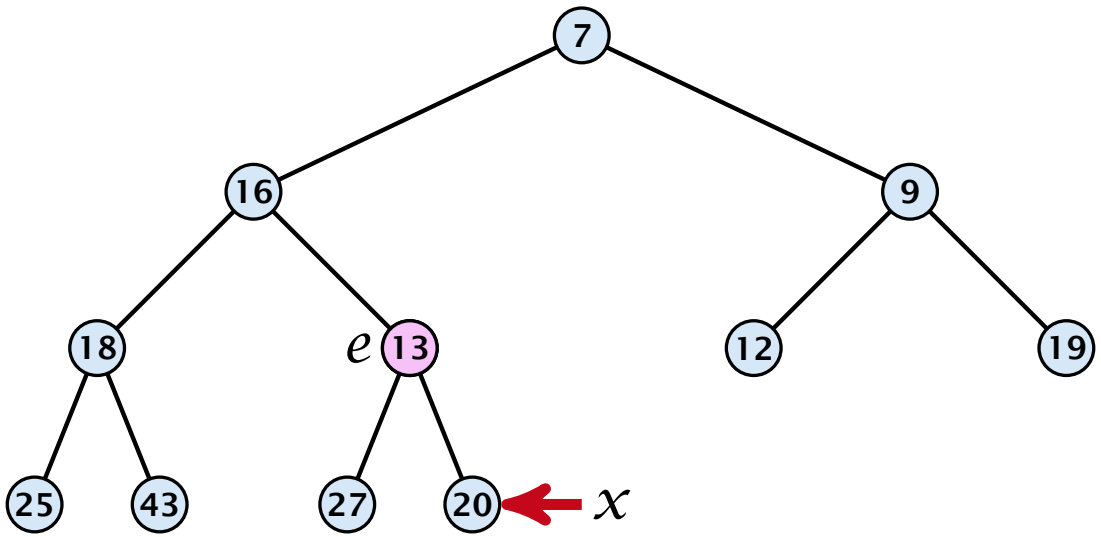

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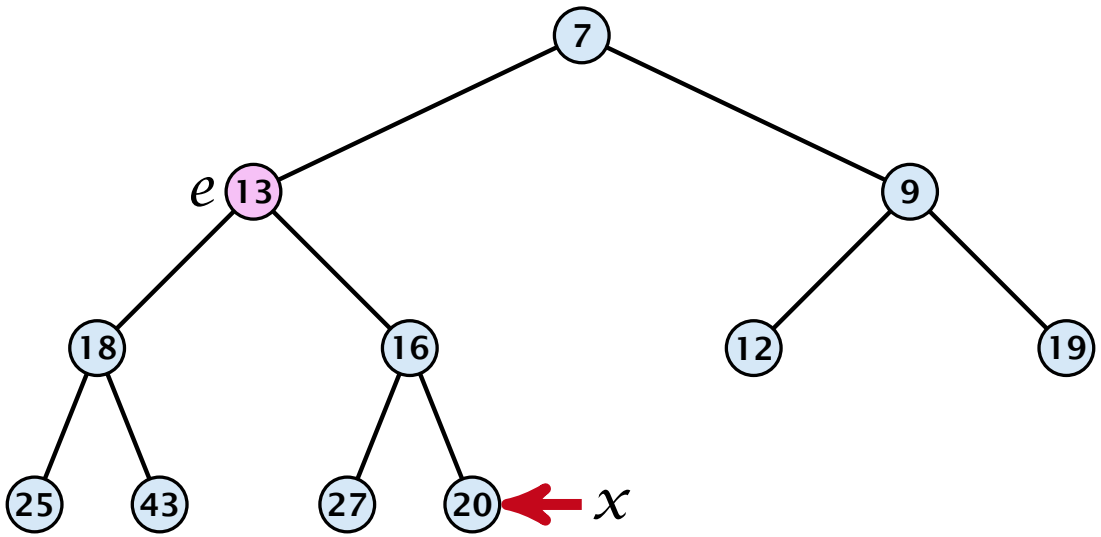

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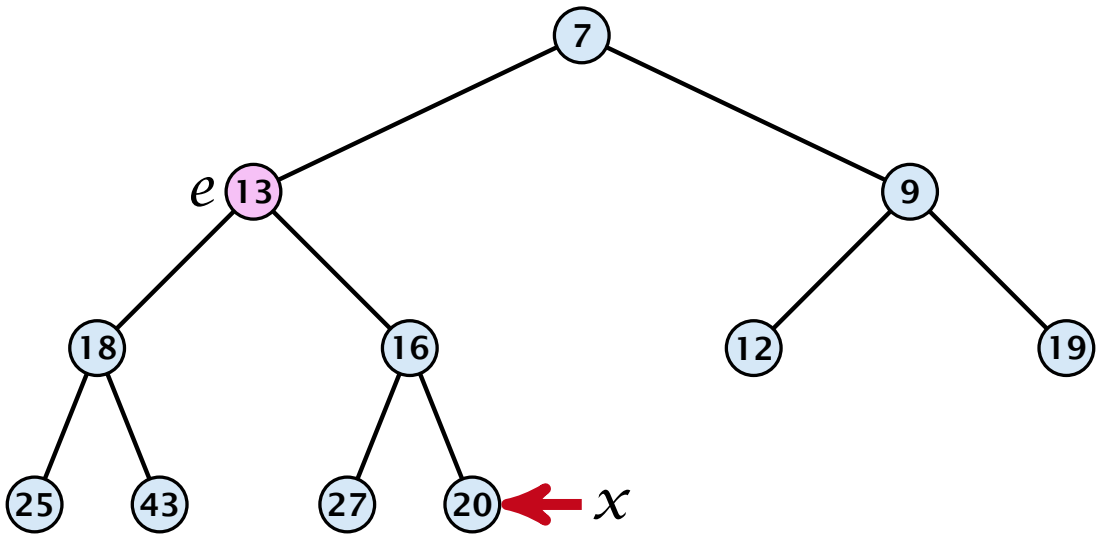

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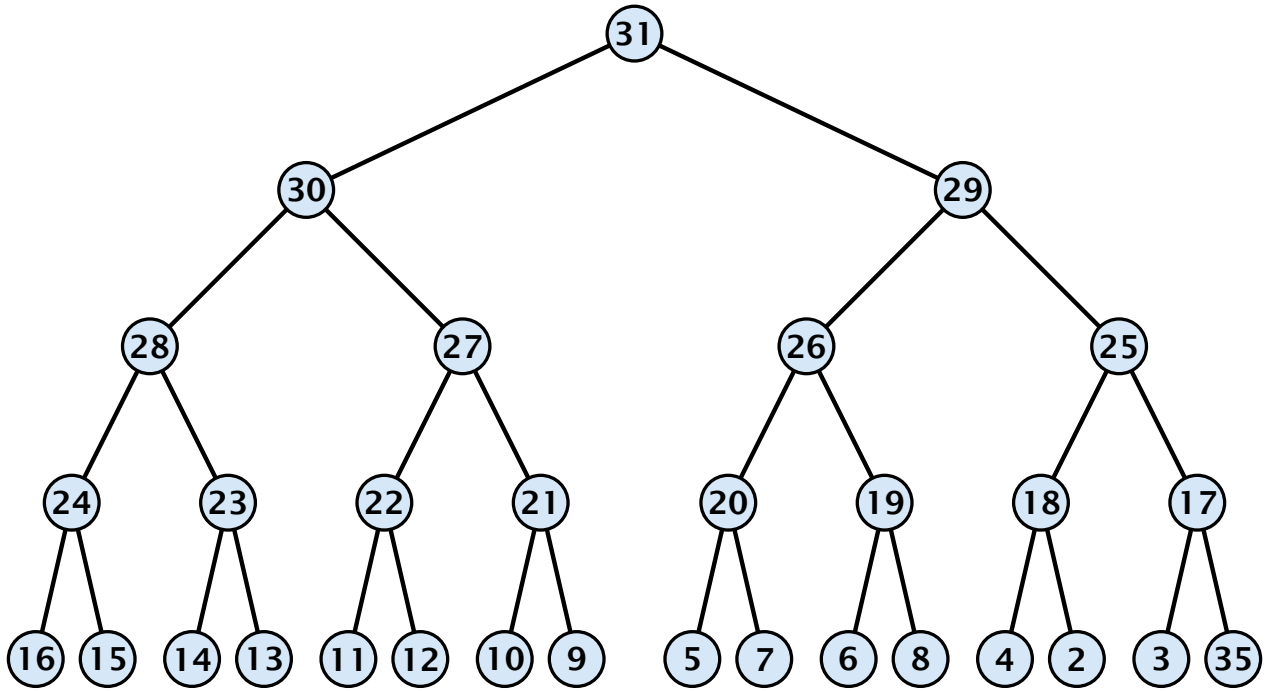
Binary Heaps

Operations:

- **minimum()**: return the root-element. Time $\mathcal{O}(1)$.
- **is-empty()**: check whether root-pointer is null. Time $\mathcal{O}(1)$.
- **insert(k)**: insert at x and bubble up. Time $\mathcal{O}(\log n)$.
- **delete(h)**: swap with x and bubble up or sift-down. Time $\mathcal{O}(\log n)$.

Build Heap

We can build a heap in linear time:



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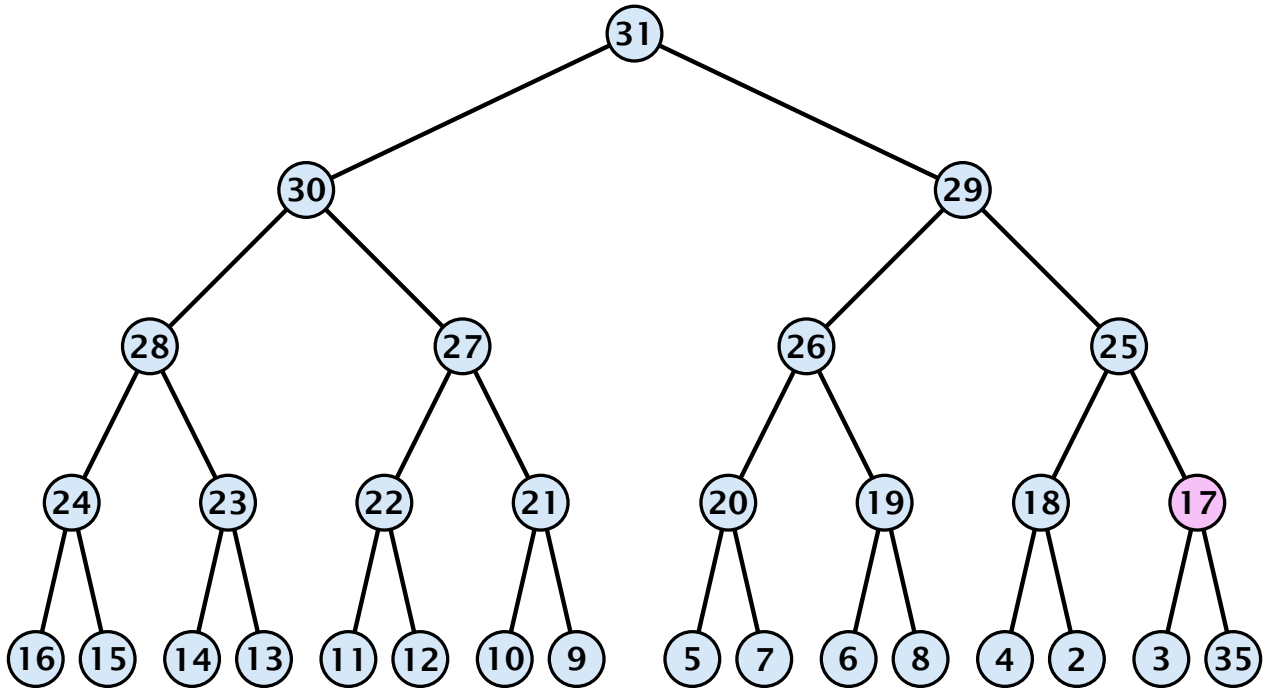

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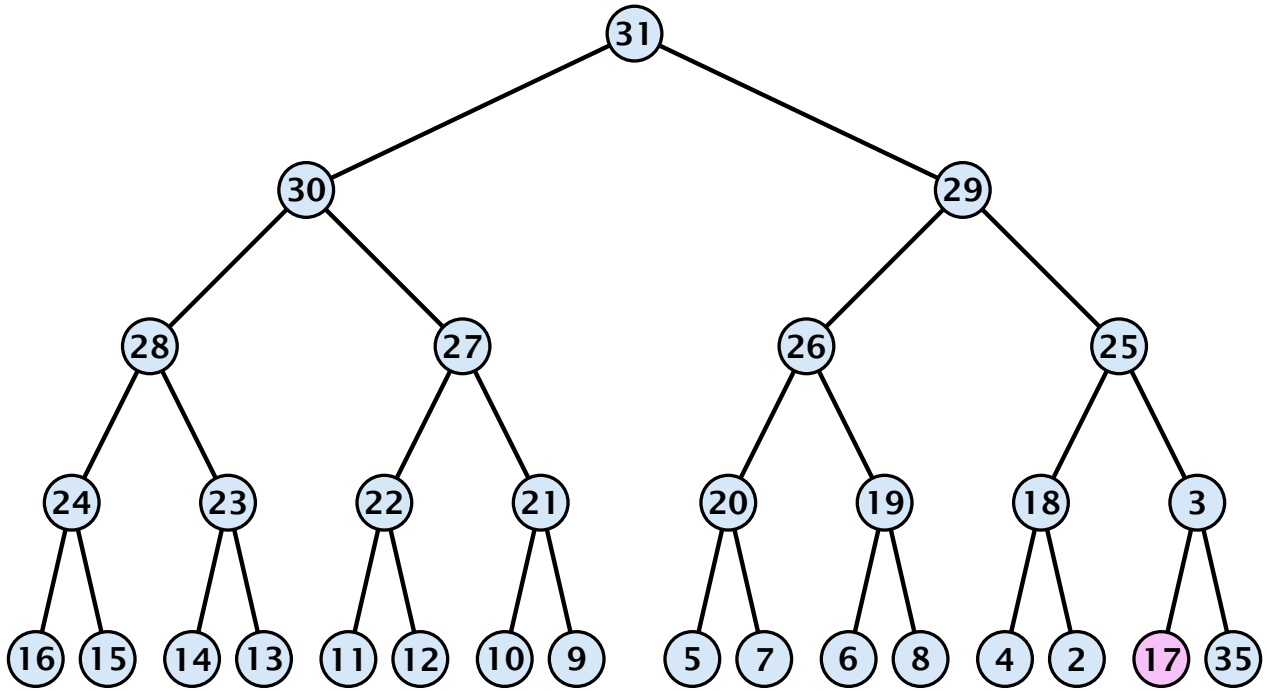

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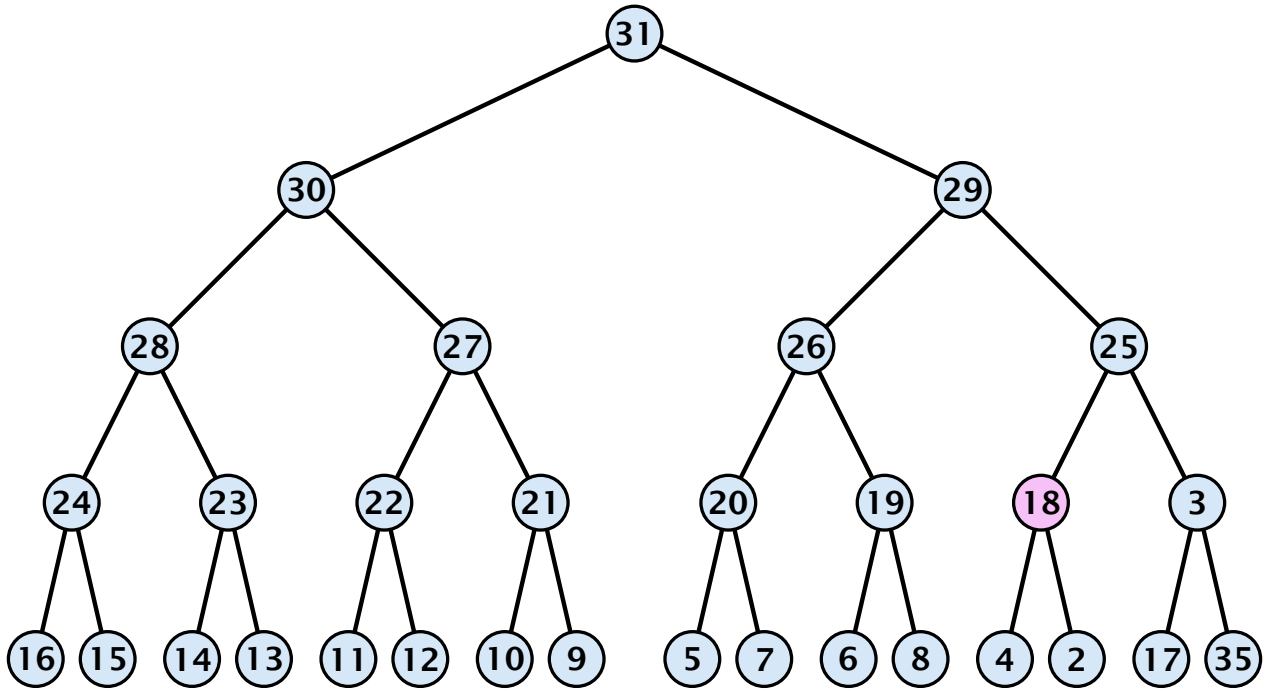

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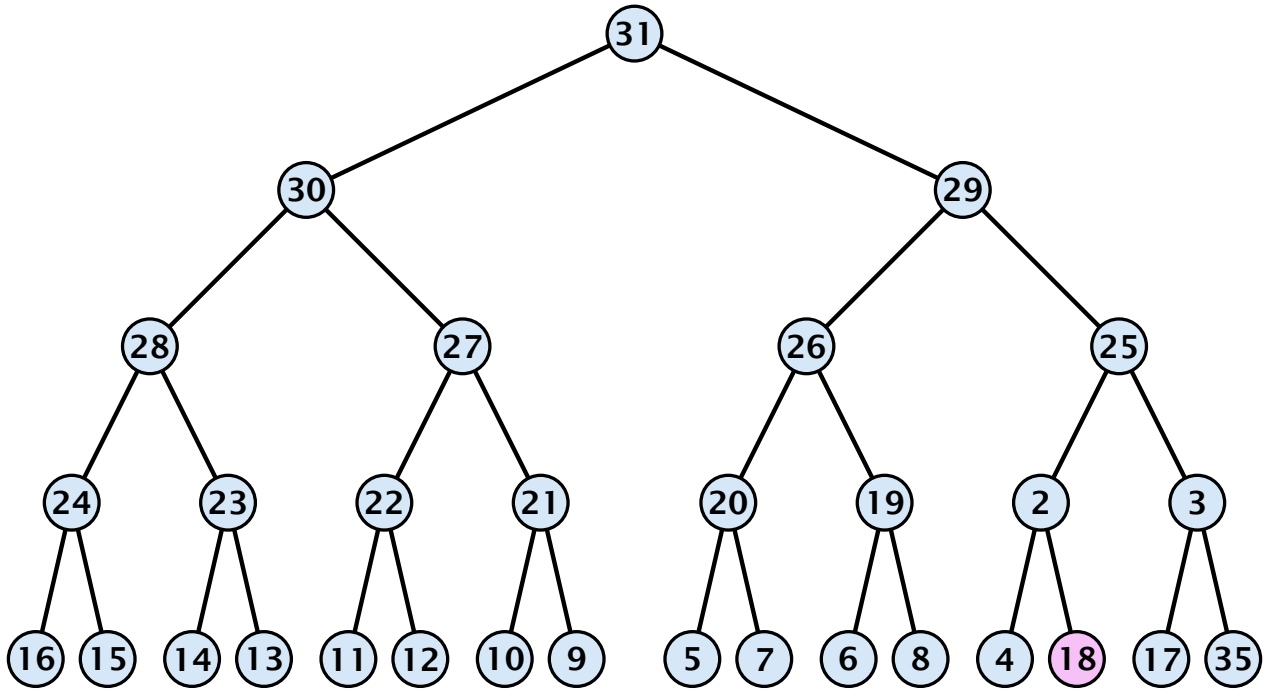

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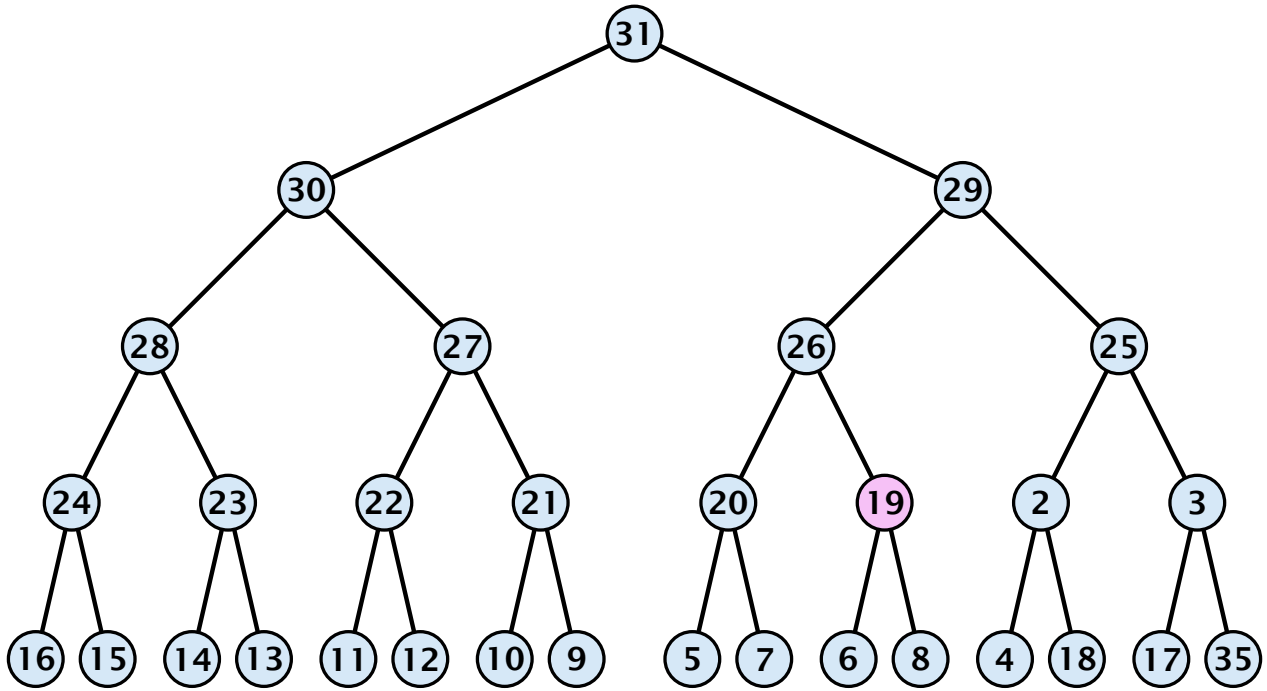

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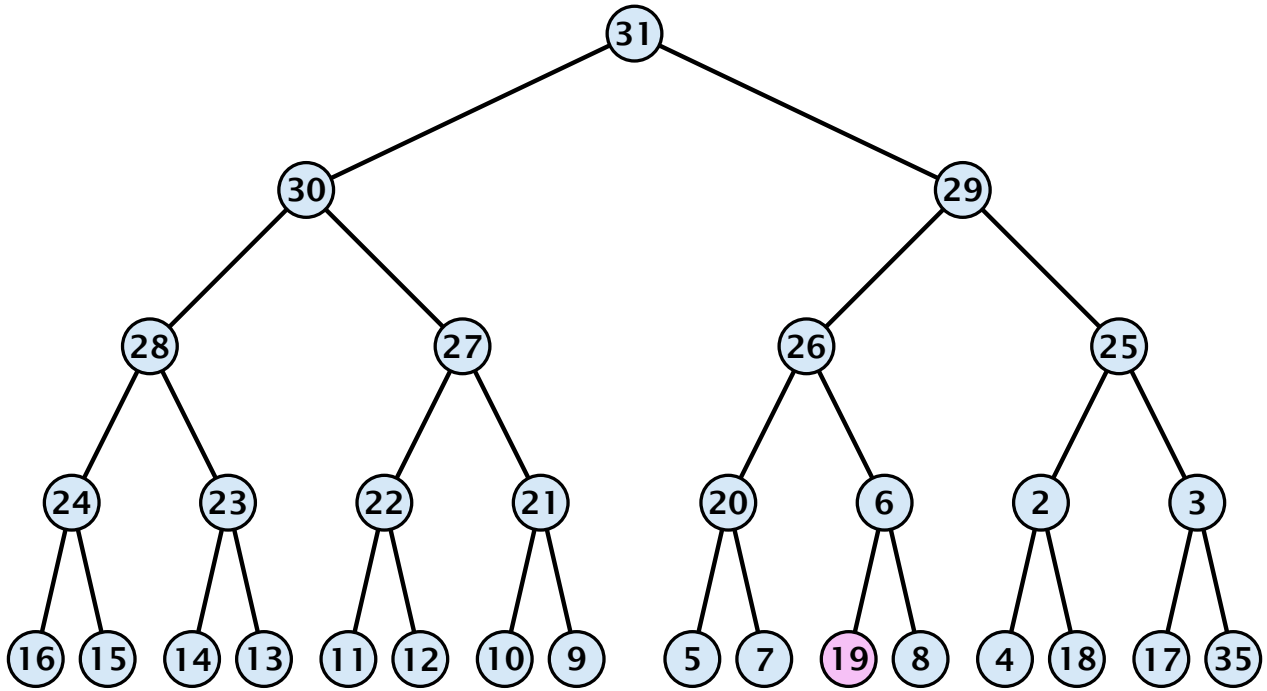

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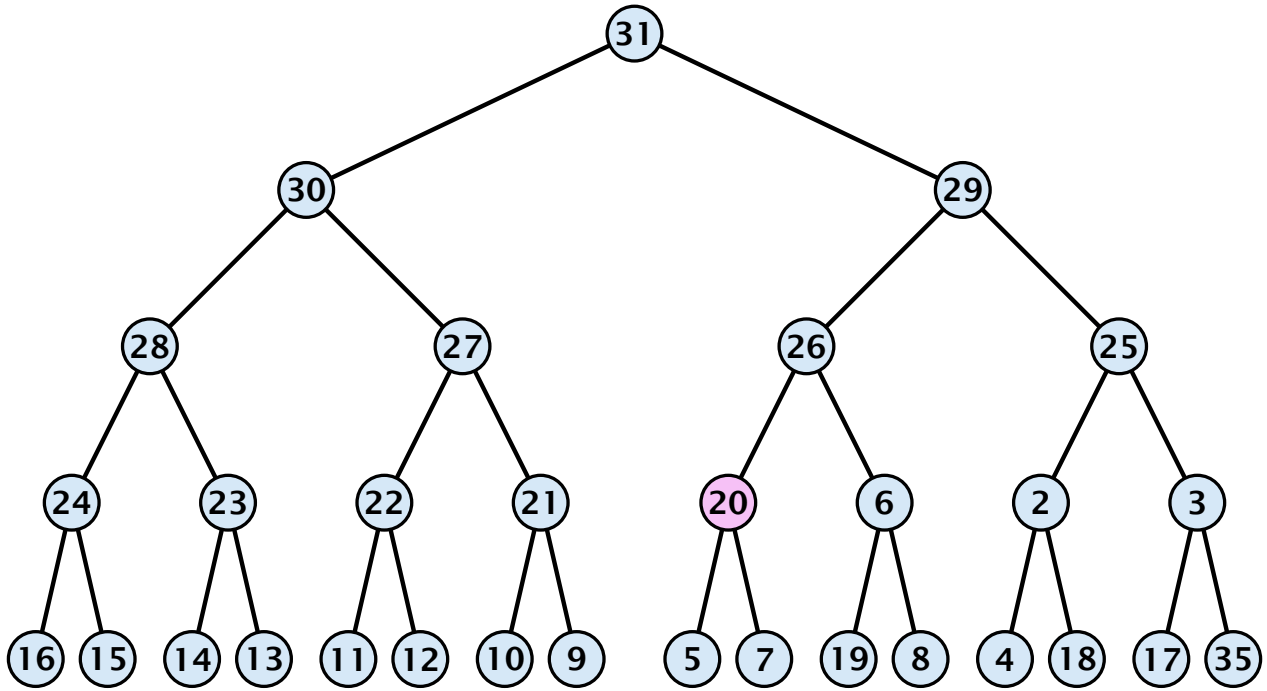

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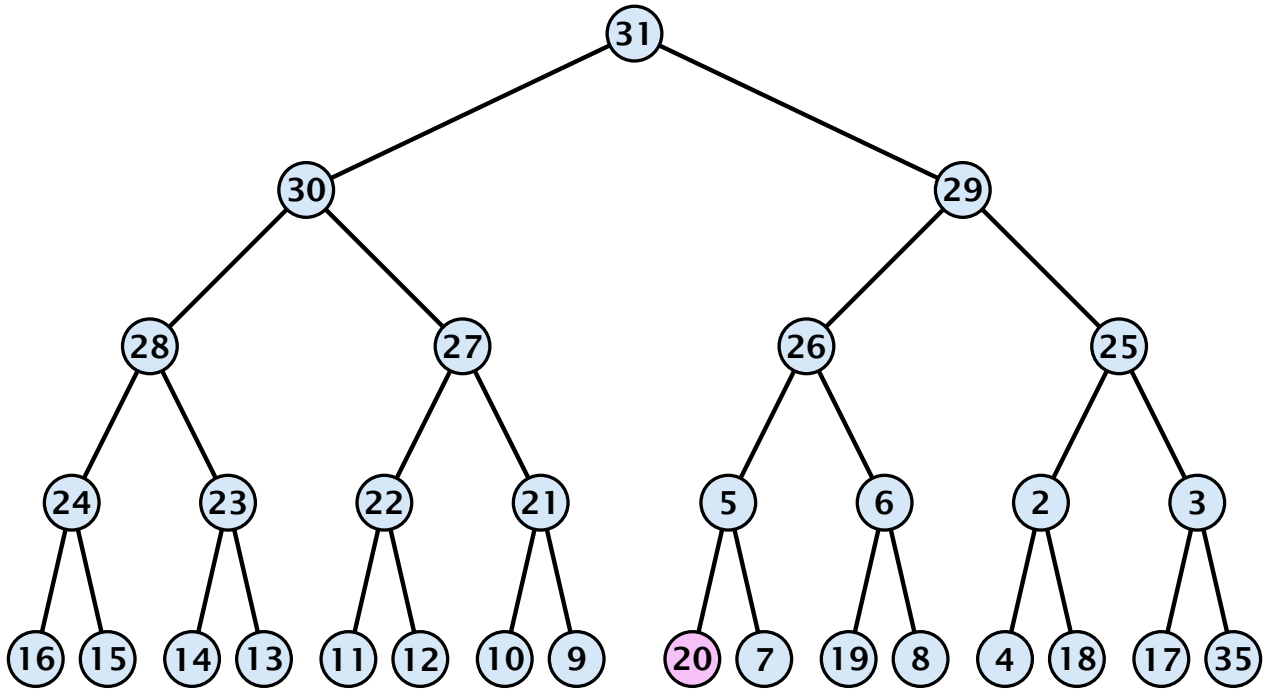

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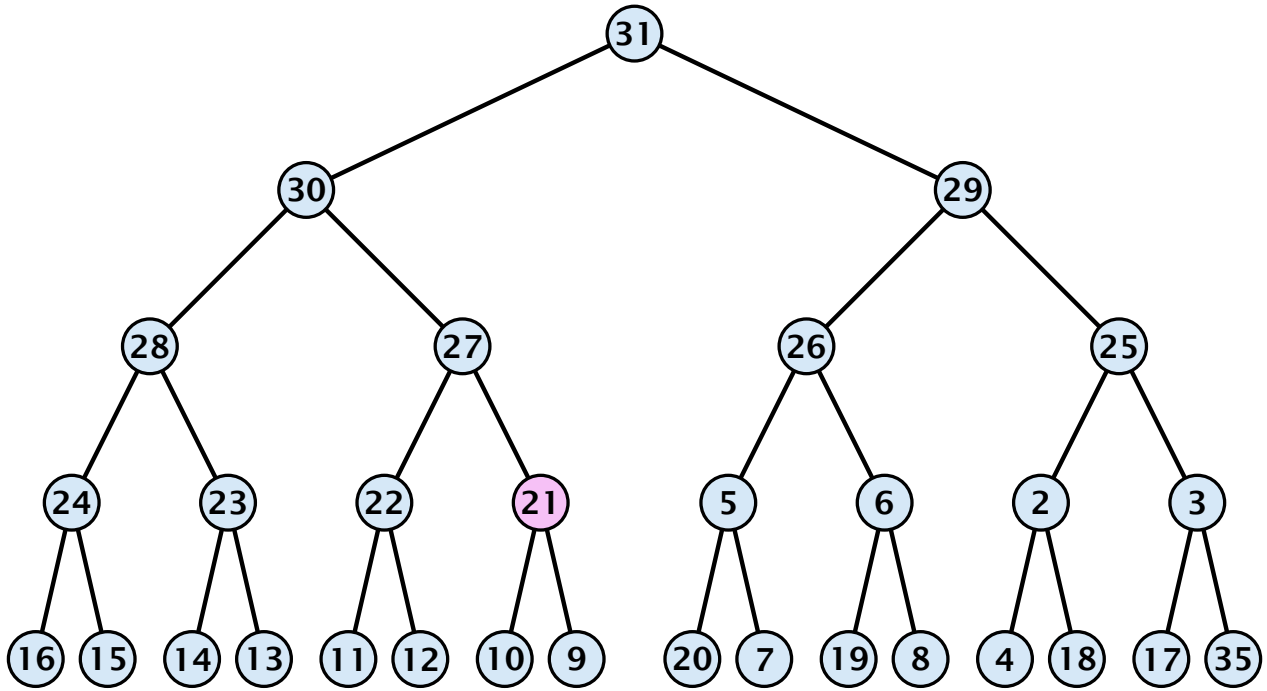

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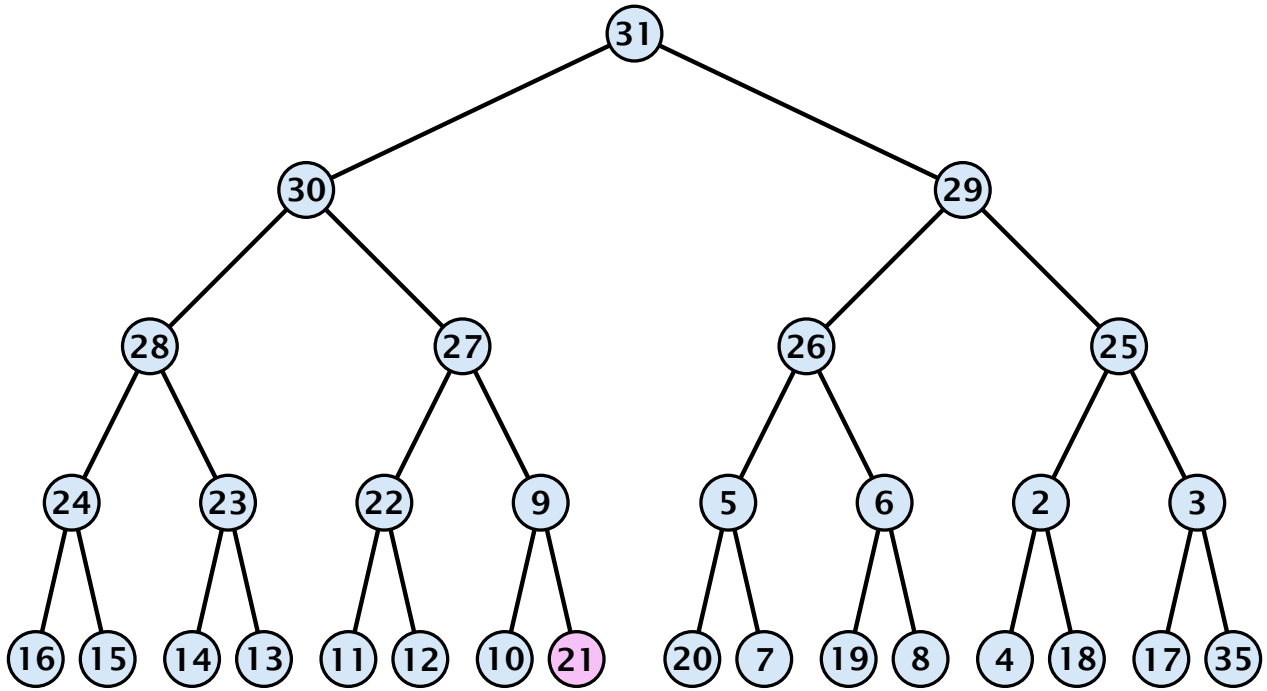

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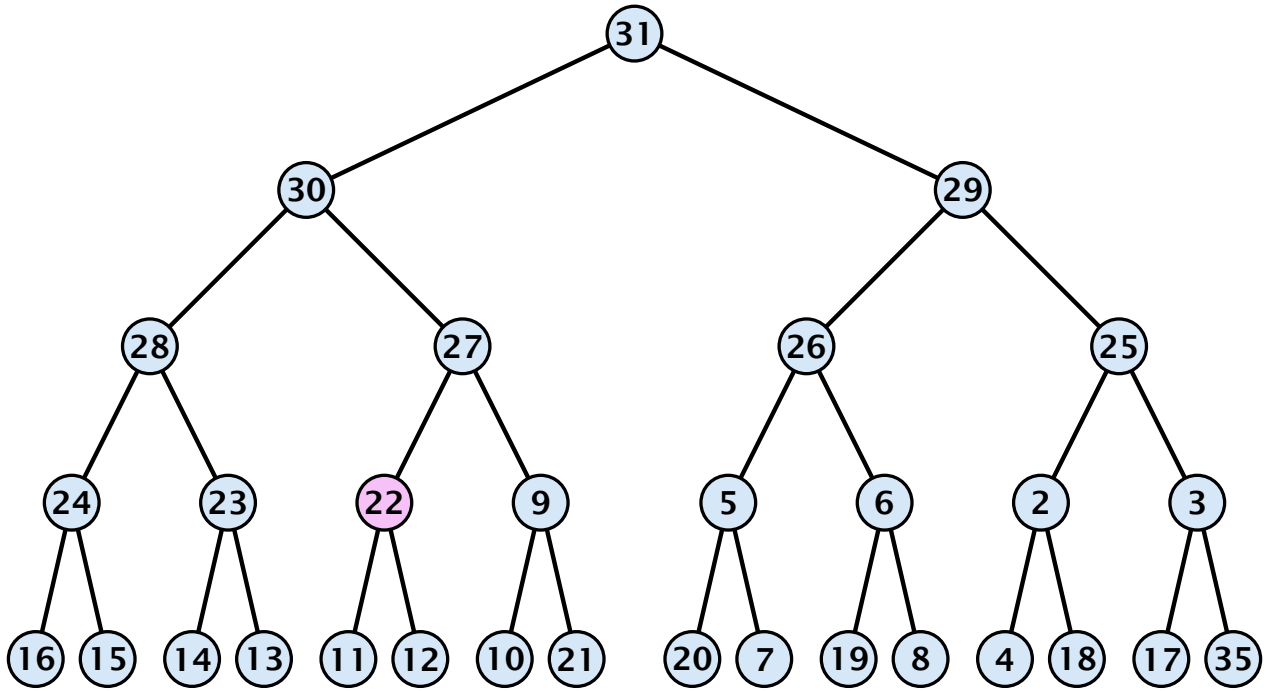

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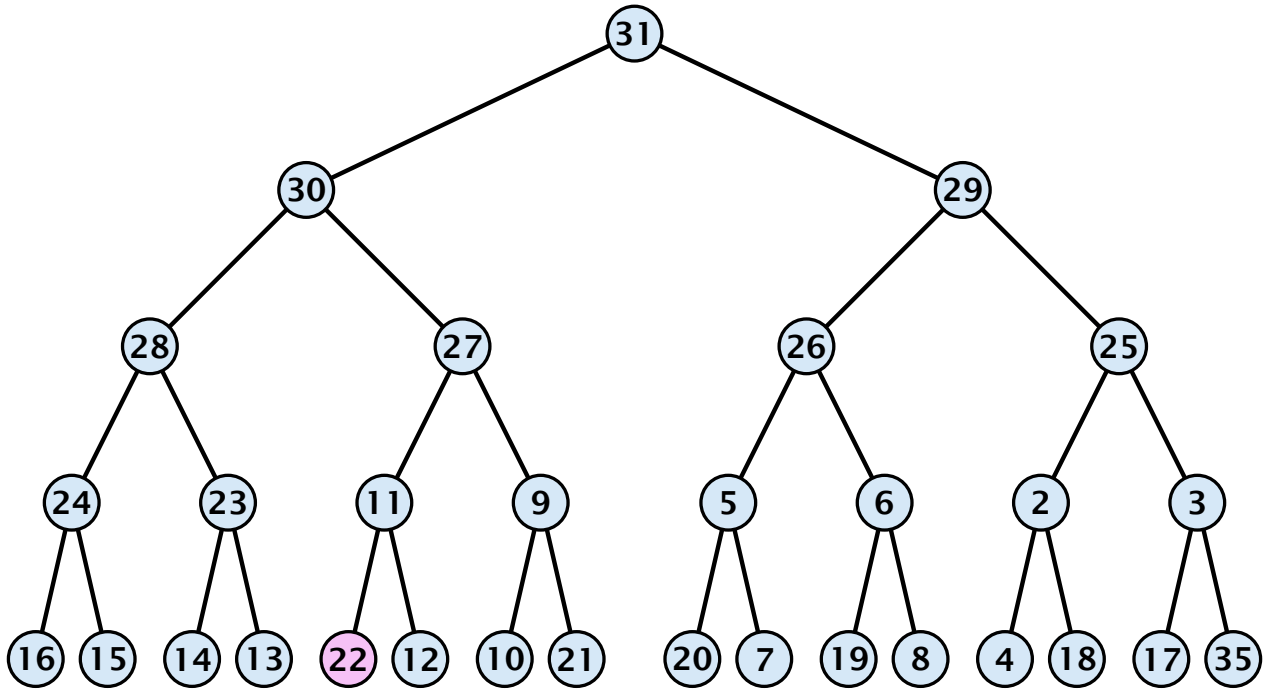

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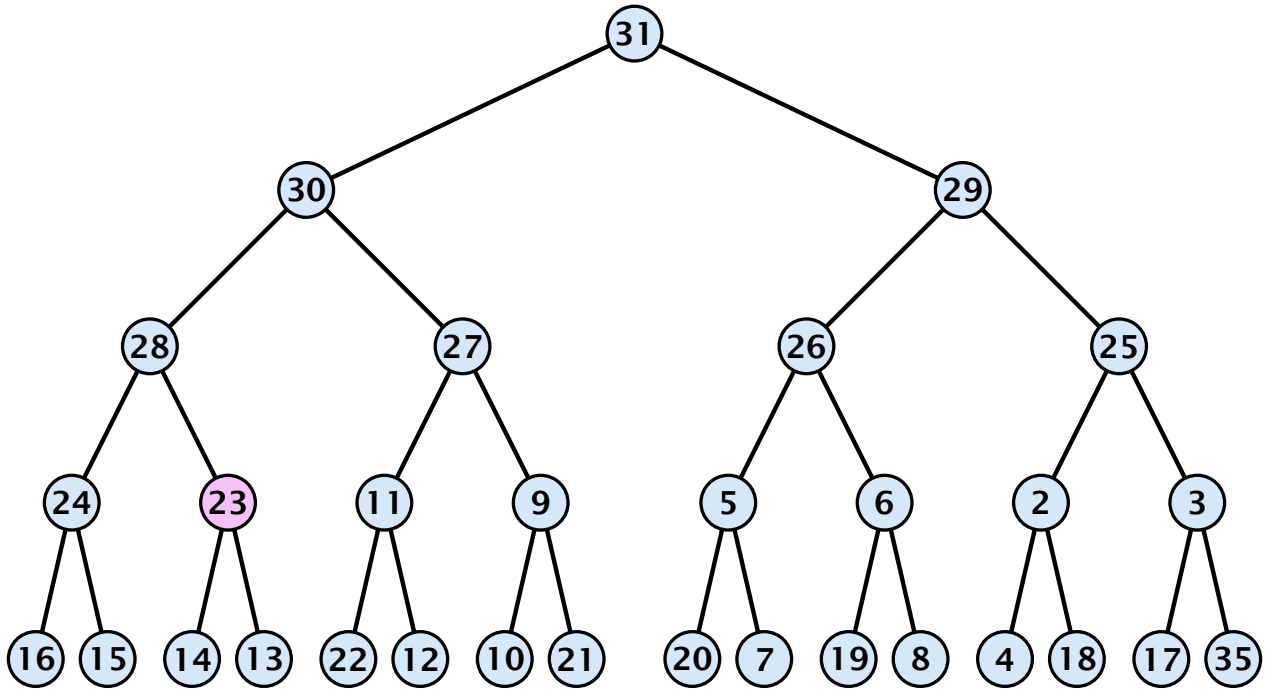

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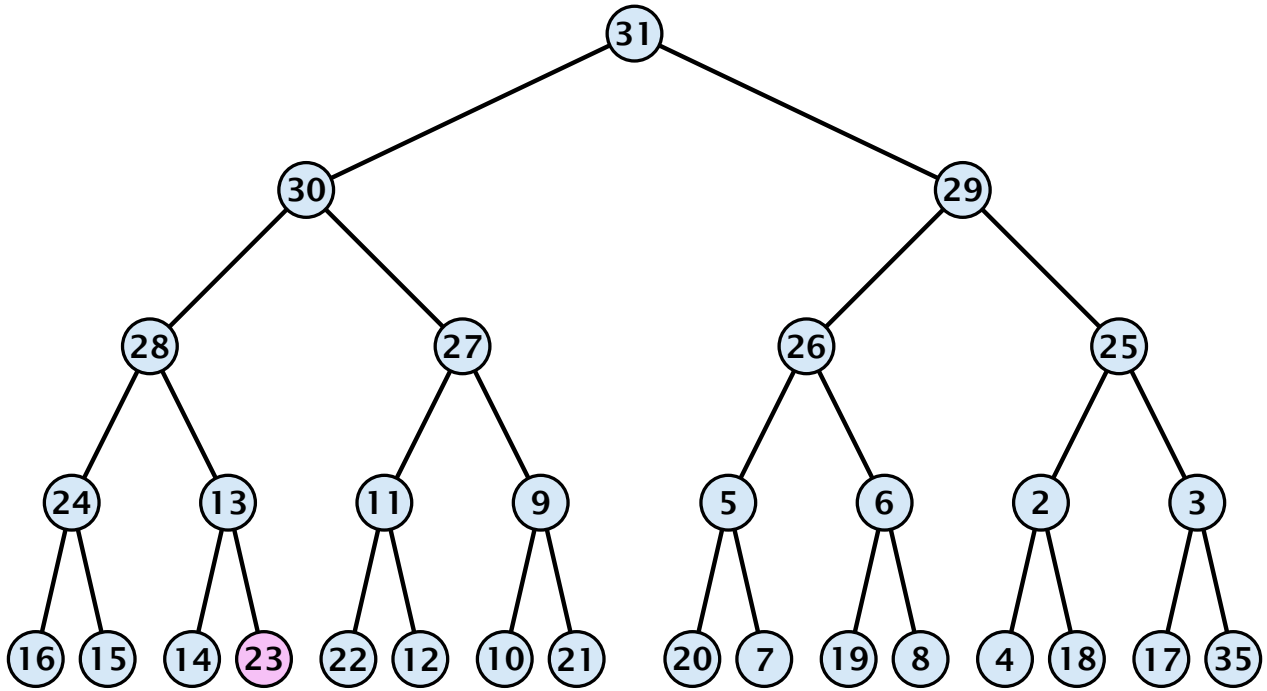

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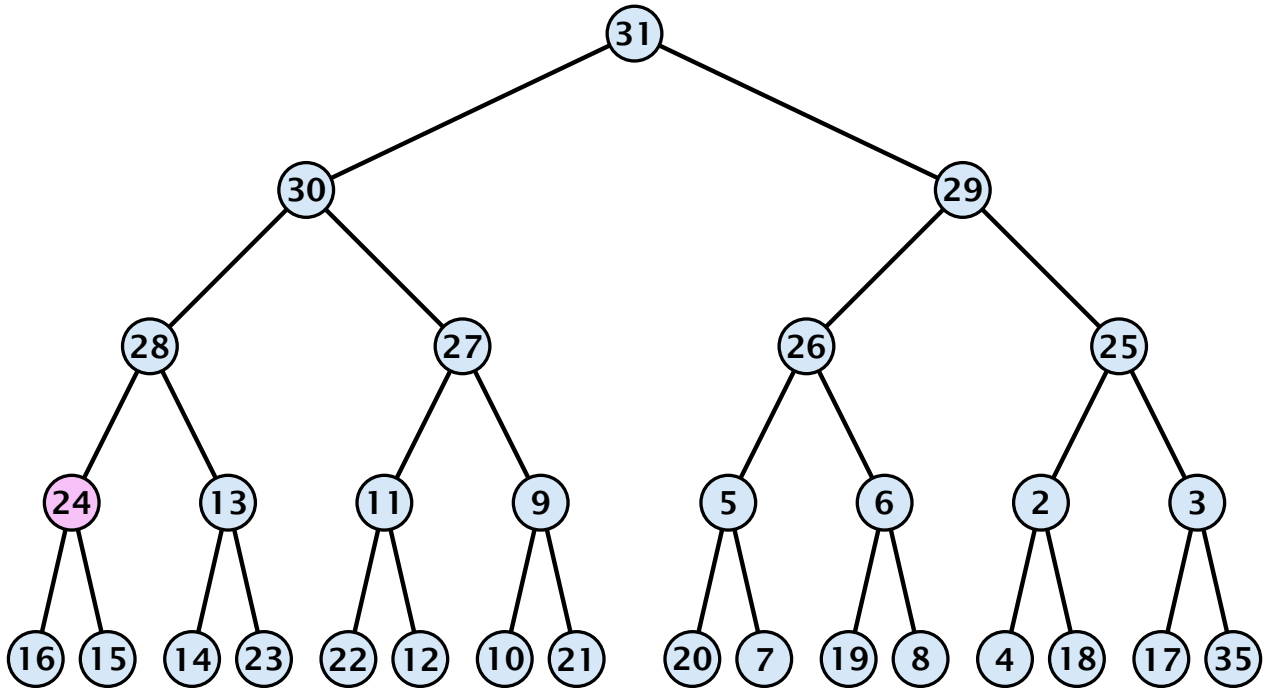

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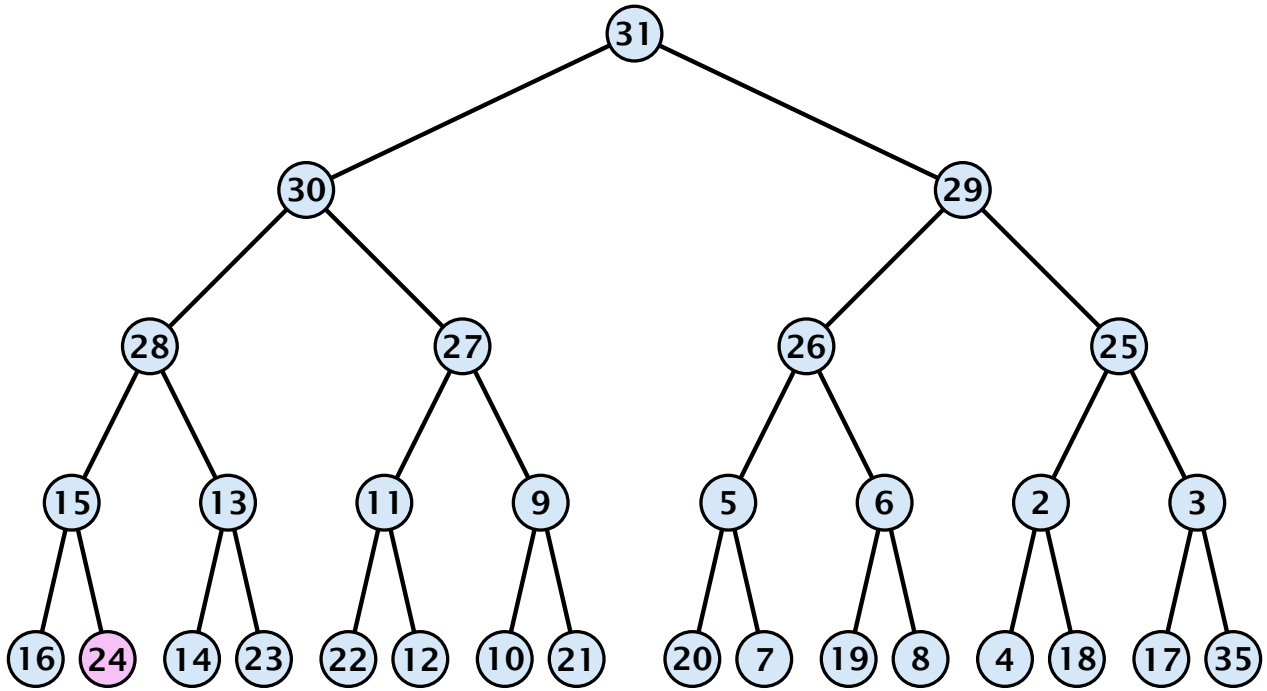

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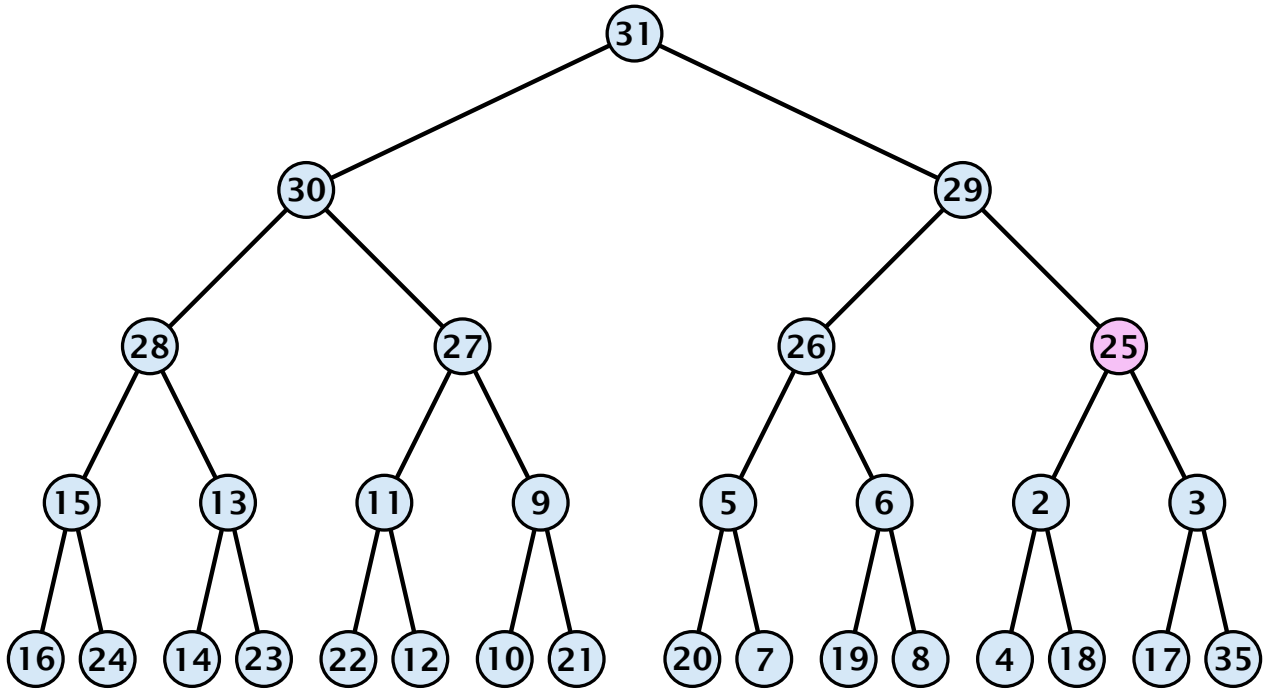

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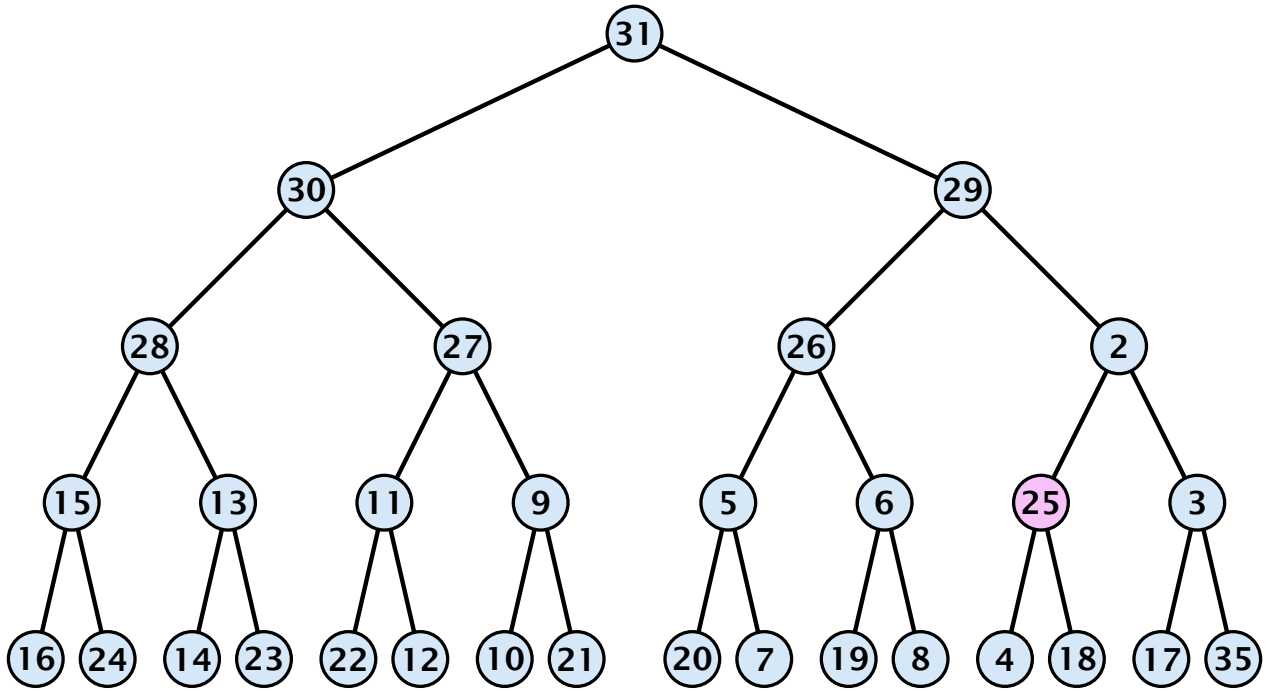

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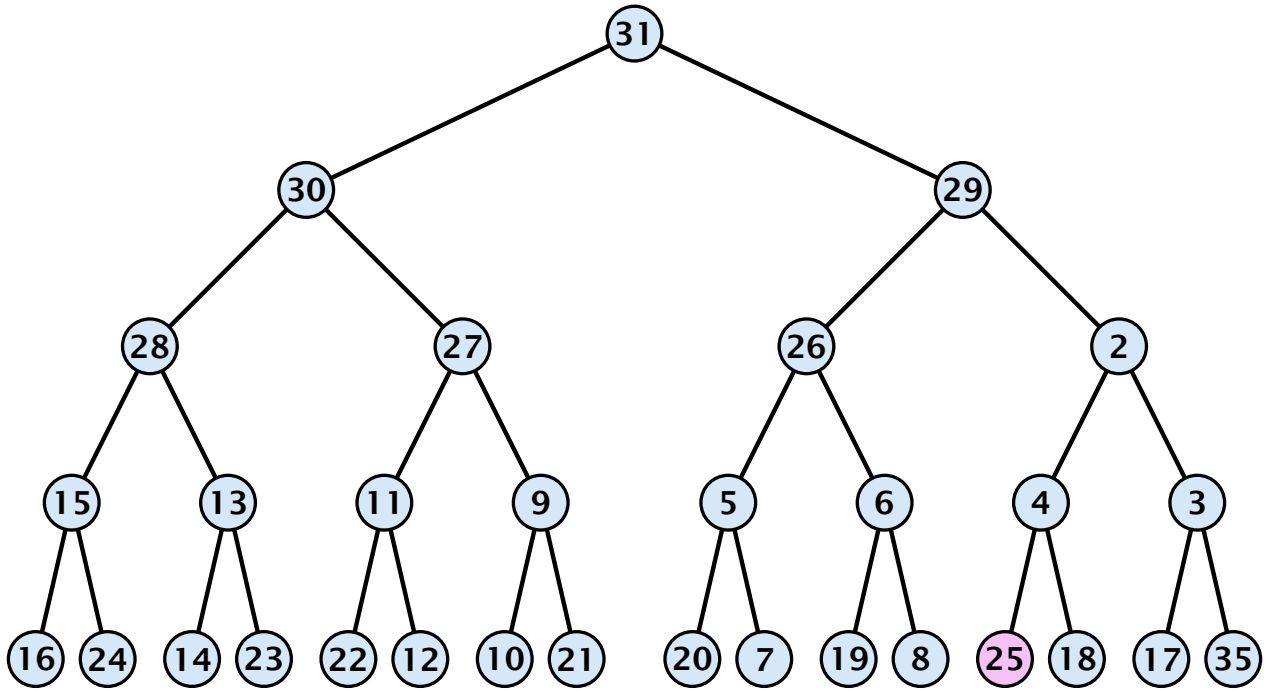

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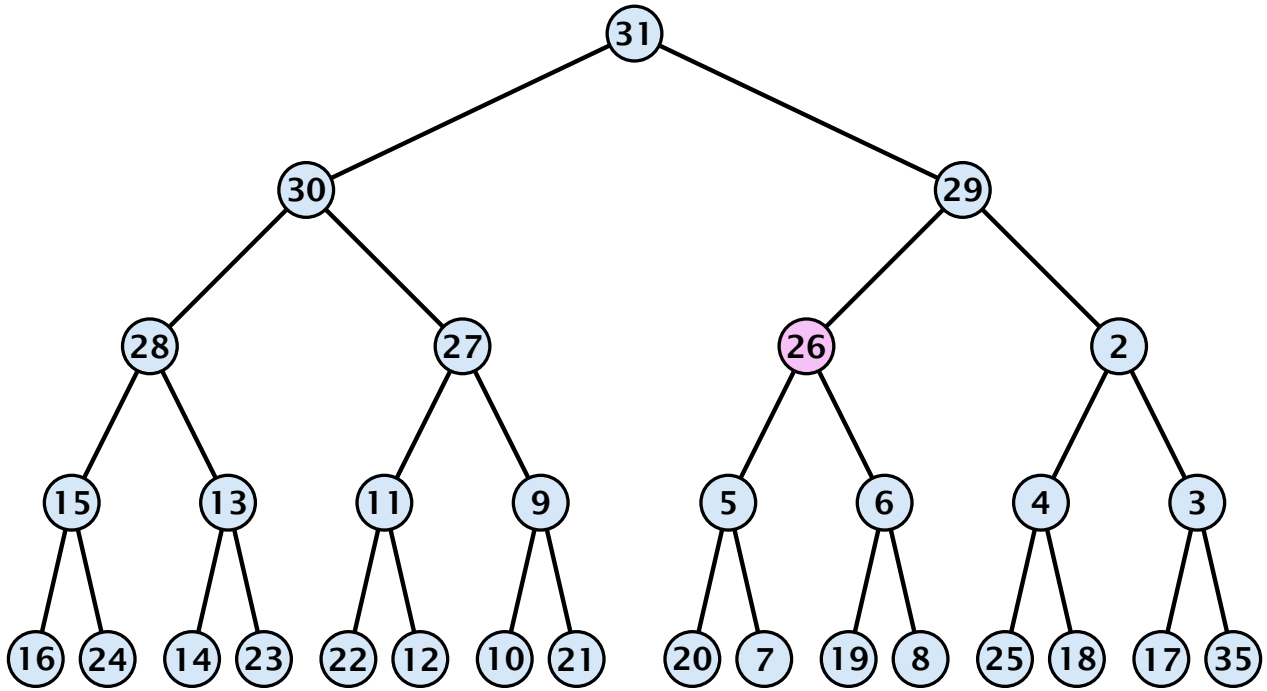

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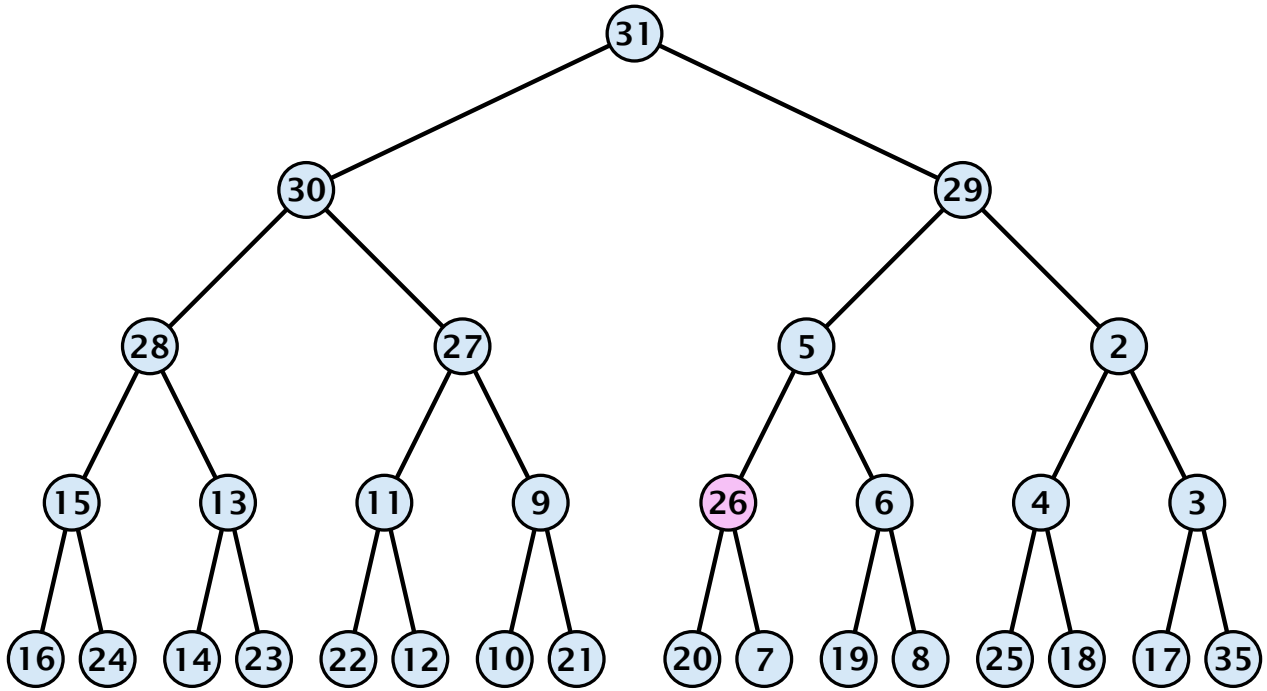

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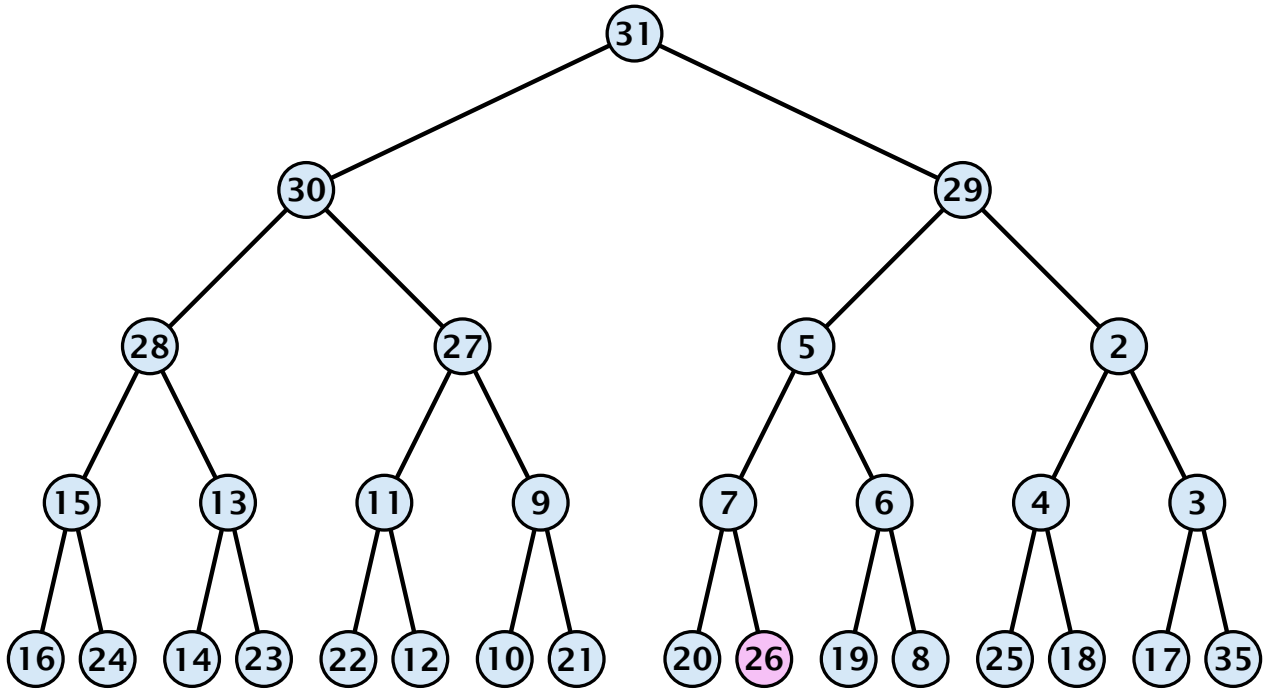

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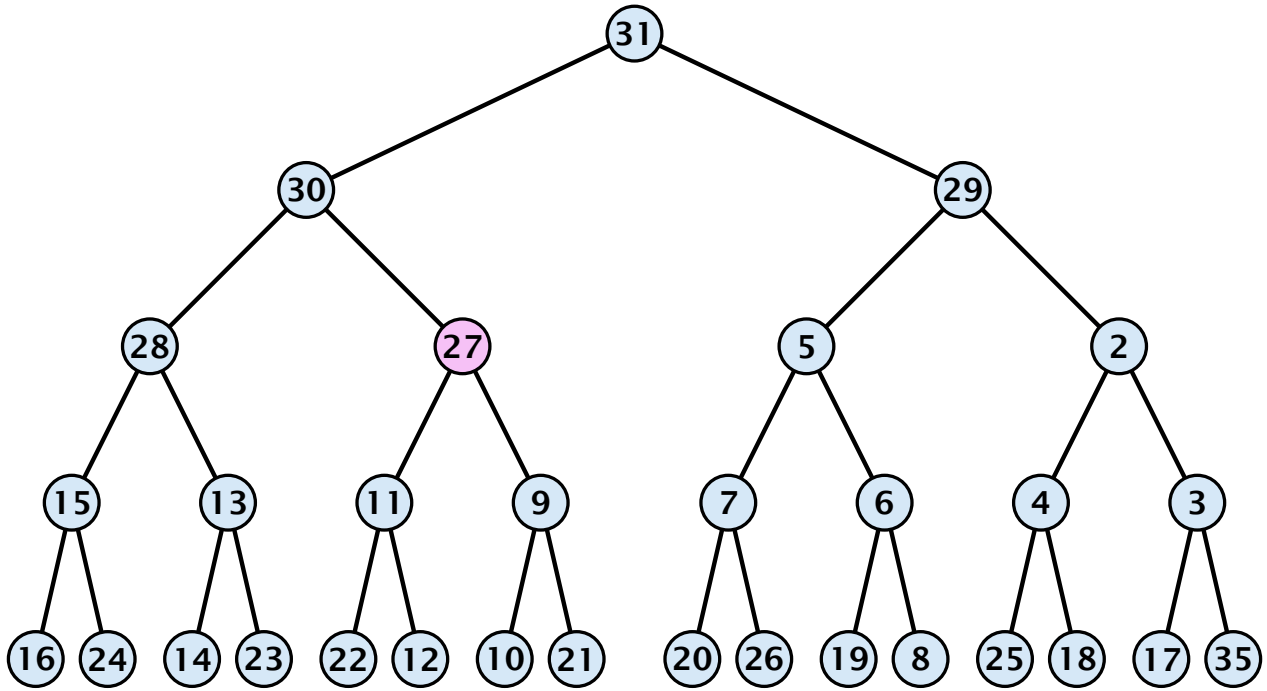

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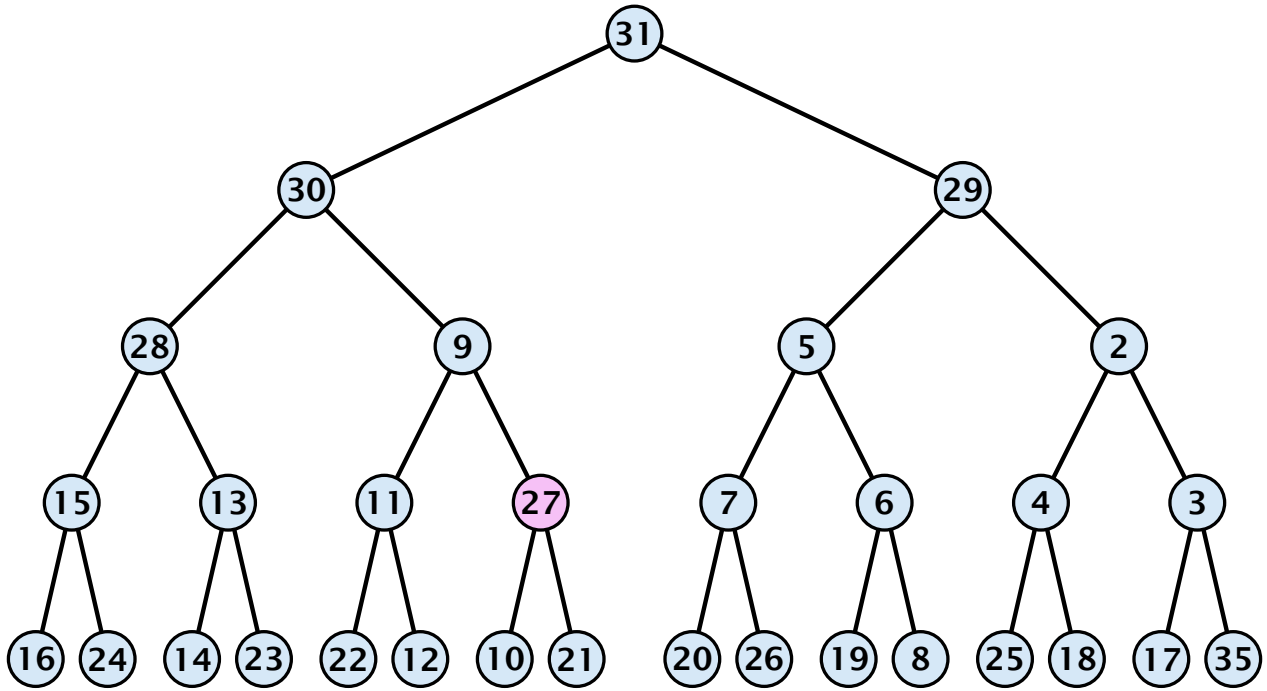

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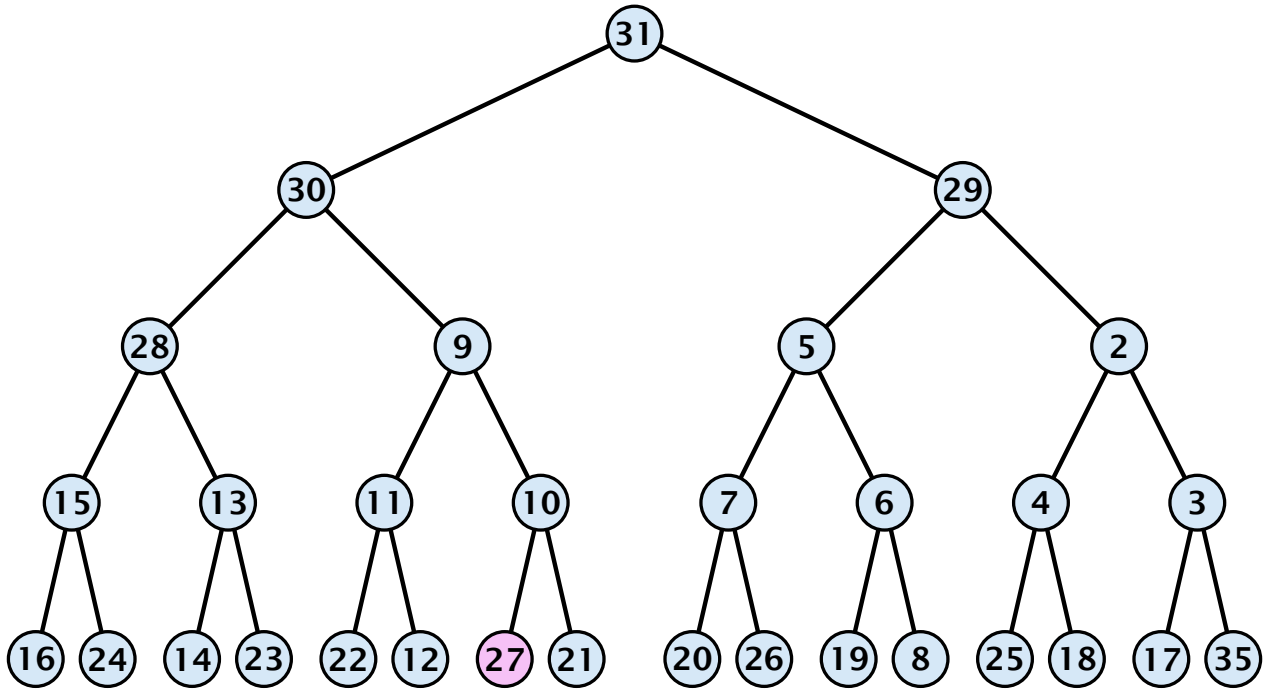

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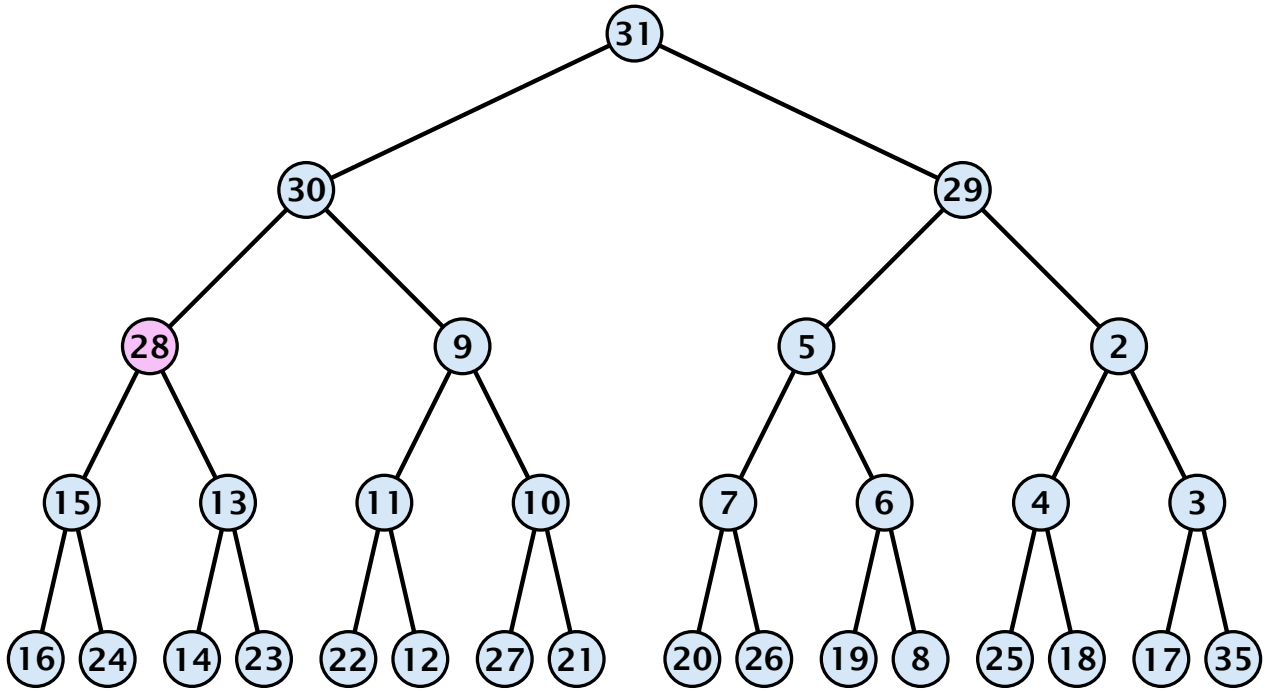

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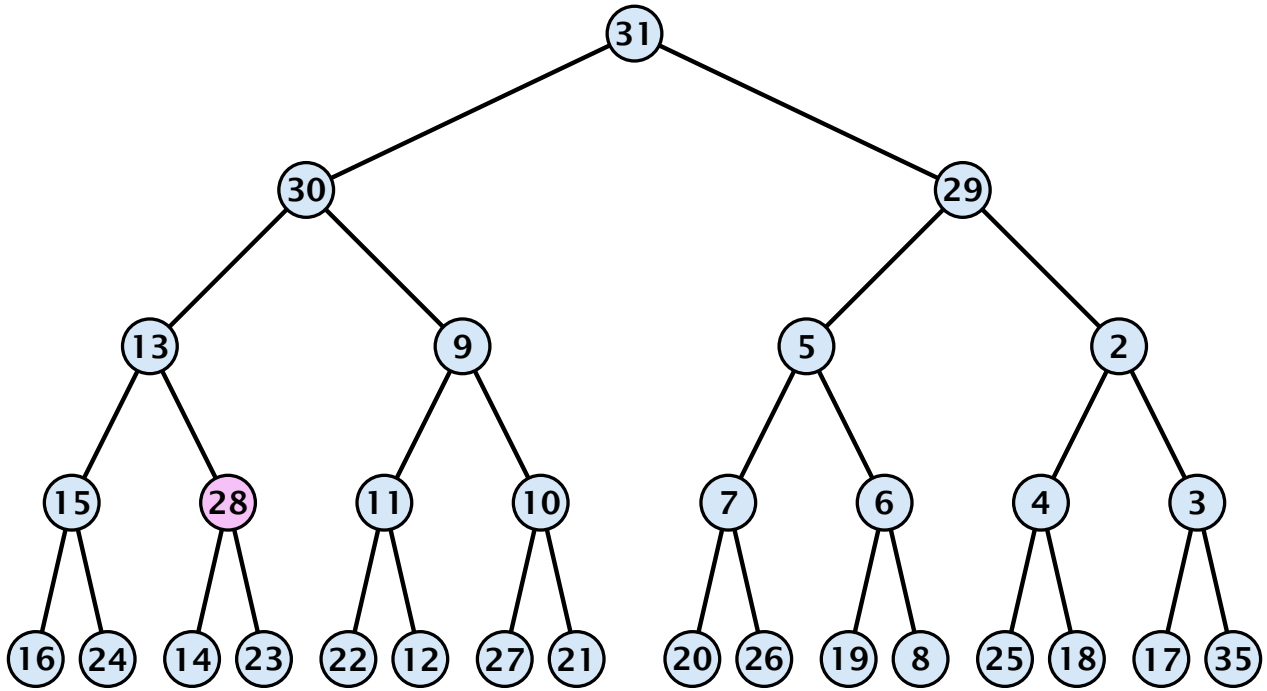

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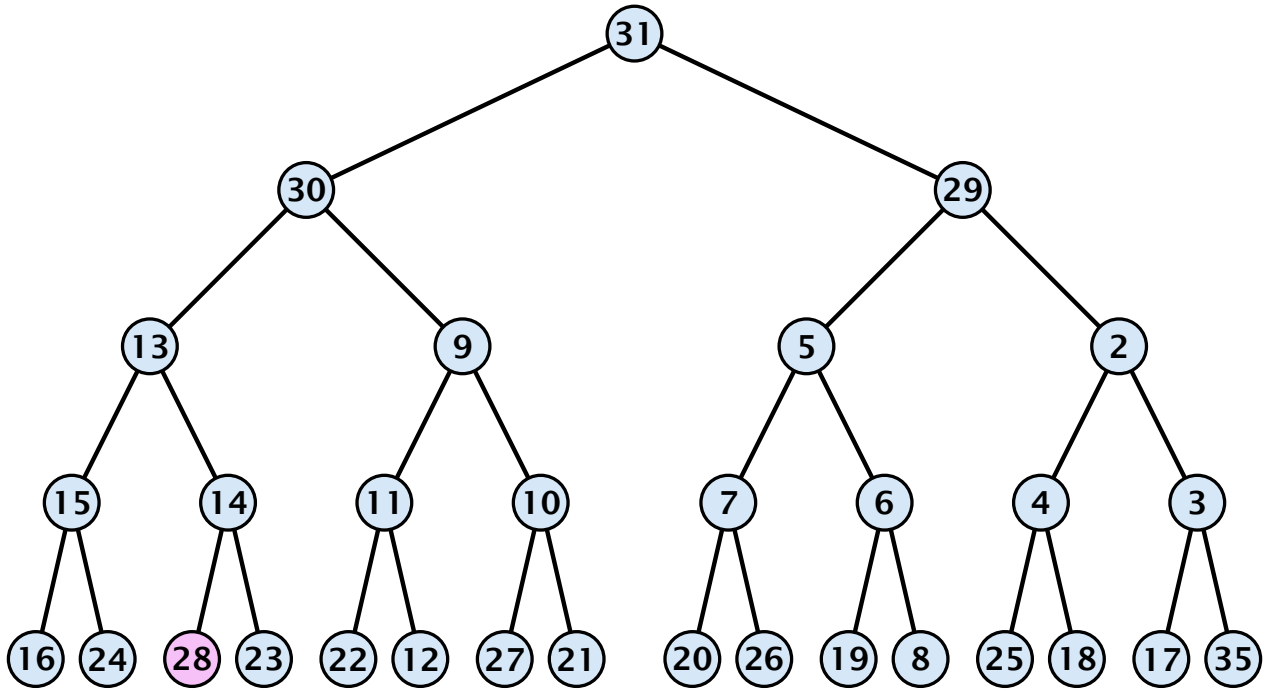

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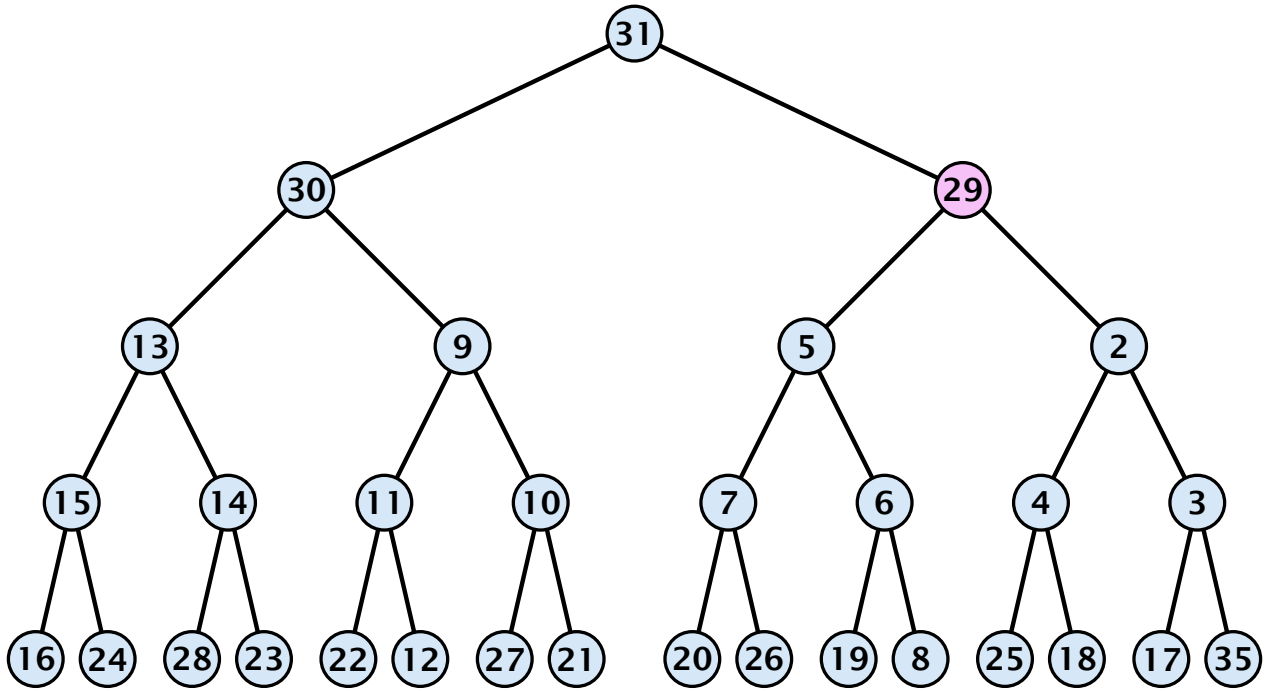

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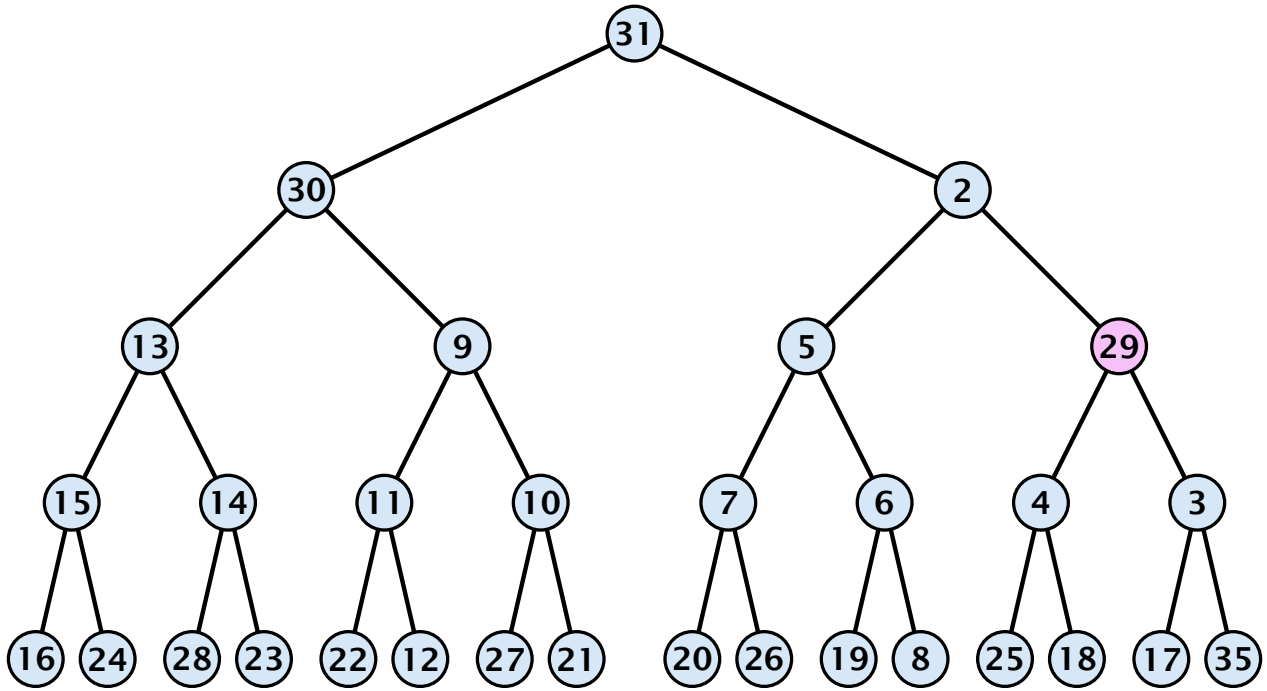

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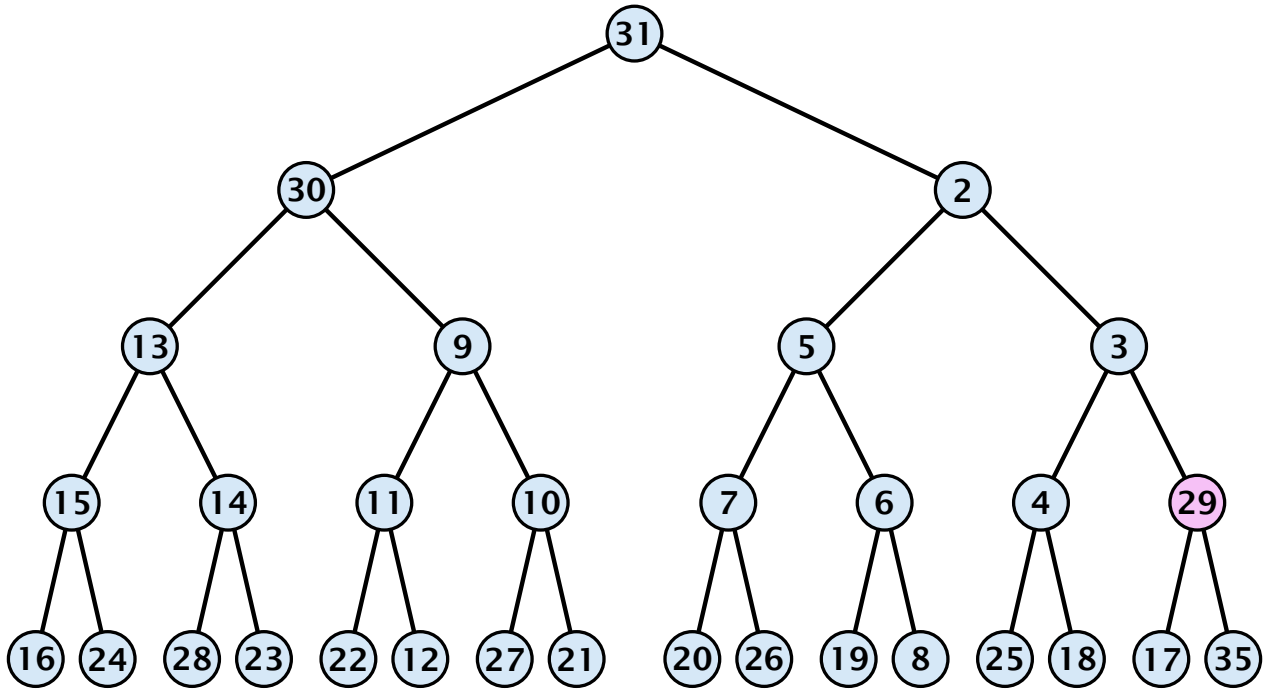

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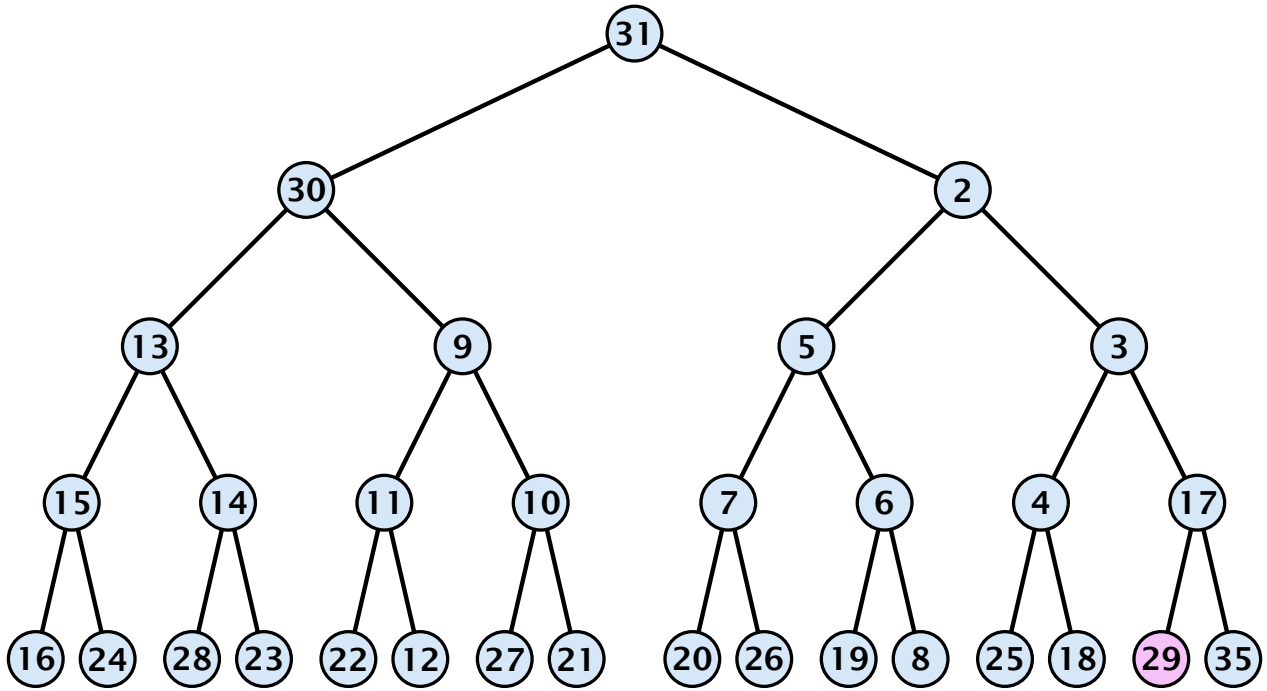

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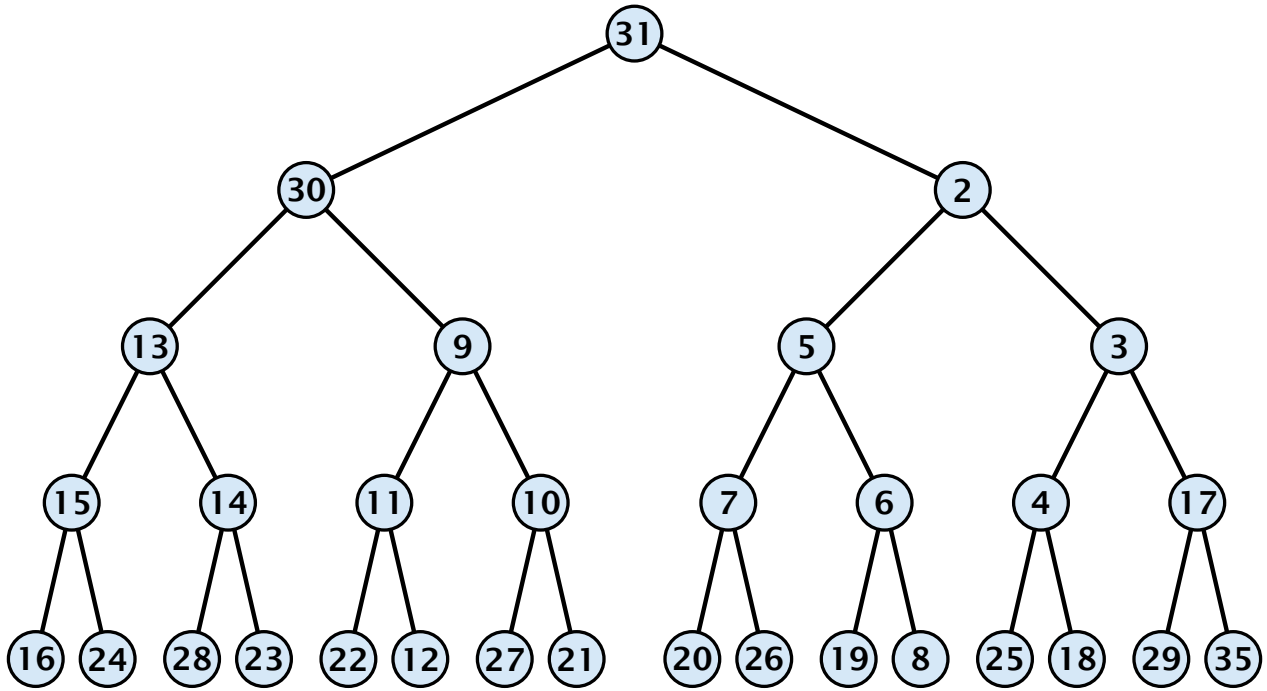

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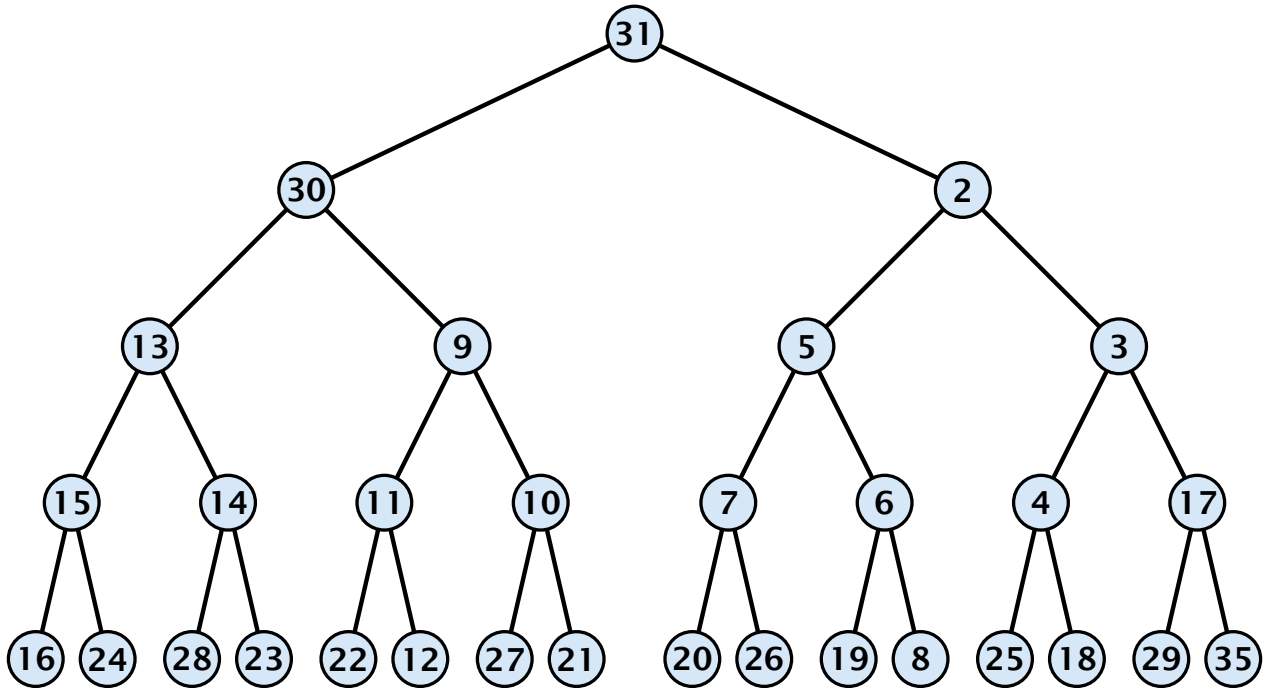

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Binary Heaps

Operations:

- **minimum()**: Return the root-element. Time $\mathcal{O}(1)$.
- **is-empty()**: Check whether root-pointer is null. Time $\mathcal{O}(1)$.
- **insert(k)**: Insert at x and bubble up. Time $\mathcal{O}(\log n)$.
- **delete(h)**: Swap with x and bubble up or sift-down. Time $\mathcal{O}(\log n)$.
- **build($x_1, \ldots, x_n$)**: Insert elements arbitrarily; then do sift-down operations starting with the lowest layer in the tree. Time $\mathcal{O}(n)$.

Binary Heaps

The standard implementation of binary heaps is via arrays. Let $A[0,\ldots,n-1]$ be an array

- The parent of i-th element is at position $\lfloor \frac{i-1}{2} \rfloor$.
- The left child of i-th element is at position $2i+1$.
- The right child of i-th element is at position $2i+2$.

Finding the successor of x is much easier than in the description on the previous slide. Simply increase or decrease x.

The resulting binary heap is not addressable. The elements don't maintain their positions and therefore there are no stable handles.

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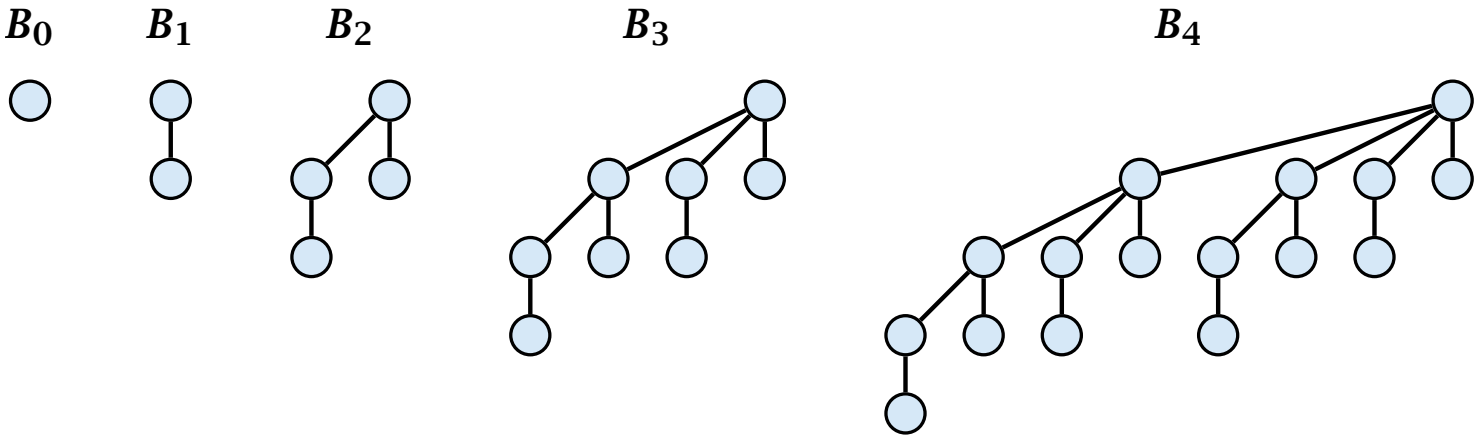
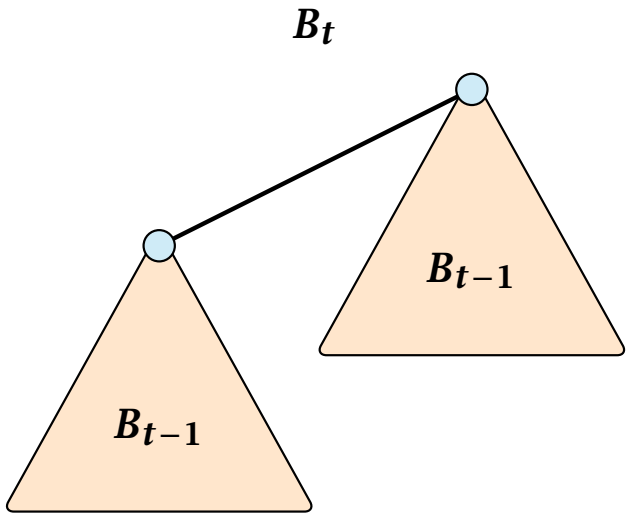
8.2 Binomial Heaps

Operation	*Binary Heap*	*BST*	*Binomial Heap*	*Fibonacci Heap*[*]
build	n	$n \log n$	$n \log n$	n
minimum	1	$\log n$	$\log n$	1
is-empty	1	1	1	1
insert	$\log n$	$\log n$	$\log n$	1
delete	$\log n$[**]	$\log n$	$\log n$	$\log n$
delete-min	$\log n$	$\log n$	$\log n$	$\log n$
decrease-key	$\log n$	$\log n$	$\log n$	1
merge	n	$n \log n$	$\log n$	1

Binomial Trees





Binomial Trees

Properties of Binomial Trees

- B_k has 2^k nodes.
- B_k has height k.
- The root of B_k has degree k.
- B_k has $\binom{k}{\ell}$ nodes on level ℓ.
- Deleting the root of B_k gives trees $B_0, B_1, \ldots, B_{k-1}$.

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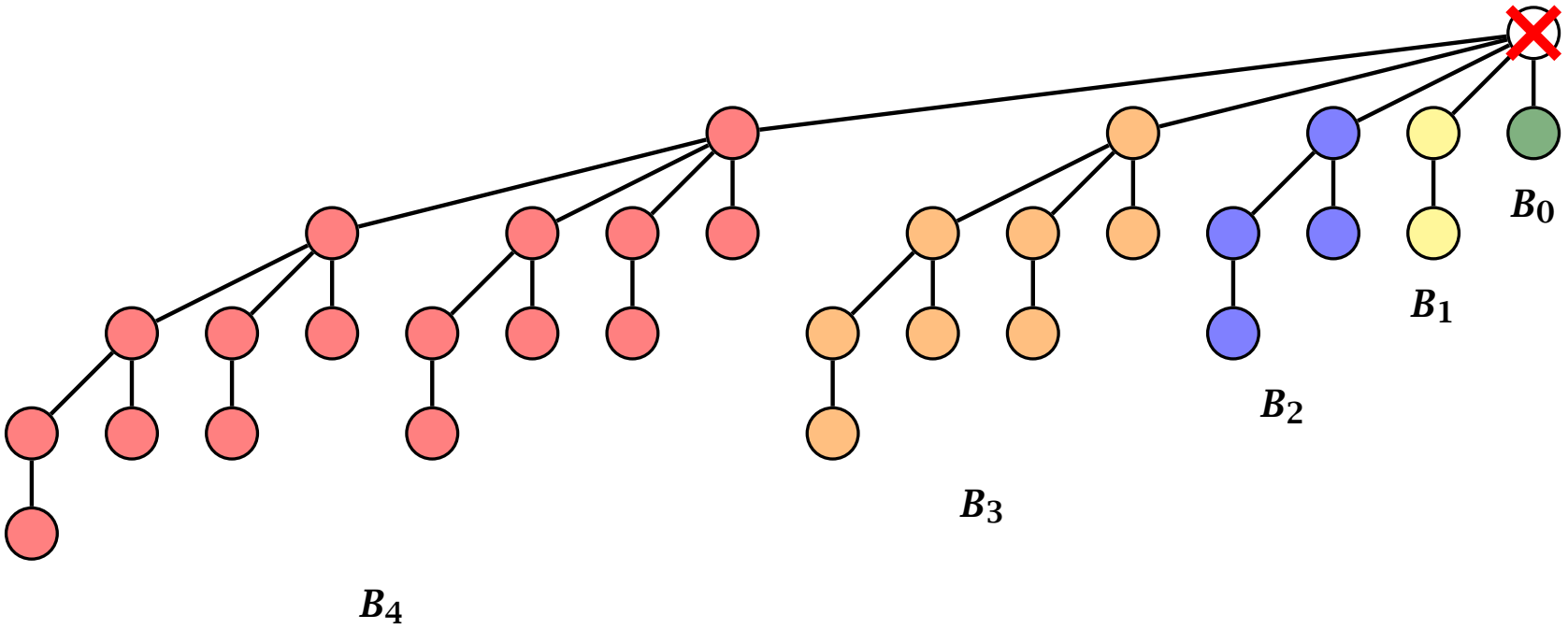
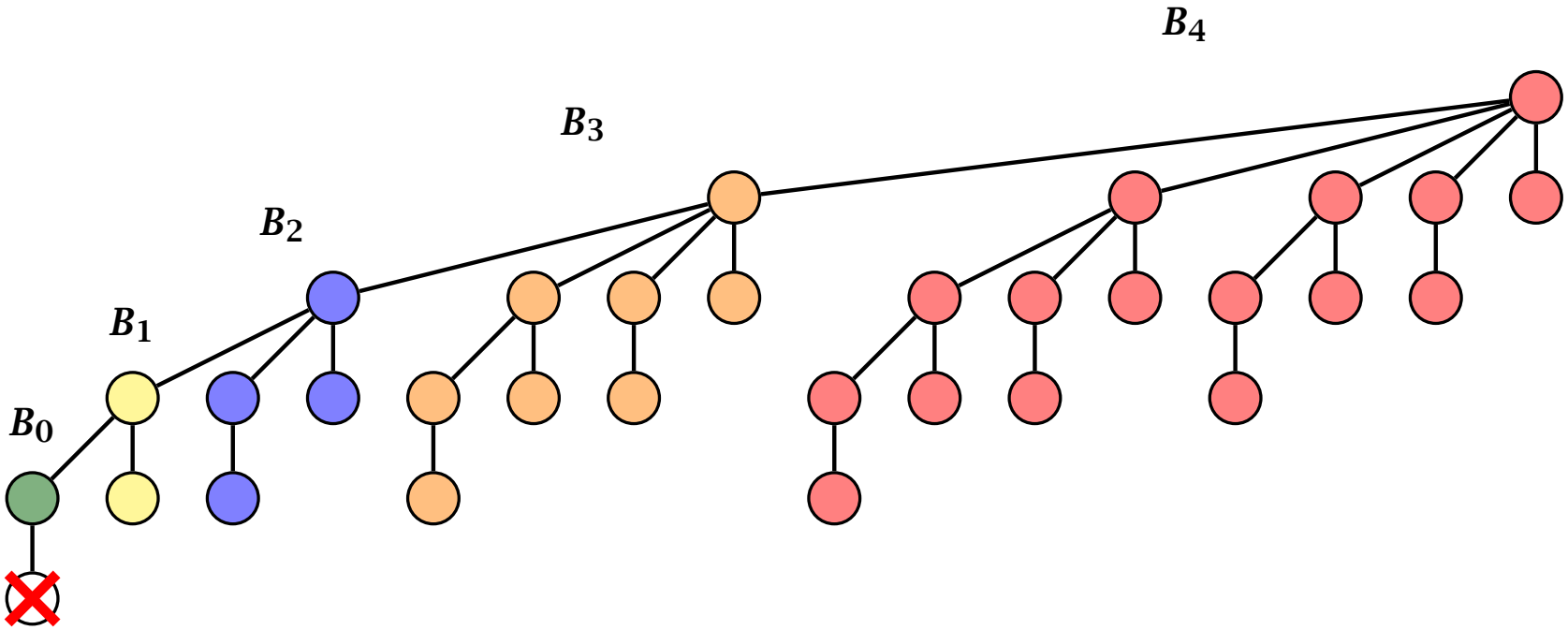


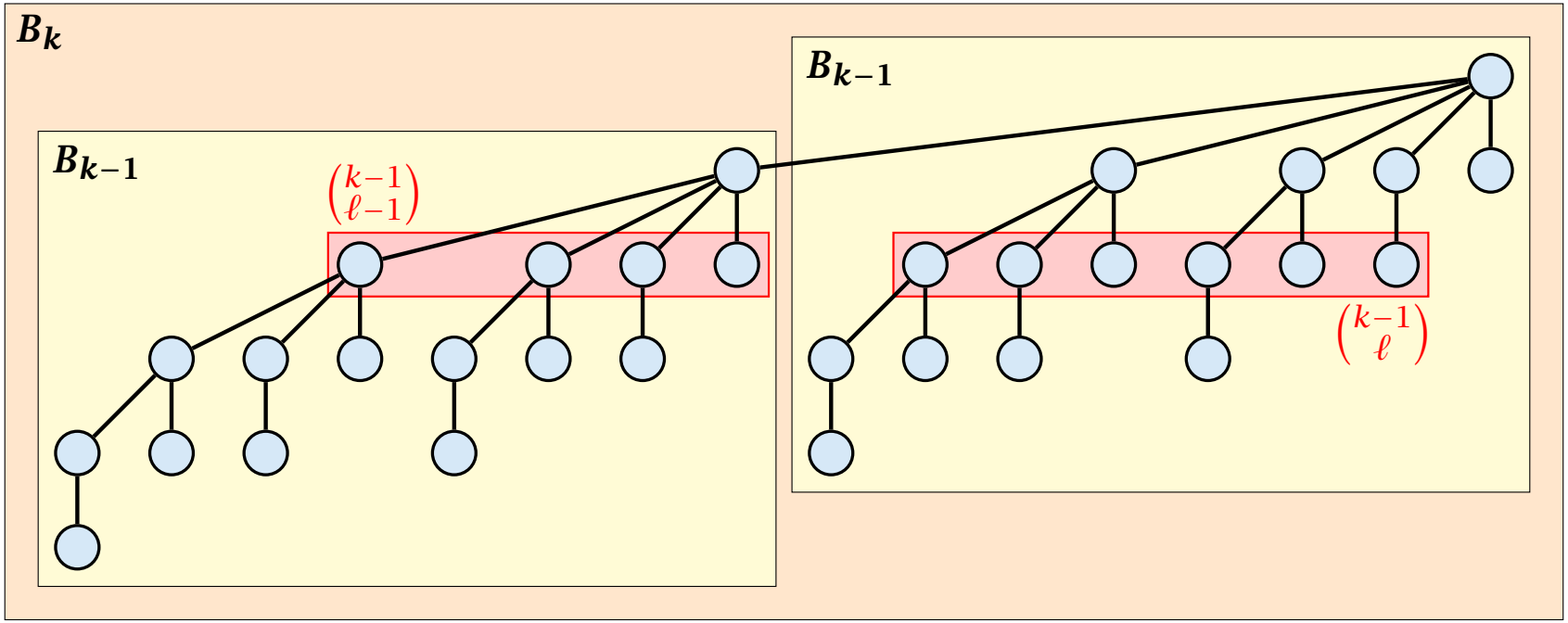
Deleting the root of B_5 leaves sub-trees B_4, B_3, B_2, B_1, and B_0.

Binomial Trees



Deleting the leaf furthest from the root (in B_5) leaves a path that connects the roots of sub-trees B_4, B_3, B_2, B_1, and B_0.

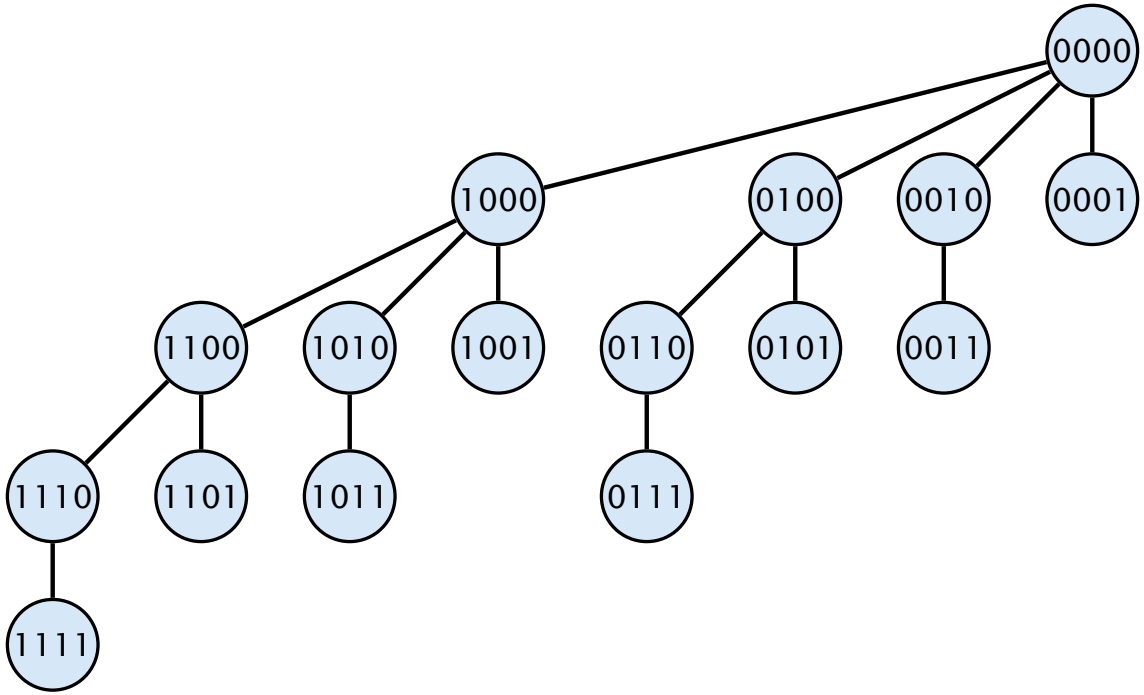
Binomial Trees



The number of nodes on level ℓ in tree B_k is therefore

$$\binom{k-1}{\ell-1} + \binom{k-1}{\ell} = \binom{k}{\ell}$$

Binomial Trees

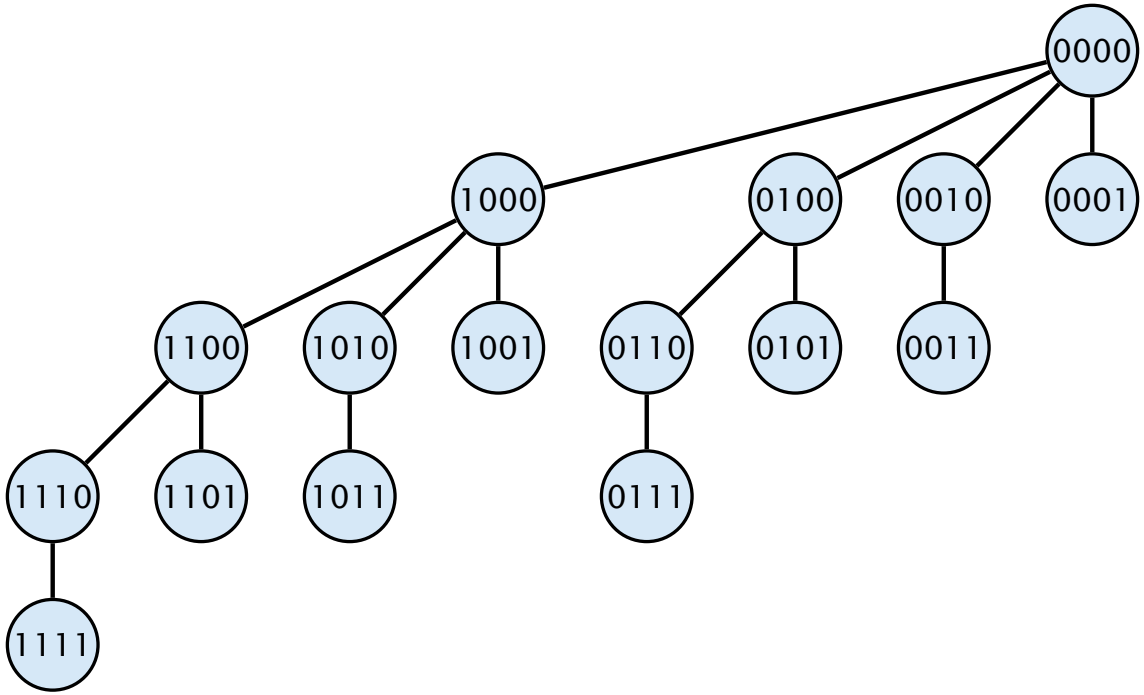


The binomial tree B_k is a sub-graph of the hypercube H_k.

The parent of a node with label $b_n, \ldots, b_1, b_0$ is obtained by setting the least significant 1-bit to 0.

The ℓ-th level contains nodes that have ℓ 1's in their label.

Binomial Trees



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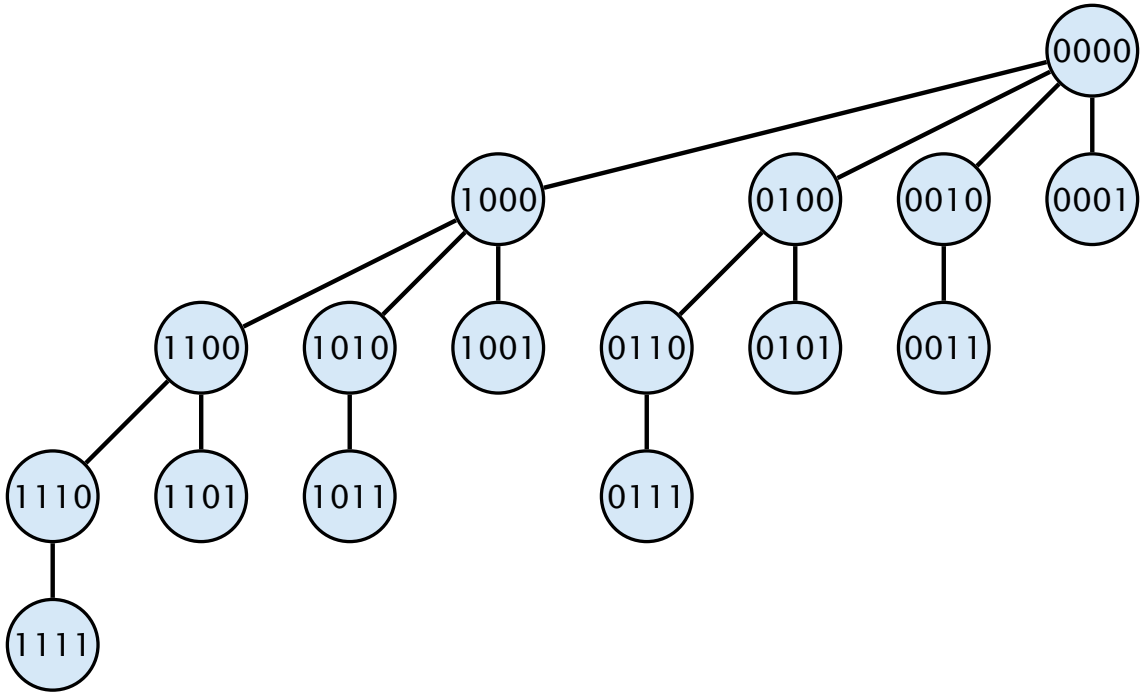

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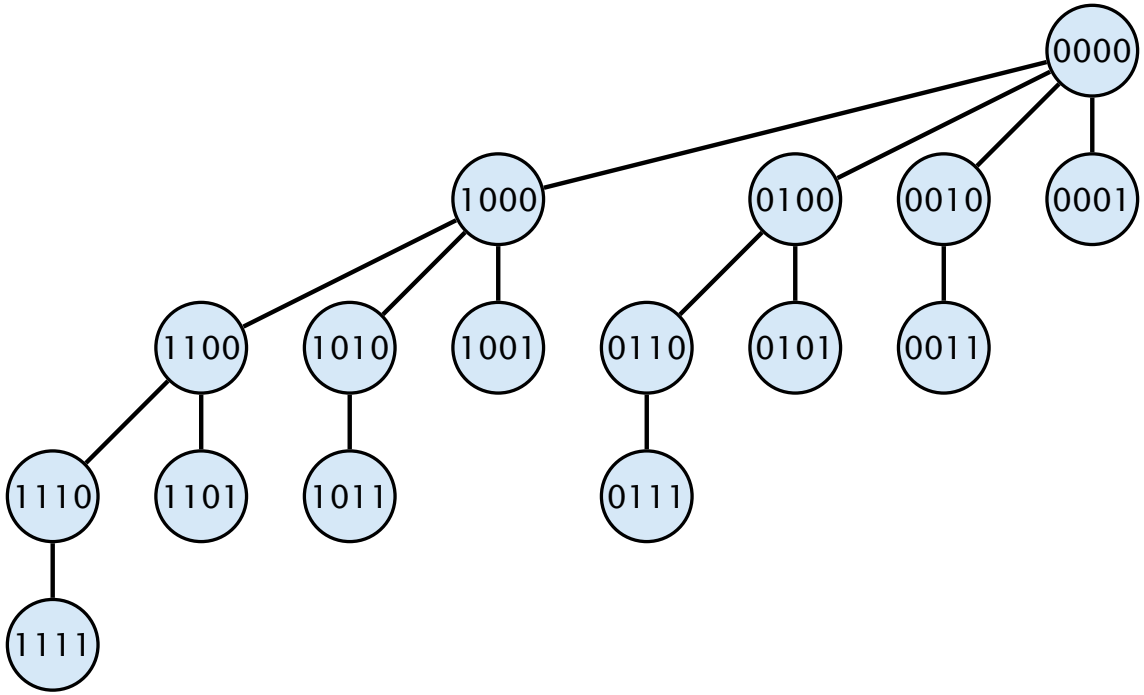

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8.2 Binomial Heaps

How do we implement trees with non-constant degree?

- The children of a node are arranged in a circular linked list.
- A child-pointer points to an arbitrary node within the list.
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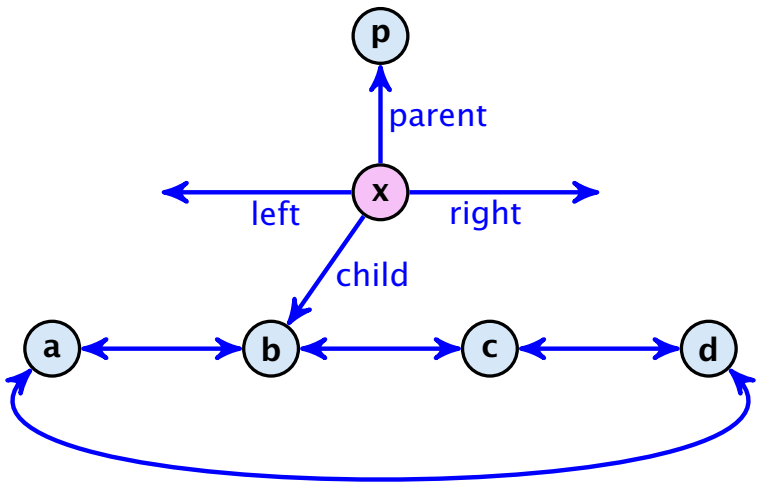
- Pointers $x.\text{left}$ and $x.\text{right}$ point to the left and right sibling of x (if x does not have siblings then $x.\text{left} = x.\text{right} = x$).



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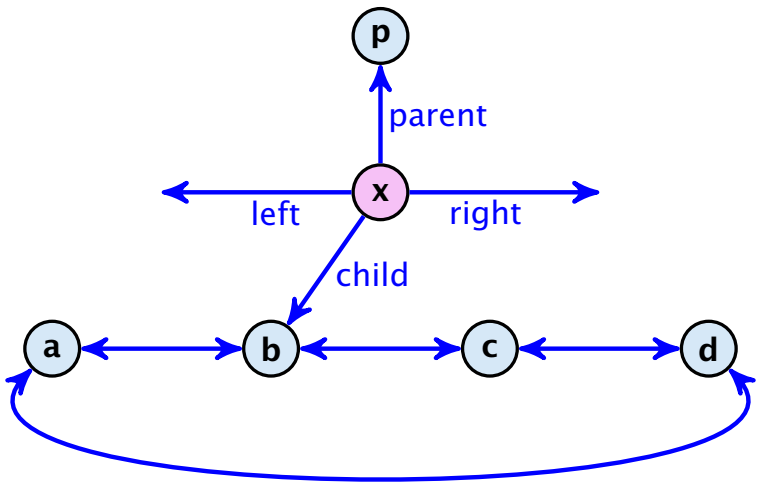

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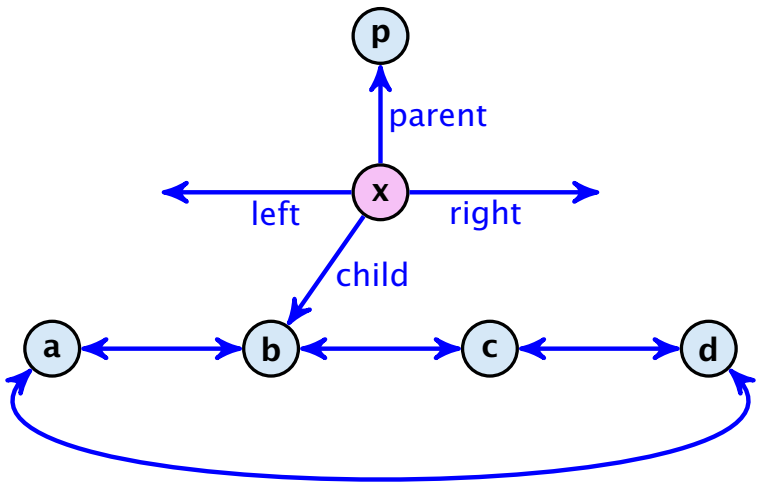

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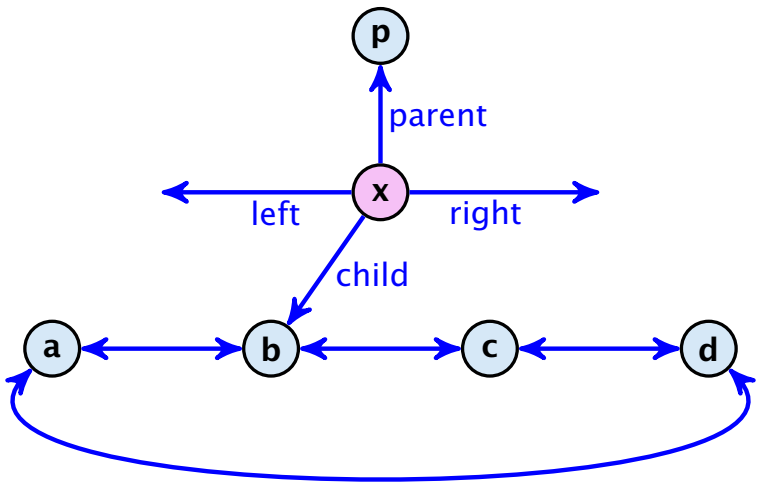
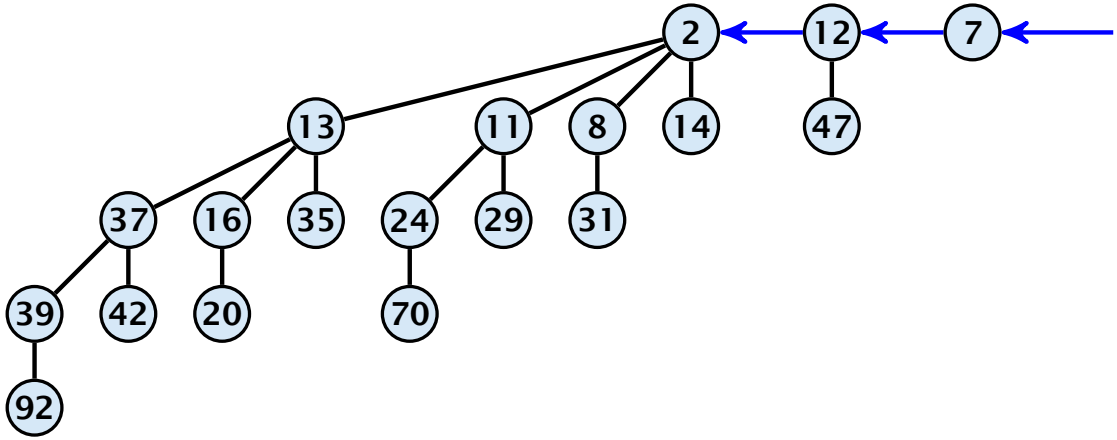


8.2 Binomial Heaps

- Given a pointer to a node x we can splice out the sub-tree rooted at x in constant time.
- We can add a child-tree T to a node x in constant time if we are given a pointer to x and a pointer to the root of T.

Binomial Heap



In a binomial heap the keys are arranged in a collection of binomial trees.

Every tree fulfills the heap-property

There is at most one tree for every dimension/order. For example the above heap contains trees B_0, B_1, and B_4.

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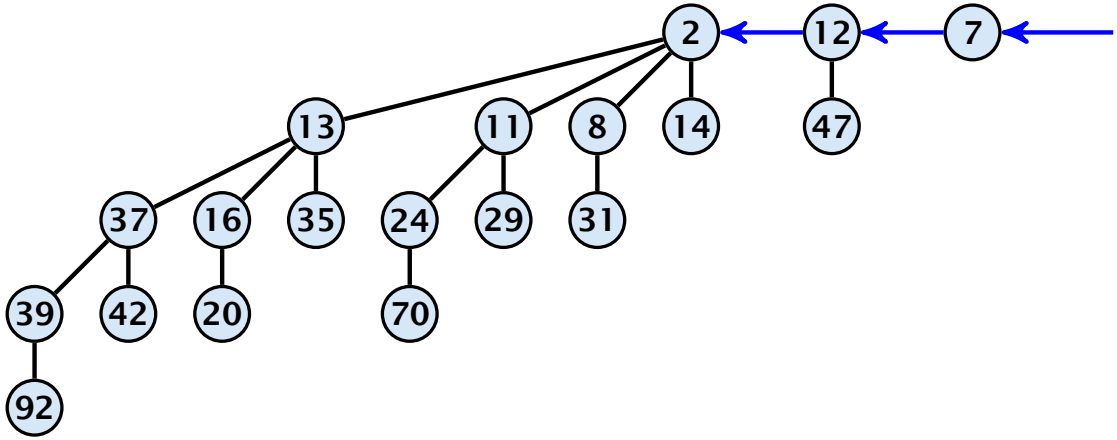

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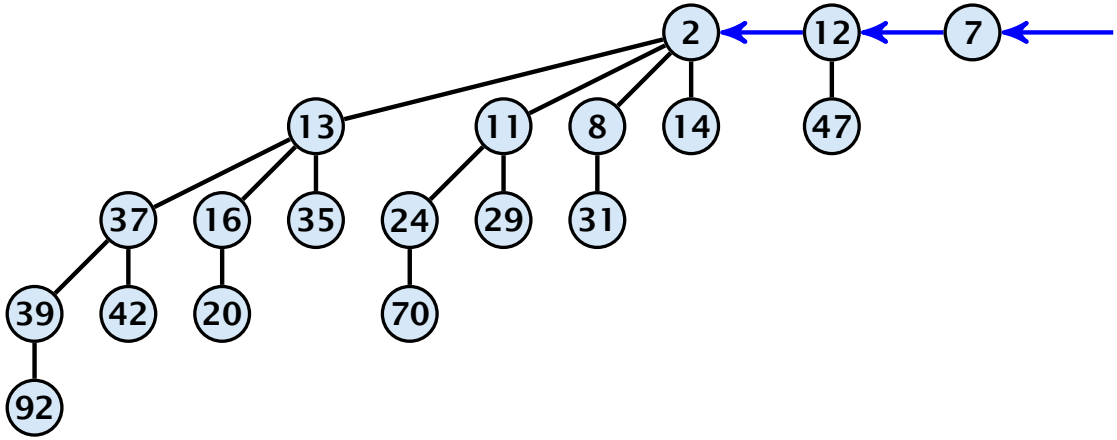

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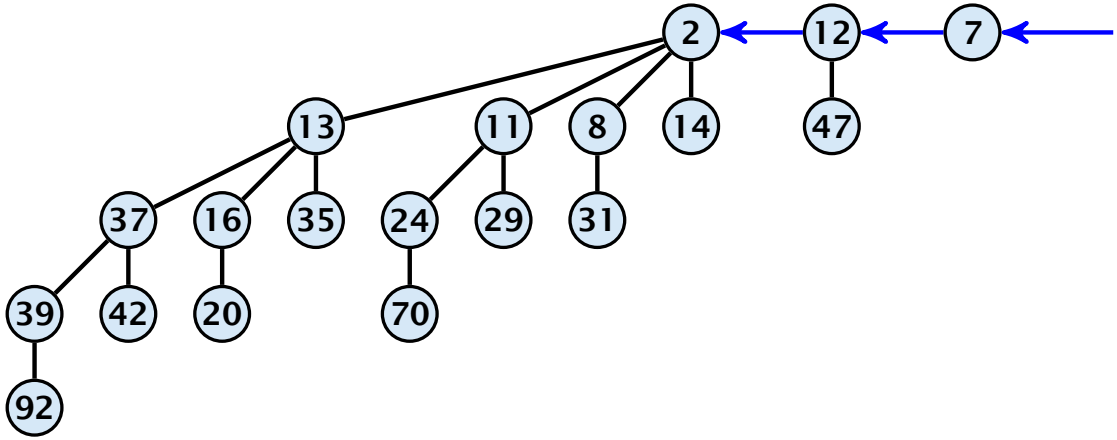

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Binomial Heap: Merge

Given the number n of keys to be stored in a binomial heap we can deduce the binomial trees that will be contained in the collection.

Let $B_{k_1}, B_{k_2}, B_{k_3}, k_i < k_{i+1}$ denote the binomial trees in the collection and recall that every tree may be contained at most once.

Then $n = \sum_i 2^{k_i}$ must hold. But since the k_i are all distinct this means that the k_i define the non-zero bit-positions in the binary representation of n.

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Properties of a heap with n keys:

- Let $n = b_d b_{d-1}, \ldots, b_0$ denote binary representation of n.
- The heap contains tree B_i iff $b_i = 1$.
- Hence, at most $\lfloor \log n \rfloor + 1$ trees.
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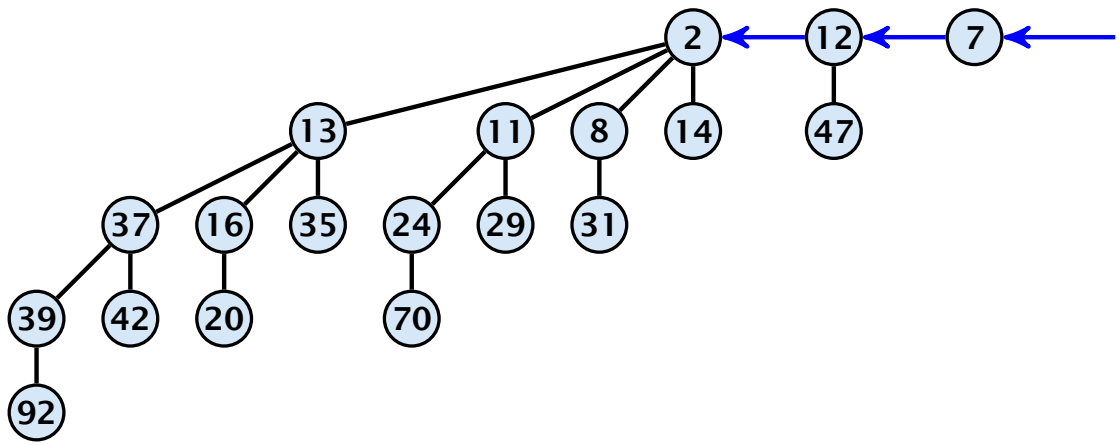

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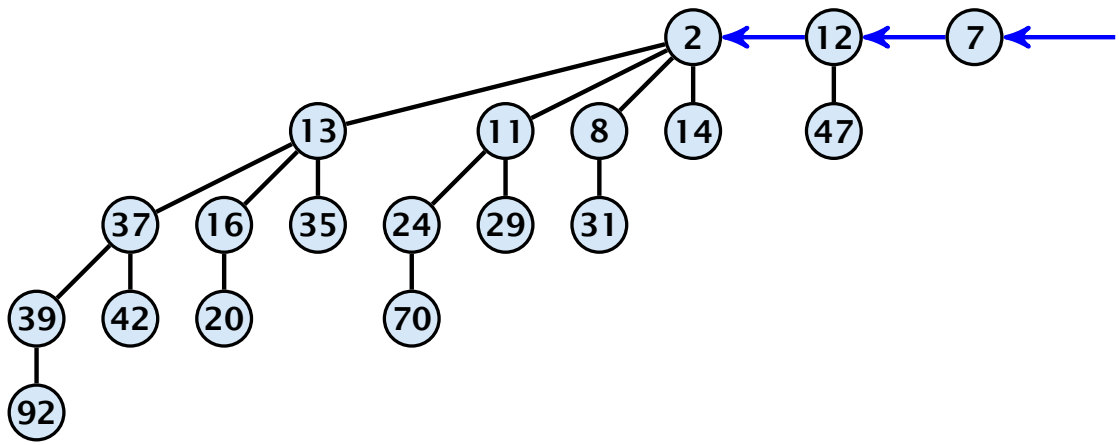

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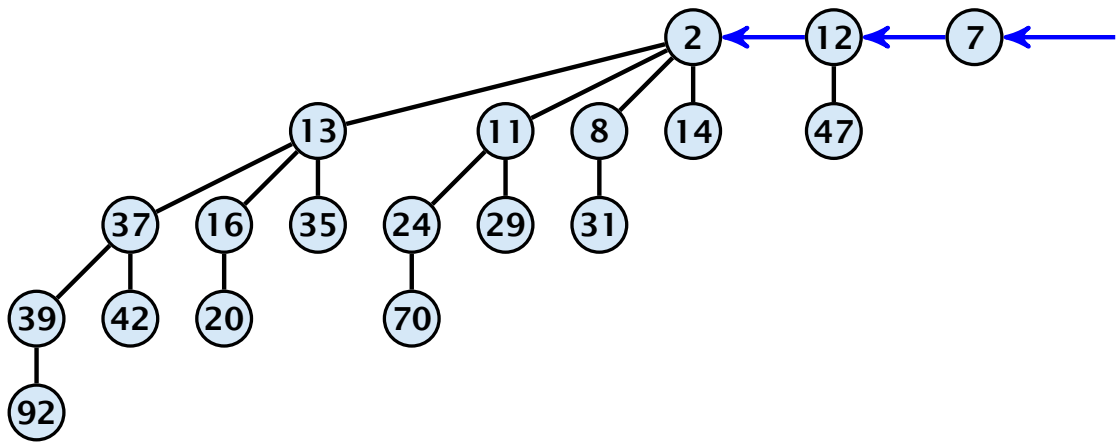

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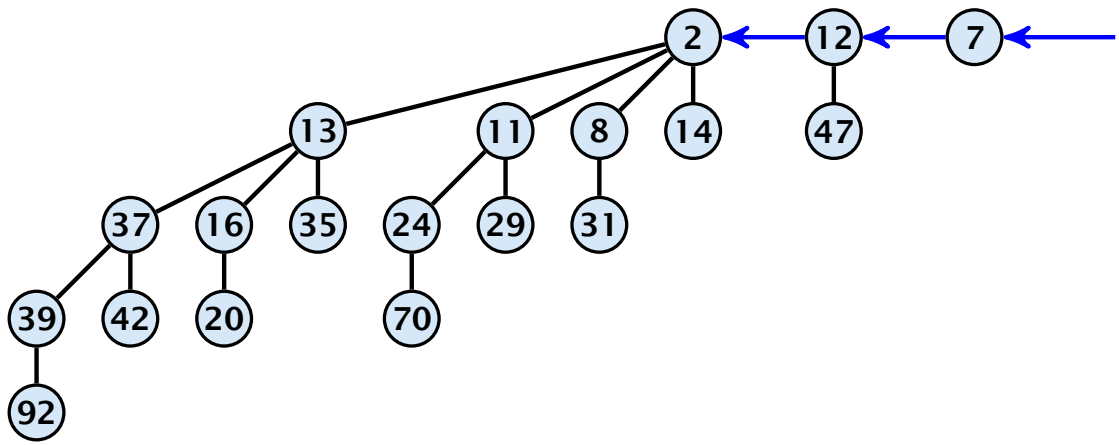

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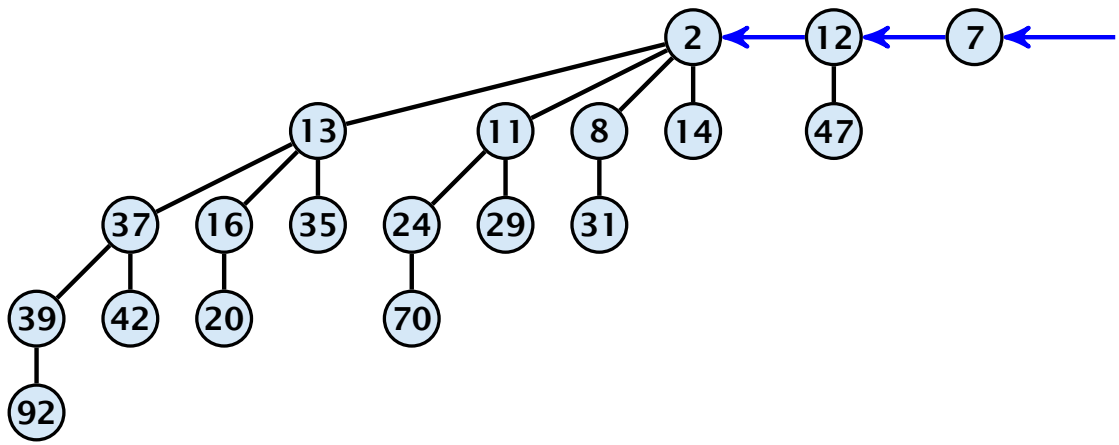

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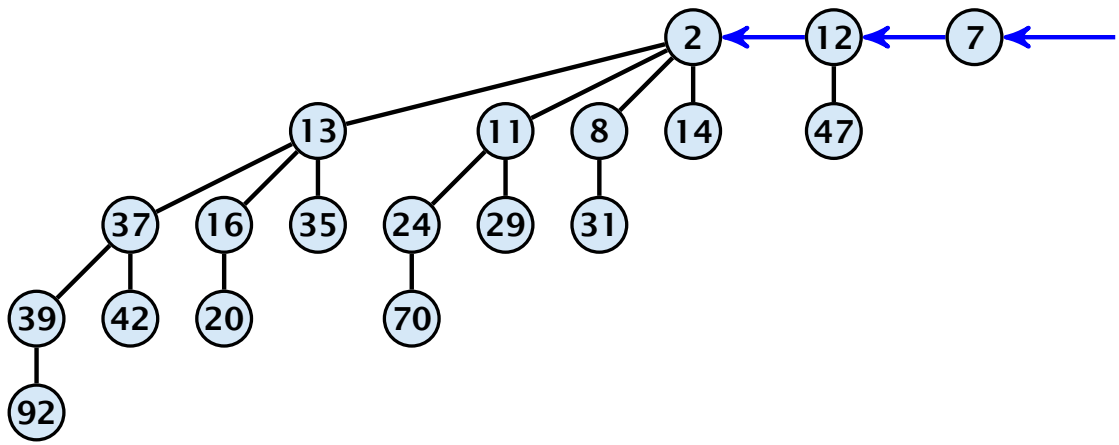

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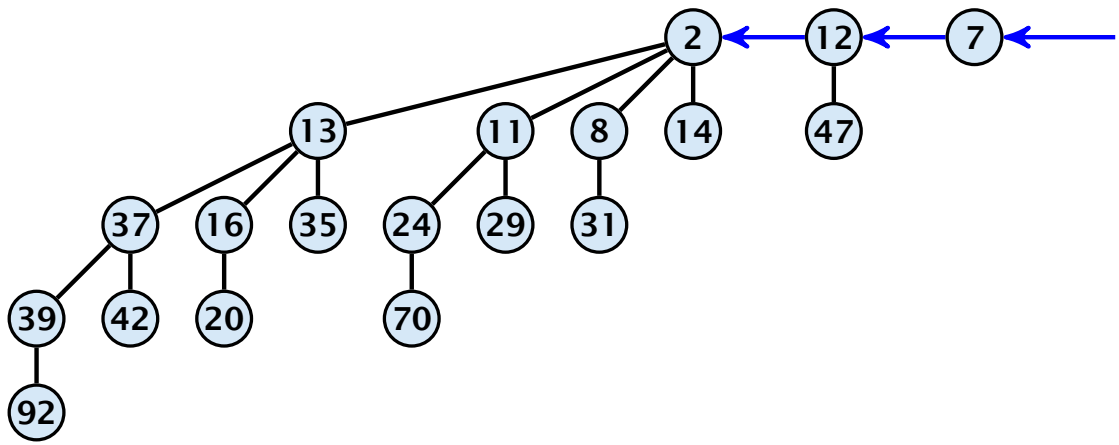

Binomial Heap: Merge

The merge-operation is instrumental for binomial heaps.

A merge is easy if we have two heaps with different binomial trees. We can simply merge the tree-lists.

Otherwise, we cannot do this because the merged heap is not allowed to contain two trees of the same order.

Merging two trees of the same size: Add the tree with larger root-value as a child to the other tree.

For more trees the technique is analogous to binary addition.

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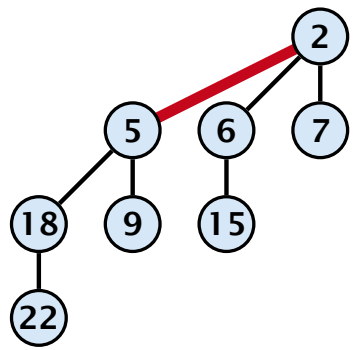

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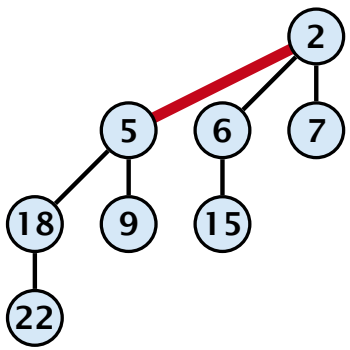

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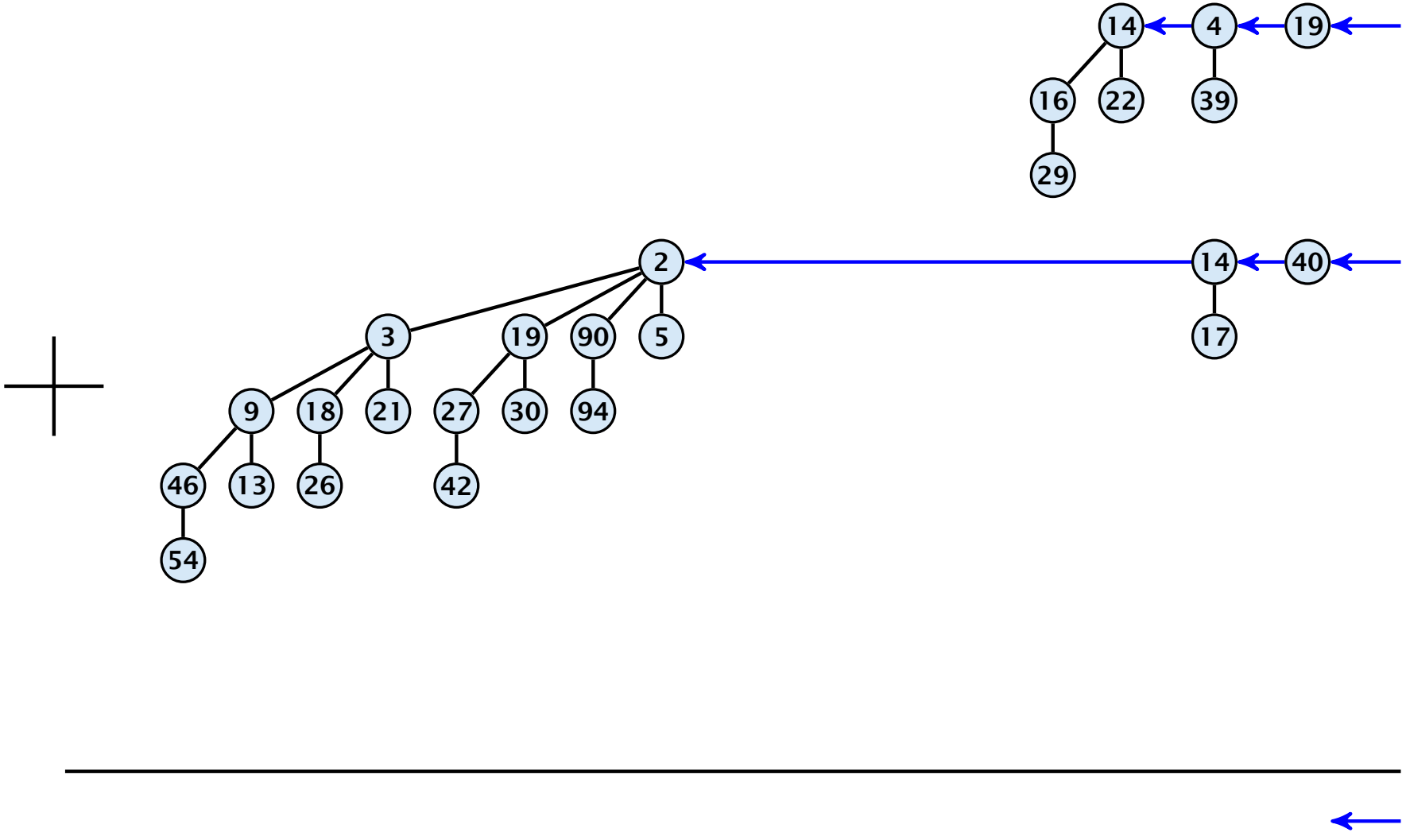
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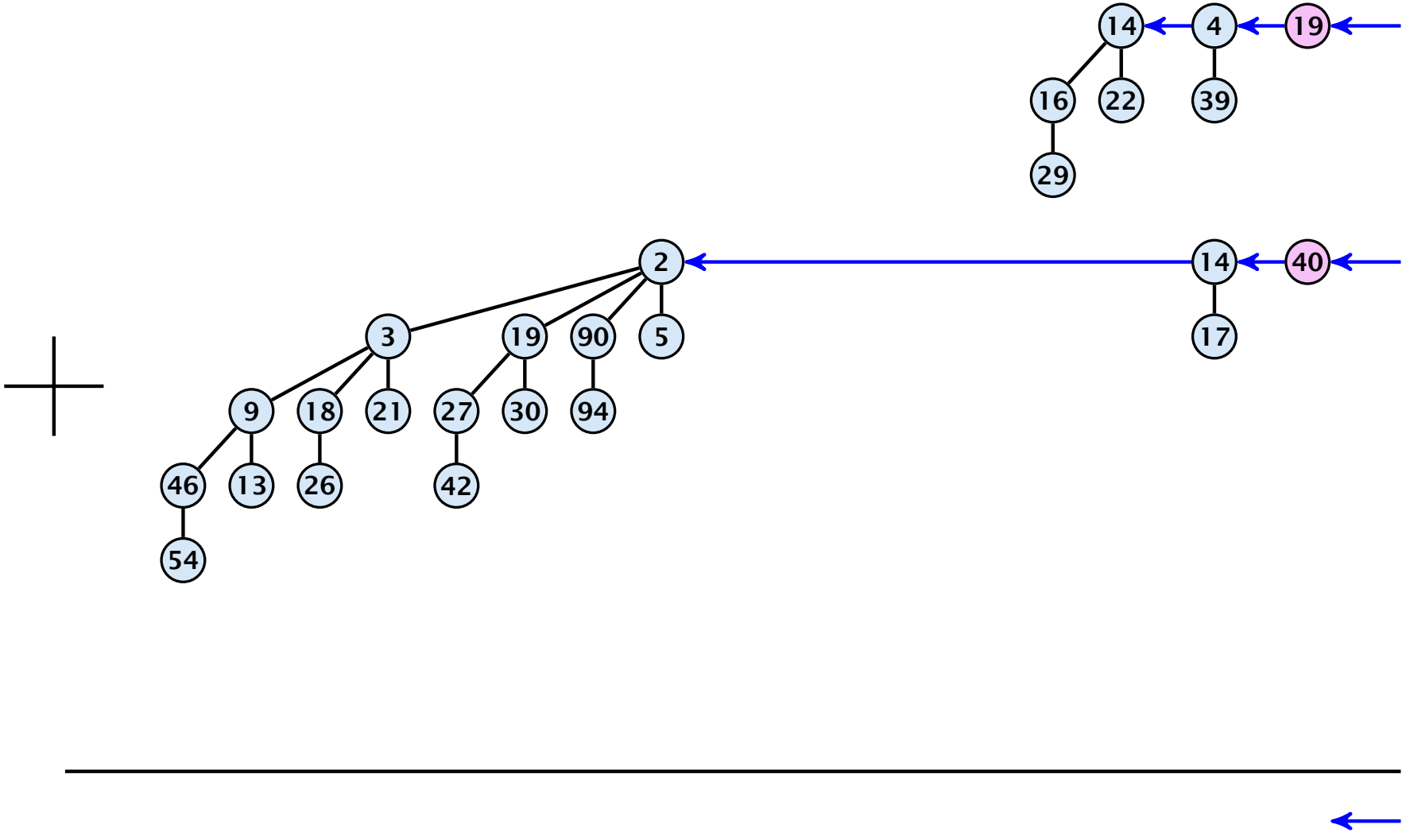
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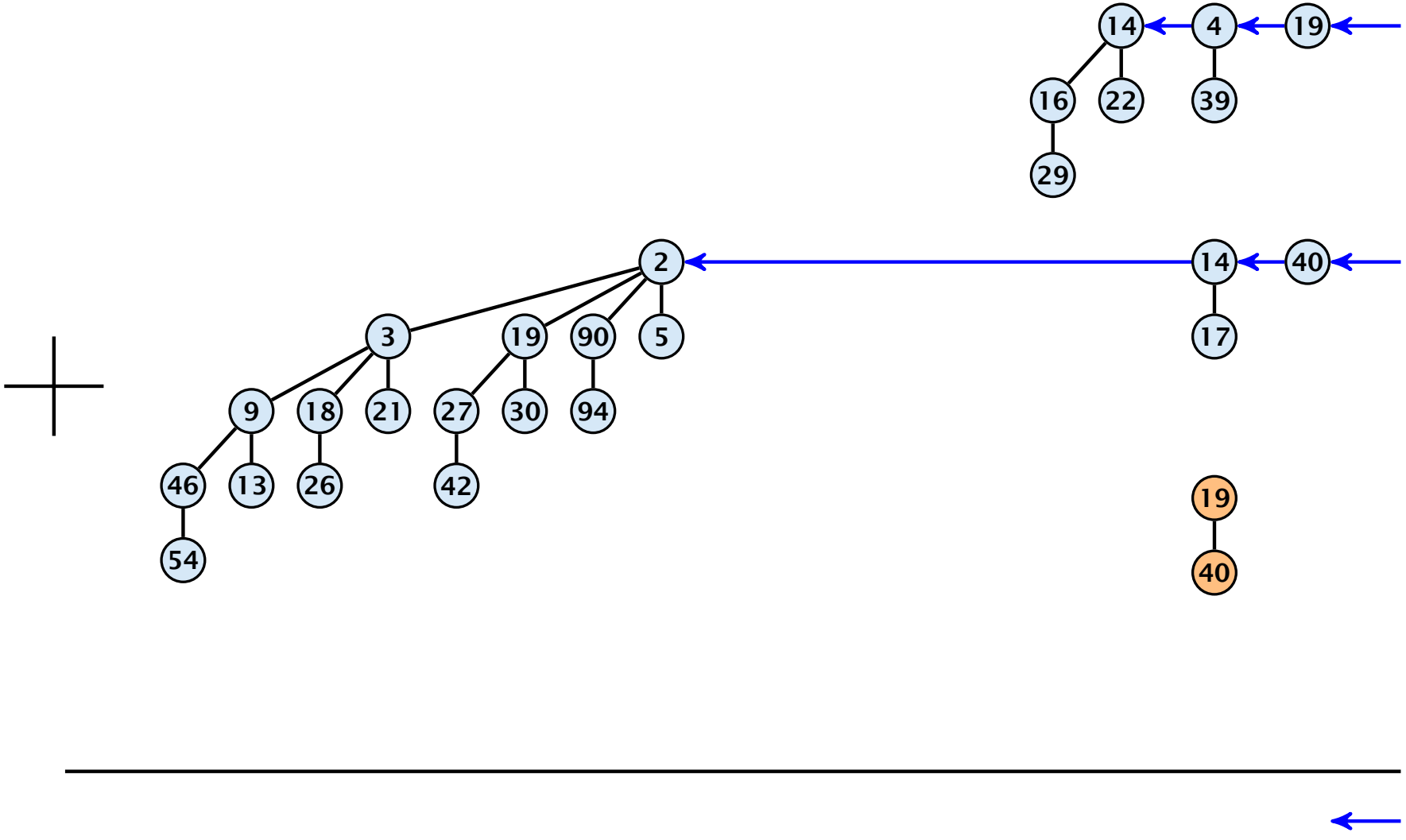
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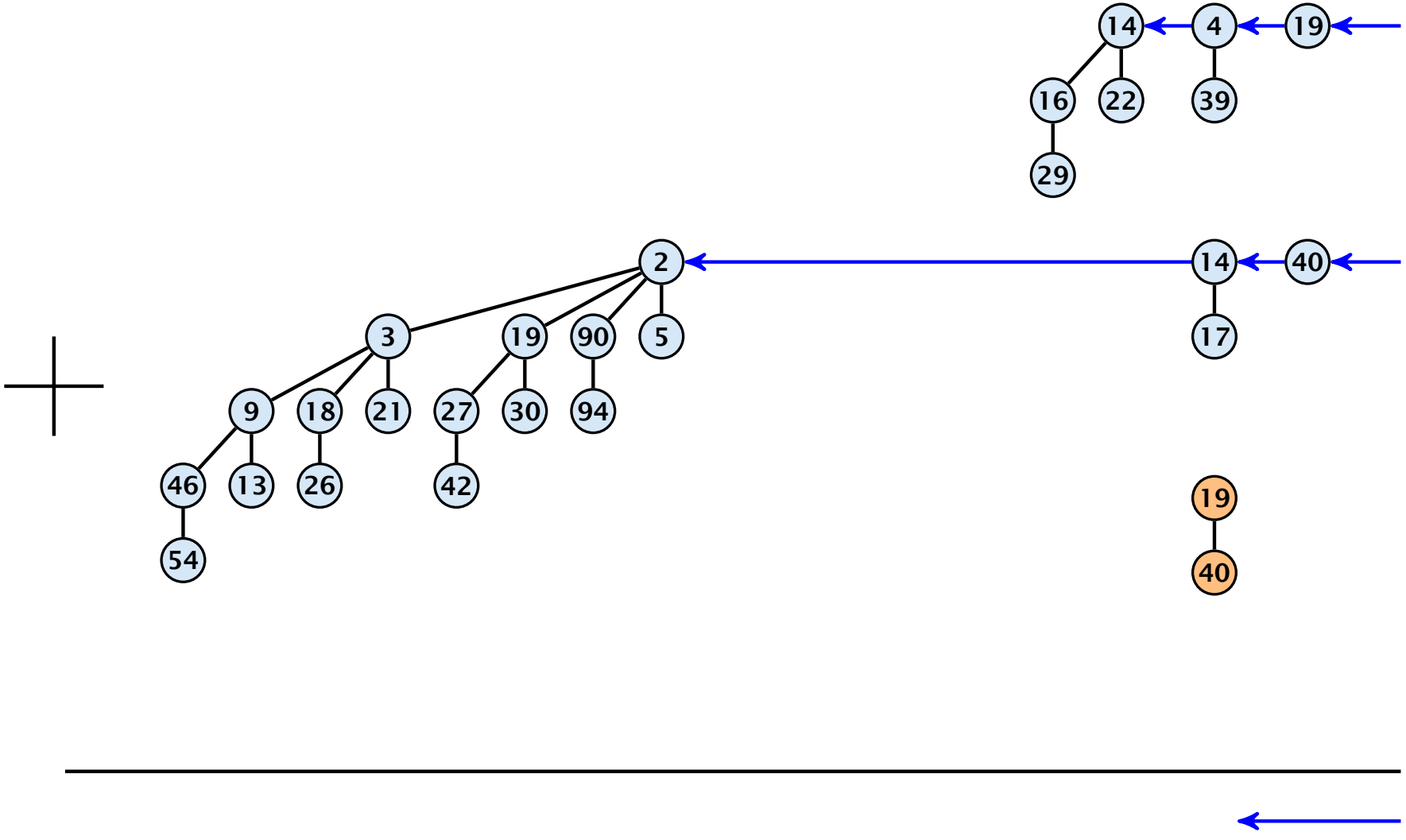
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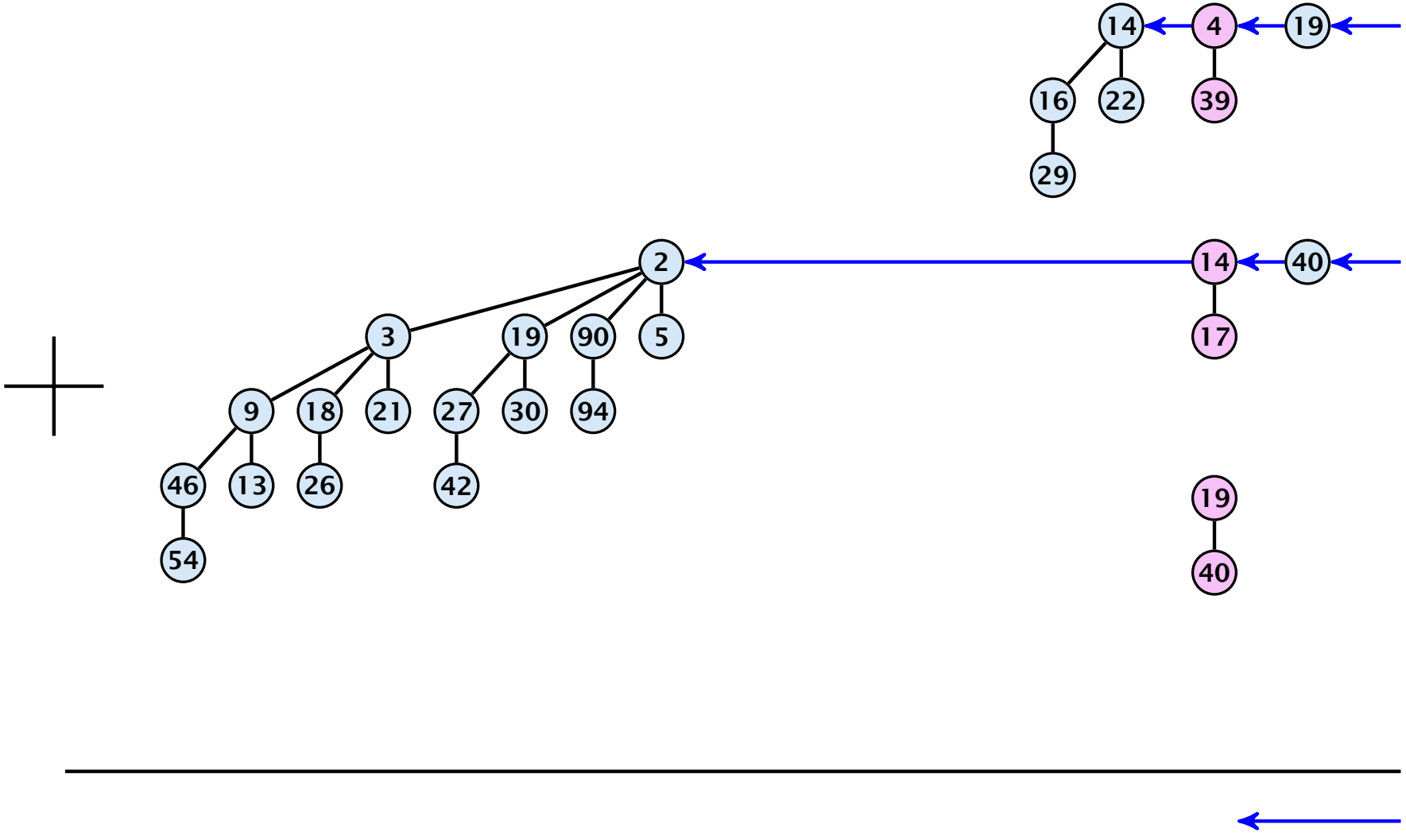
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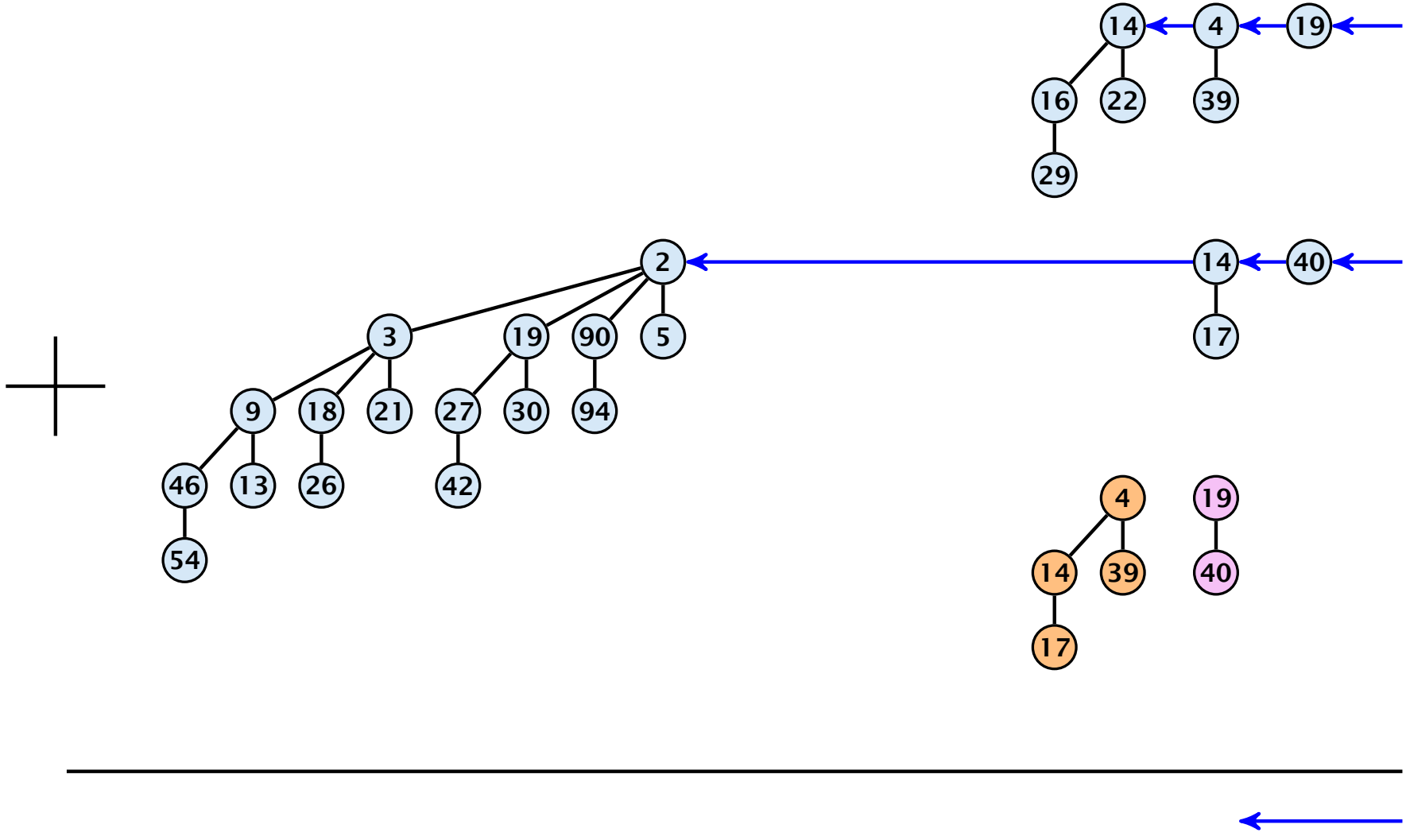
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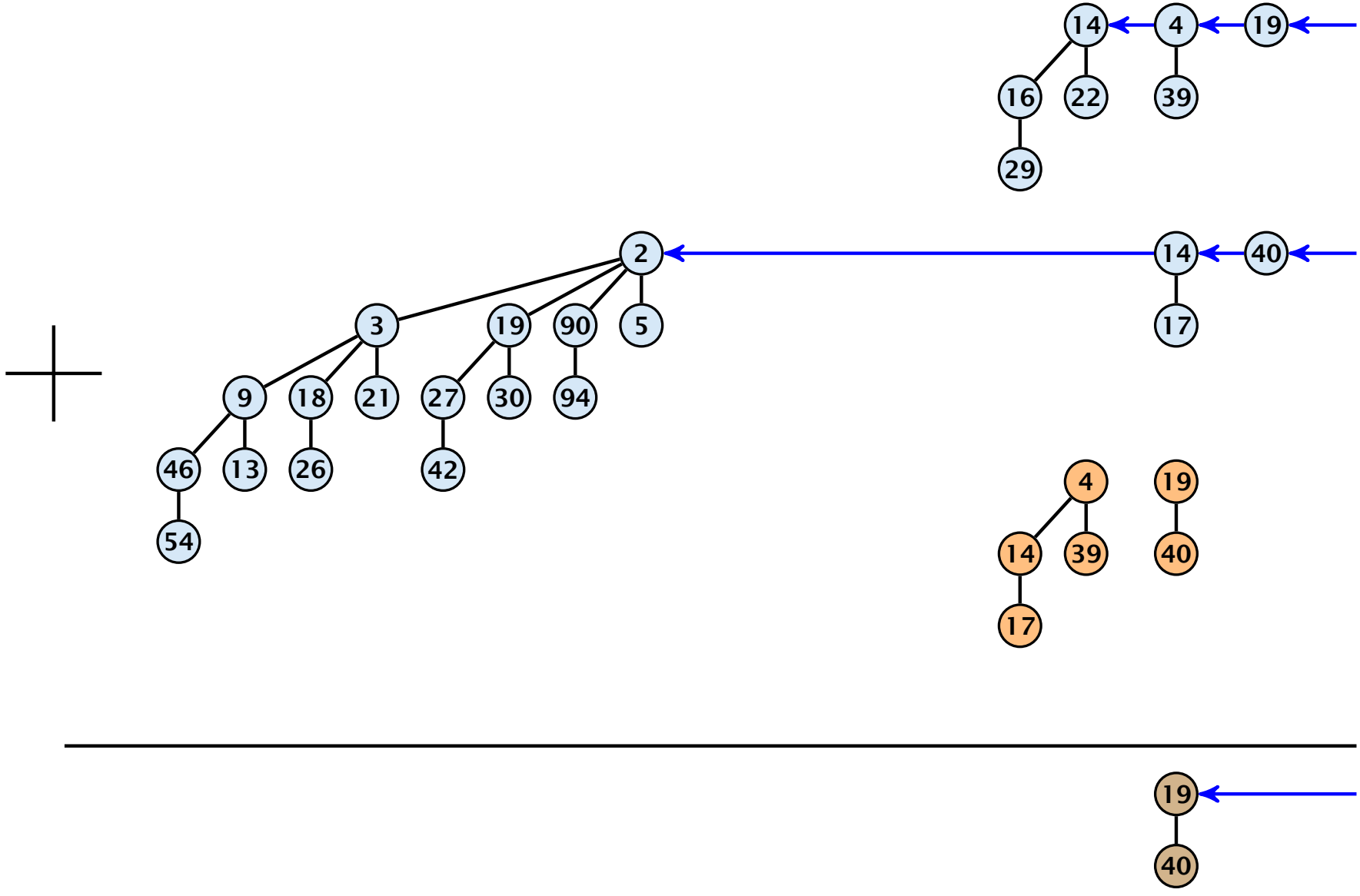
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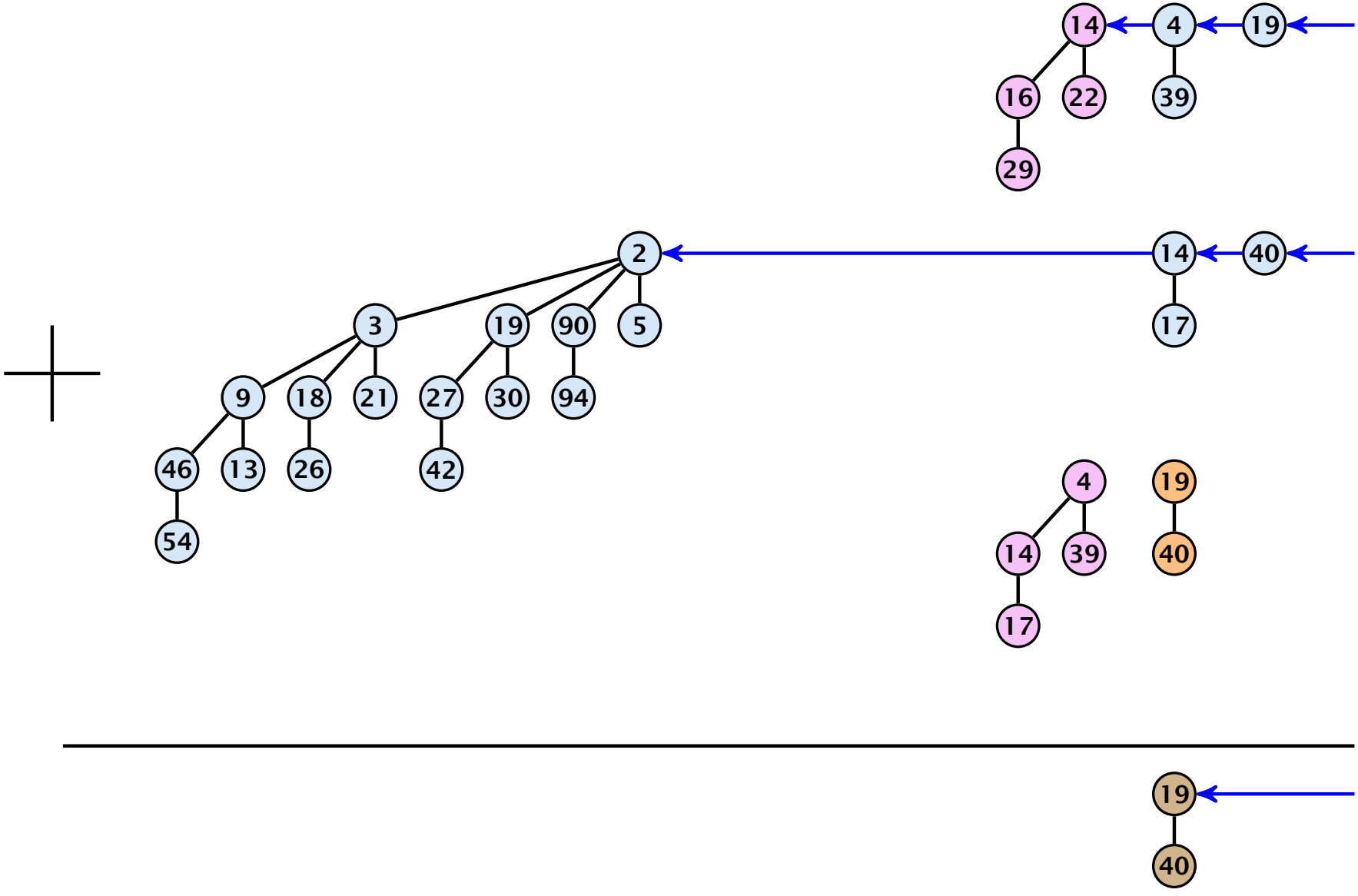
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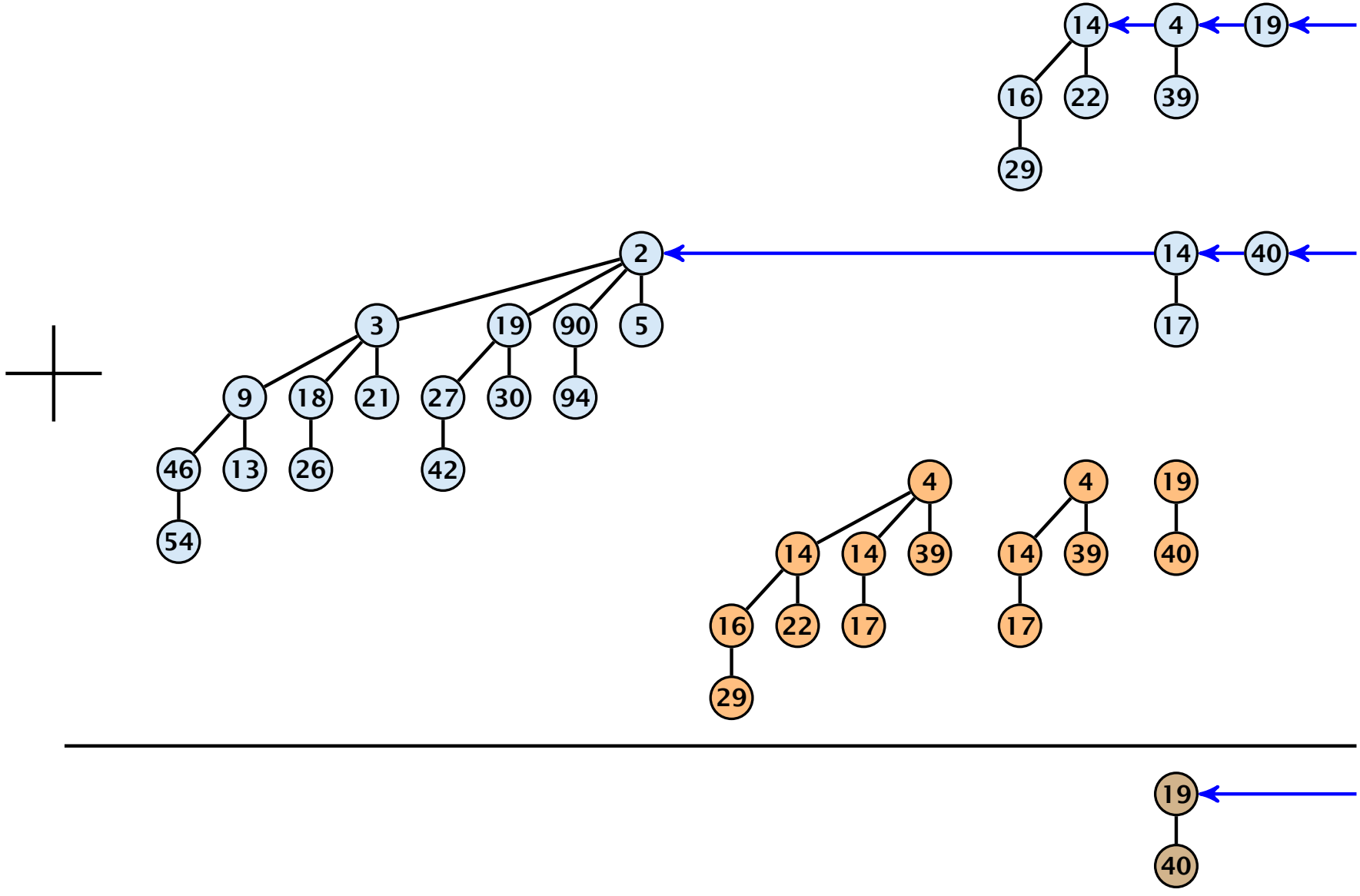
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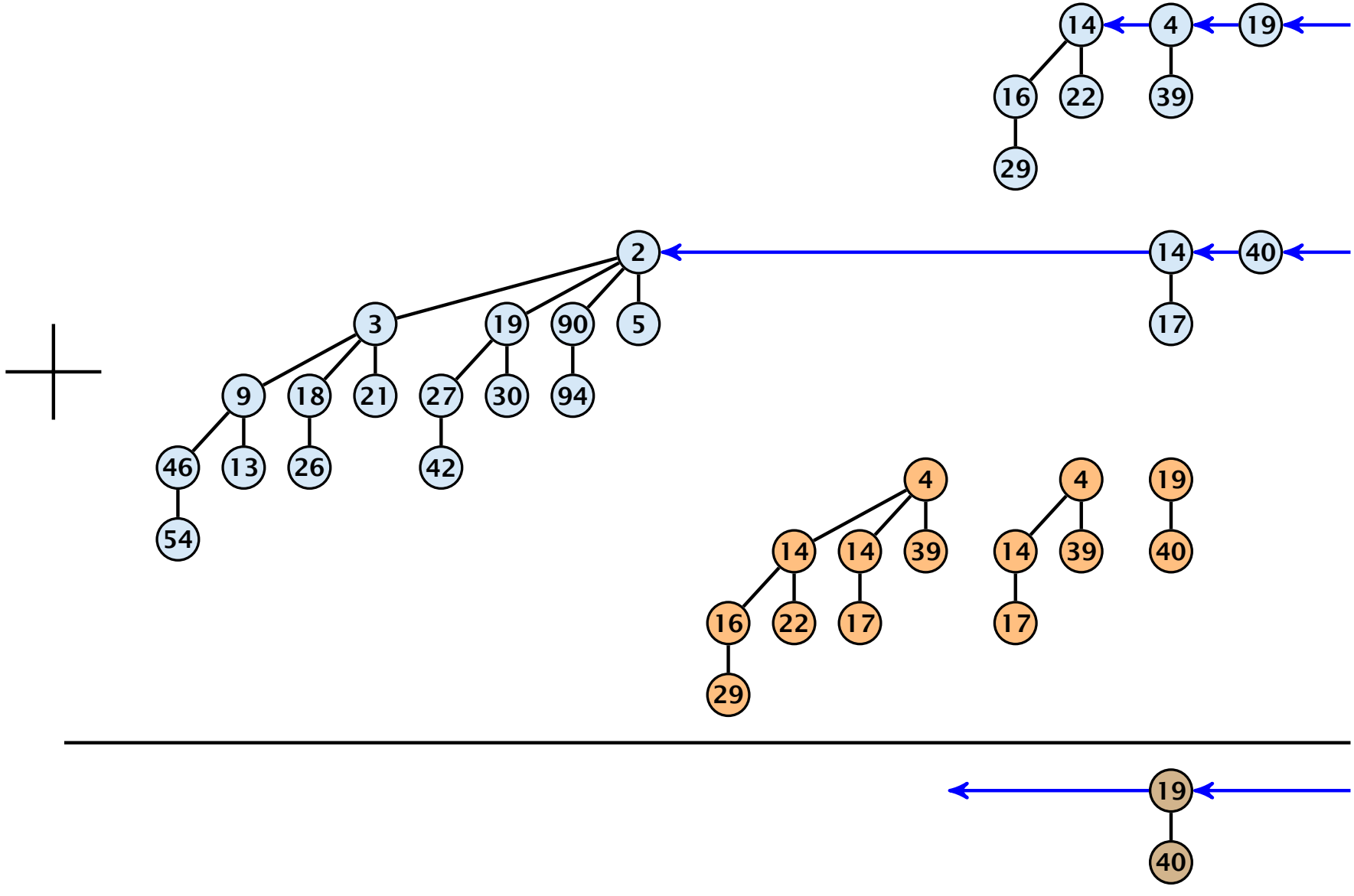
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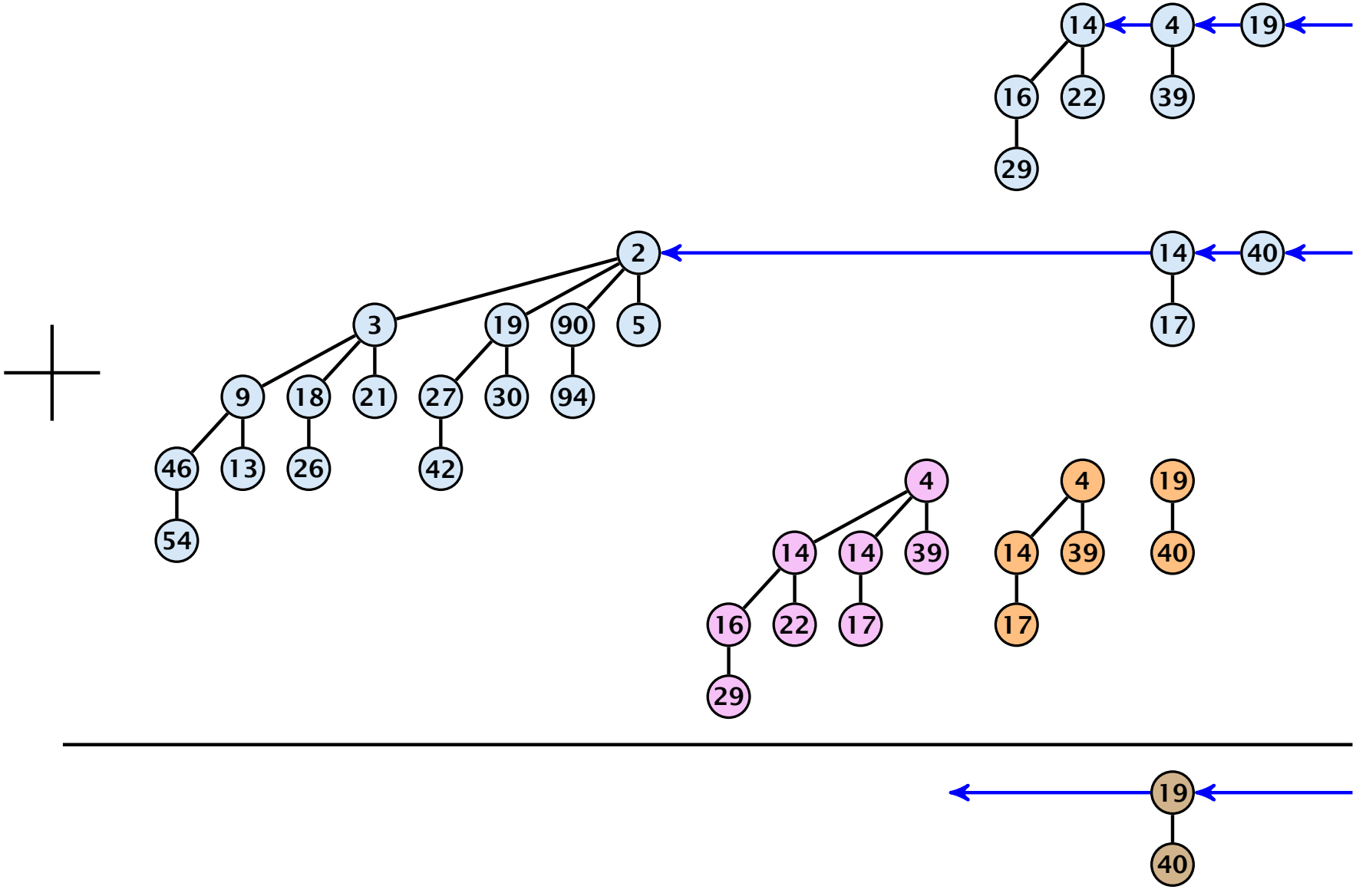
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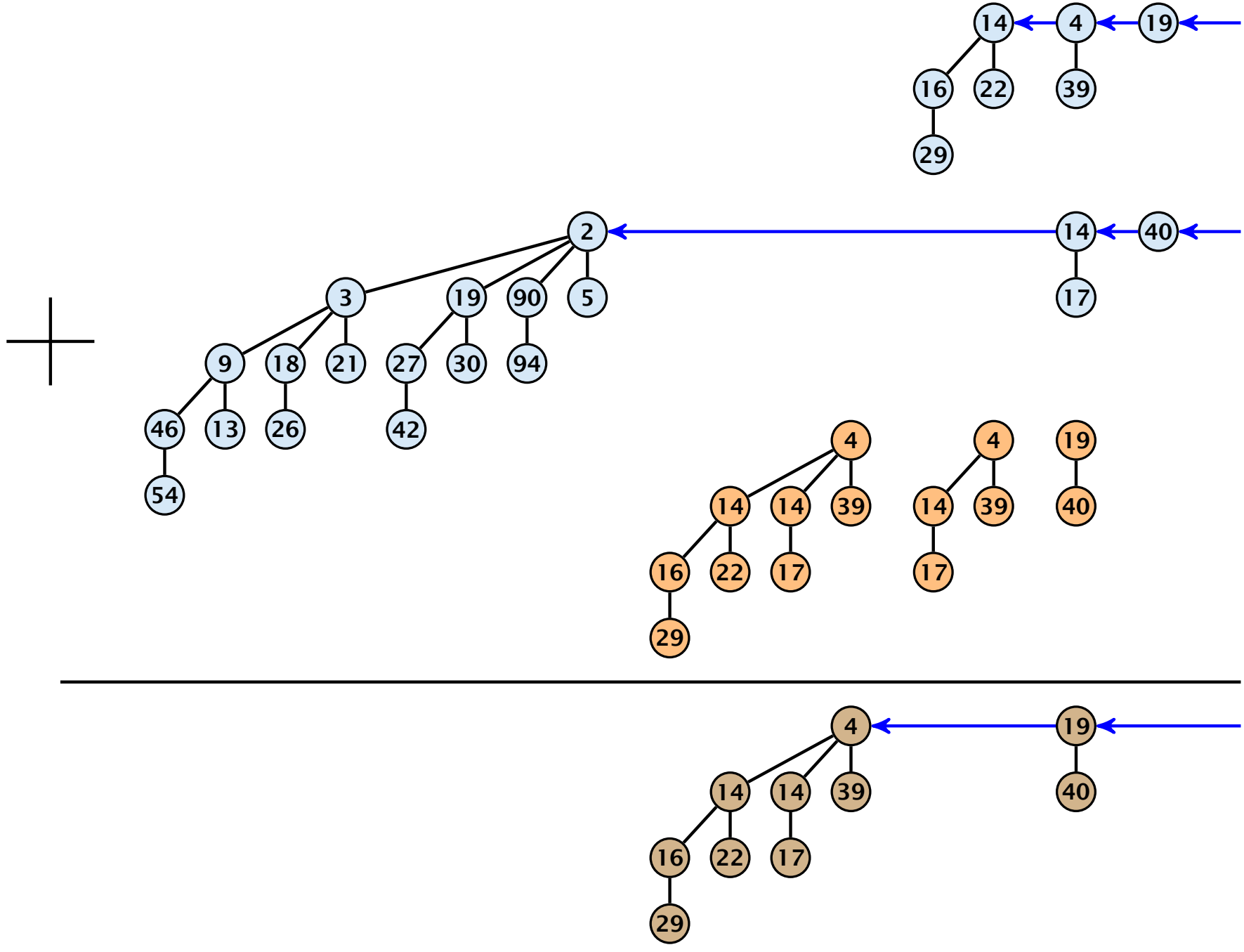
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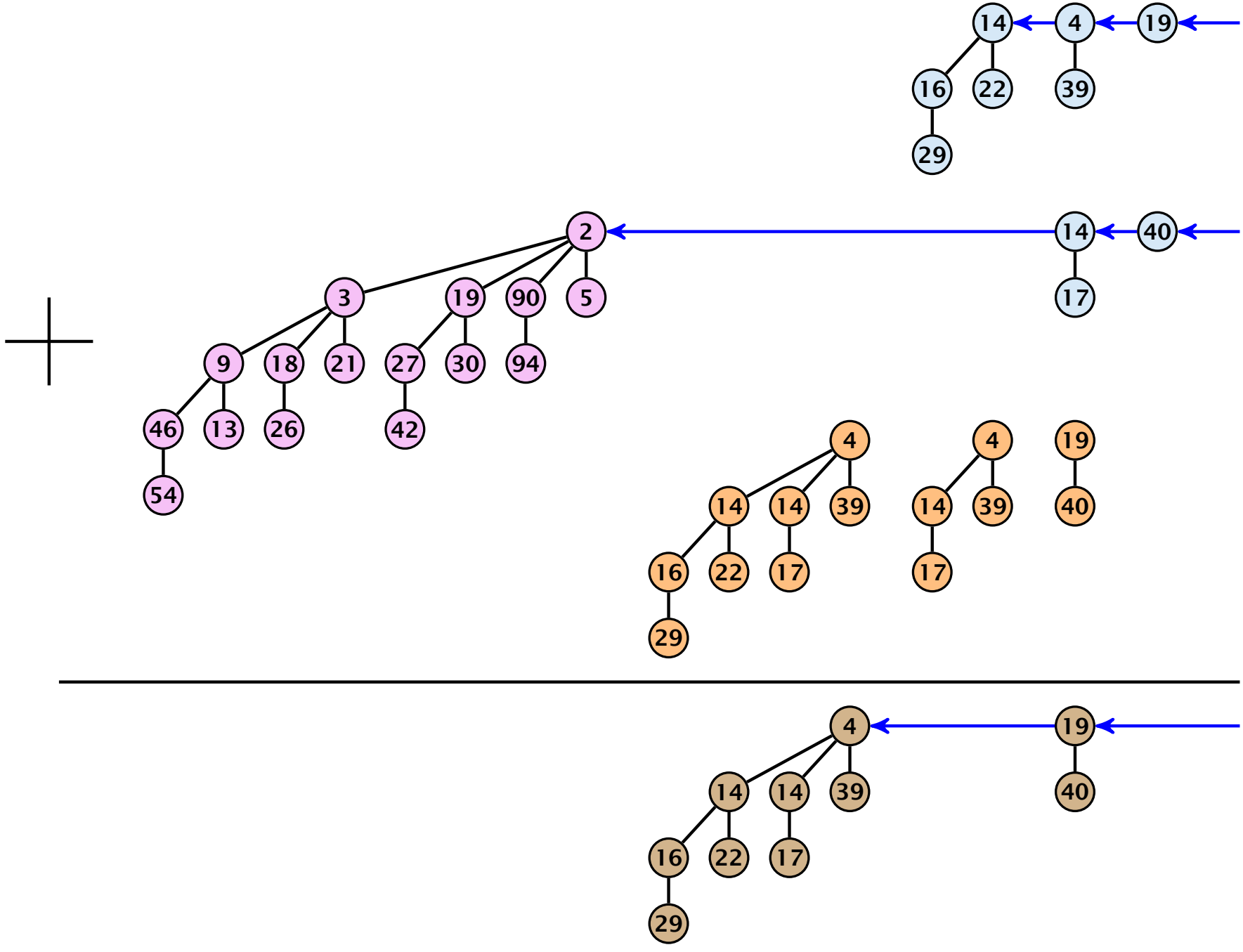
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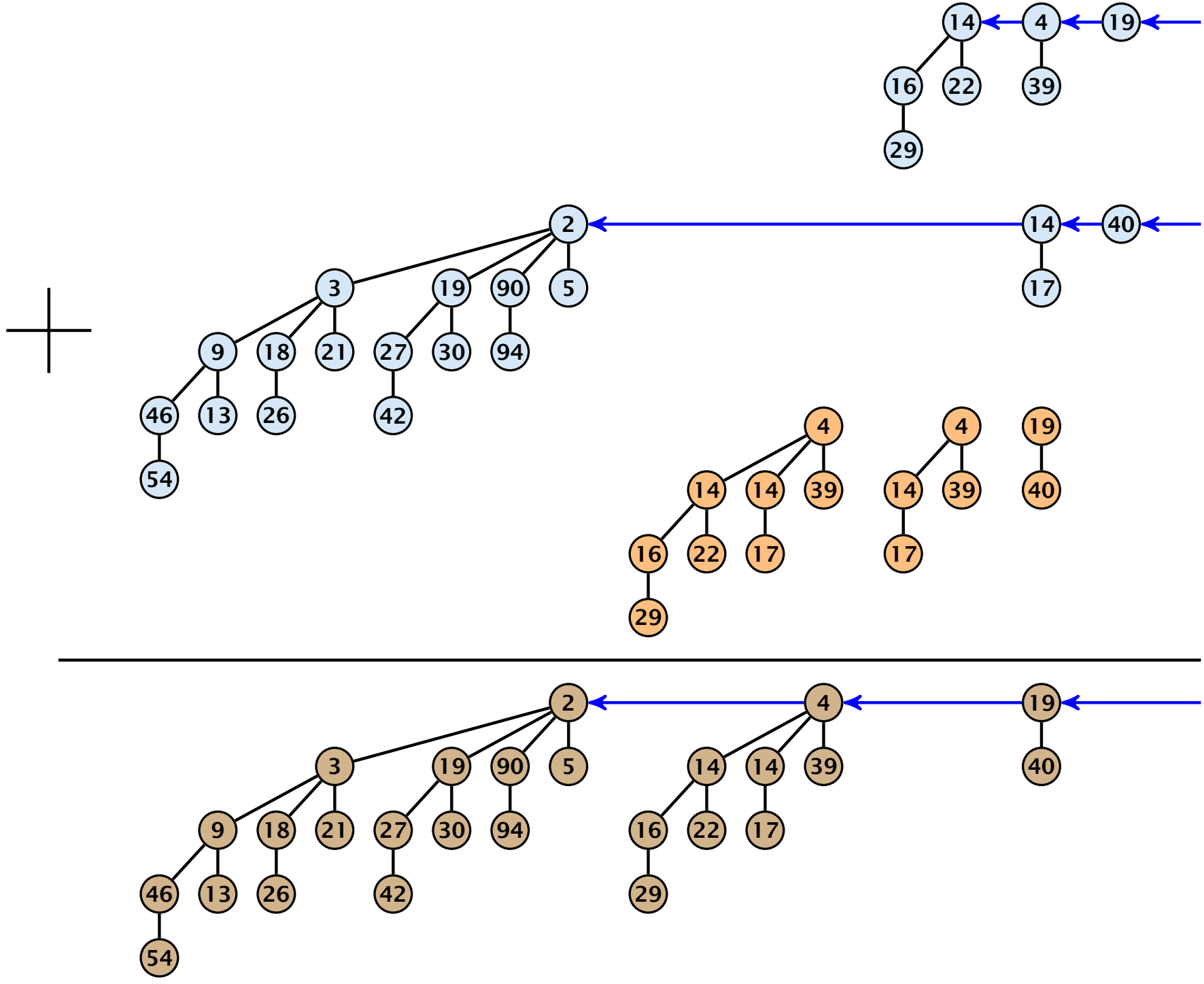
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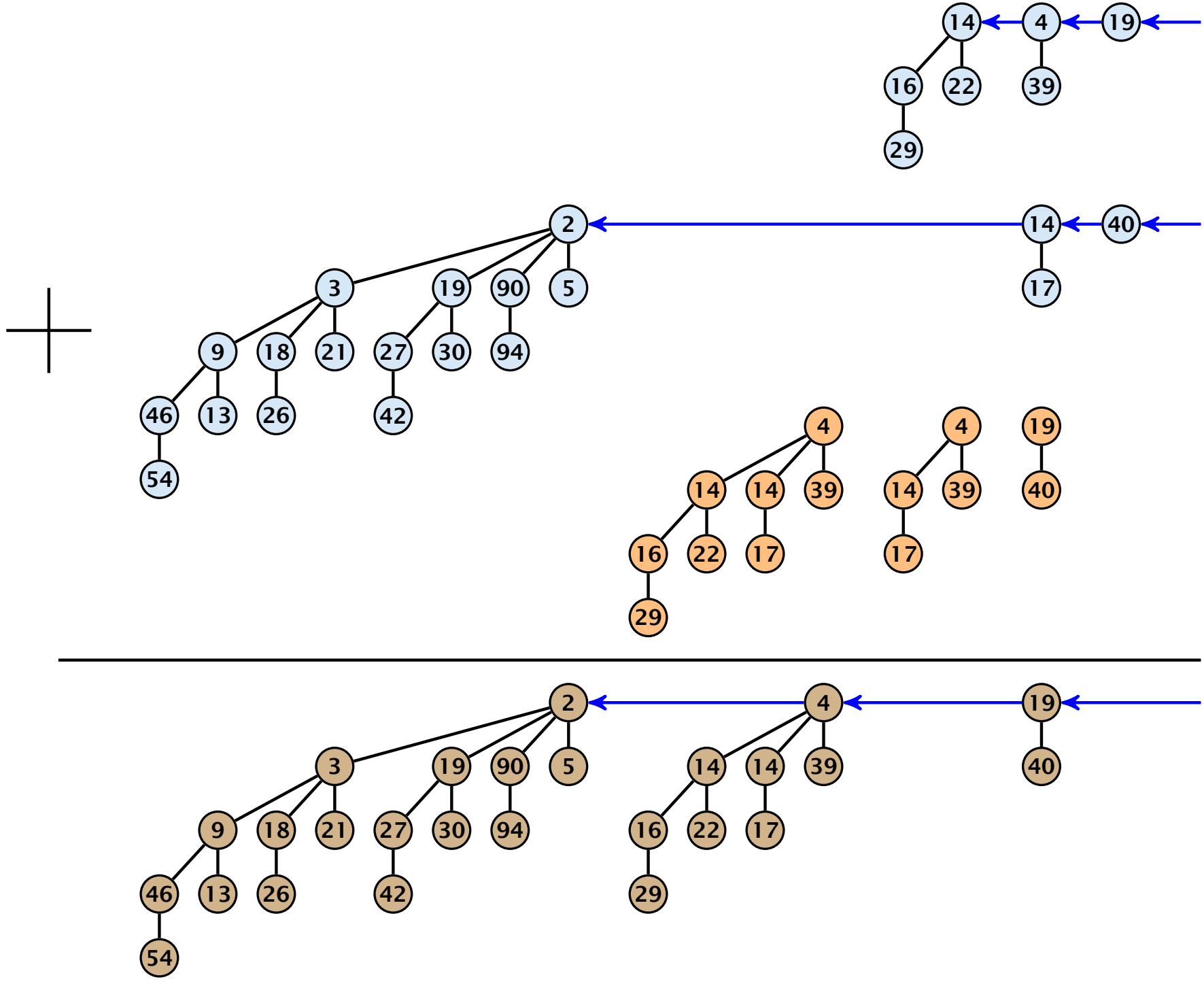
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8.2 Binomial Heaps

$S_1.\mathbf{merge}(S_2)$:

- Analogous to binary addition.
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S. delete-min():

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- Remove the corresponding tree $T_{\min}$ from the heap.
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S. decrease-key(handle h):

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S. delete(handle h):

- Execute S.decrease-key$(h, -\infty)$.
- Execute S.delete-min().
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Amortized Analysis

Definition 34

A data structure with operations $\mathrm{op}_1(), \ldots, \mathrm{op}_k()$ has amortized running times $t_1, \ldots, t_k$ for these operations if the following holds.

Suppose you are given a sequence of operations (starting with an empty data-structure) that operate on at most n elements, and let k_i denote the number of occurences of $\mathrm{op}_i()$ within this sequence. Then the actual running time must be at most $\sum_i k_i \cdot t_i(n)$.

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This means the amortized costs can be used to derive a bound on the total cost.

Example: Stack

Stack

- $S.\,\mathbf{push}()$
- $S.\,\mathbf{pop}()$
- $S.\,\mathbf{multipop}(k)$: removes k items from the stack. If the stack currently contains less than k items it empties the stack.
- The user has to ensure that pop and multipop do not generate an underflow.

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Use potential function $\Phi(S) =$ number of elements on the stack.

Amortized cost:

- PUSH: push(x) cost
- POP: pop() cost
- MULTIPOP: multipop(k) cost

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Example: Binary Counter

Incrementing a binary counter:
Consider a computational model where each bit-operation costs one time-unit.

Incrementing an n-bit binary counter may require to examine n-bits, and maybe change them.

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- Changing bit from 0 to 1: cost 1.
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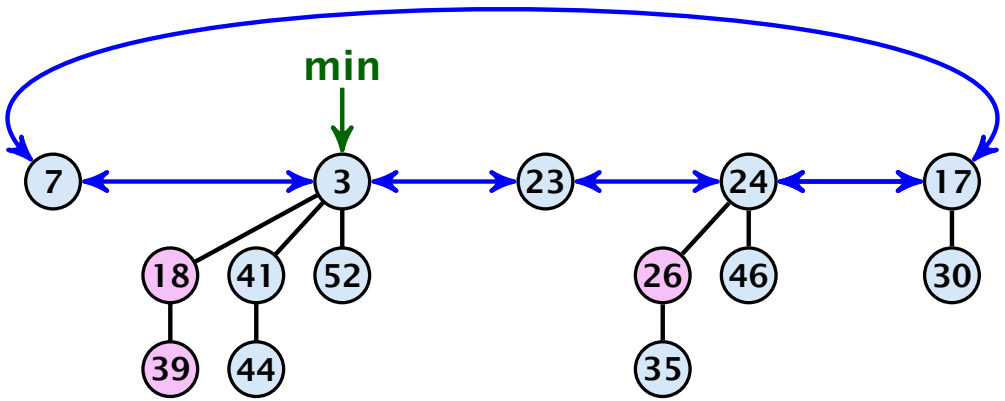
8.3 Fibonacci Heaps

Collection of trees that fulfill the heap property.

Structure is much more relaxed than binomial heaps.



8.3 Fibonacci Heaps

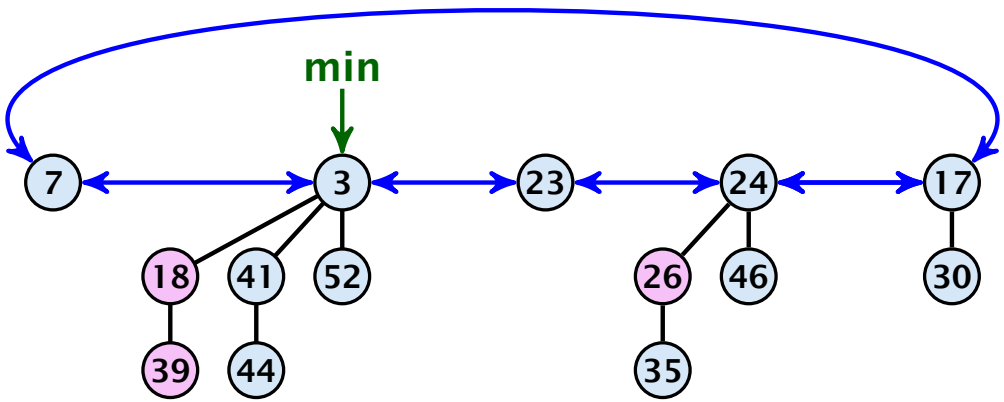
Additional implementation details:

- Every node x stores its degree in a field $x.\mathrm{degree}$. Note that this can be updated in constant time when adding a child to x.
- Every node stores a boolean value $x.\mathrm{marked}$ that specifies whether x is marked or not.

8.3 Fibonacci Heaps

The potential function:

- $t(S)$ denotes the number of trees in the heap.
- $m(S)$ denotes the number of marked nodes.
- We use the potential function $\Phi(S) = t(S) + 2m(S)$.



The potential is $\Phi(S) = 5 + 2 \cdot 3 = 11$.

8.3 Fibonacci Heaps

We assume that one unit of potential can pay for a constant amount of work, where the constant is chosen “big enough” (to take care of the constants that occur).

To make this more explicit we use c to denote the amount of work that a unit of potential can pay for.

8.3 Fibonacci Heaps

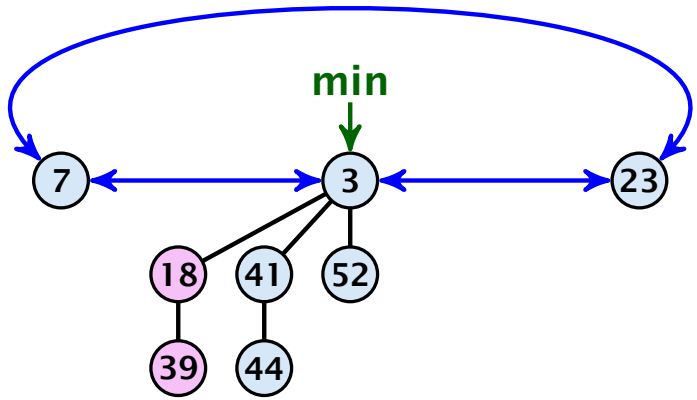
S.minimum()

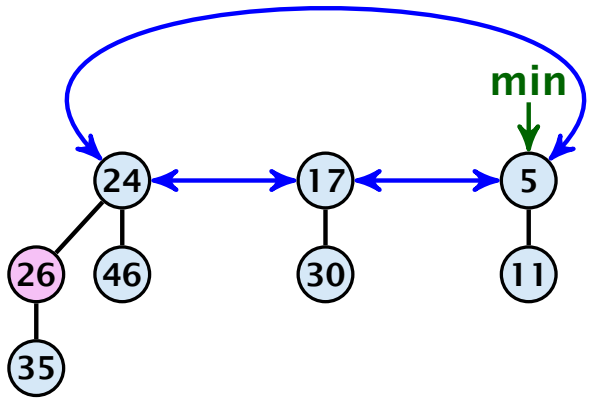
- Access through the min-pointer.
- Actual cost $\mathcal{O}(1)$.
- No change in potential.
- Amortized cost $\mathcal{O}(1)$.

8.3 Fibonacci Heaps

$S.\mathrm{merge}(S')$

- Merge the root lists.
- Adjust the min-pointer

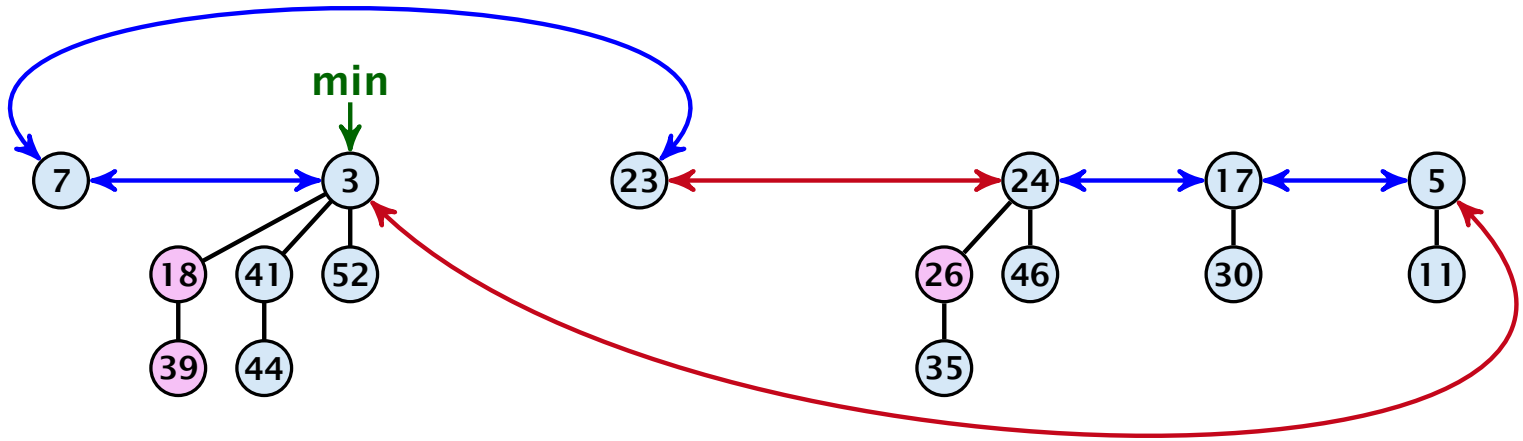




8.3 Fibonacci Heaps

$S.\mathrm{merge}(S')$

- Merge the root lists.
- Adjust the min-pointer



Running time:

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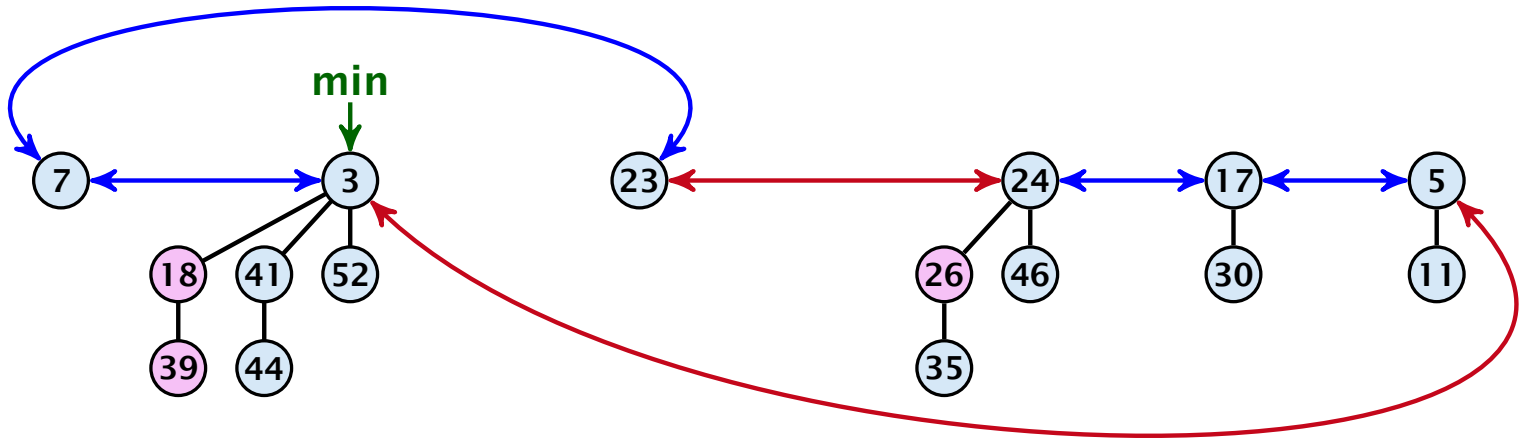

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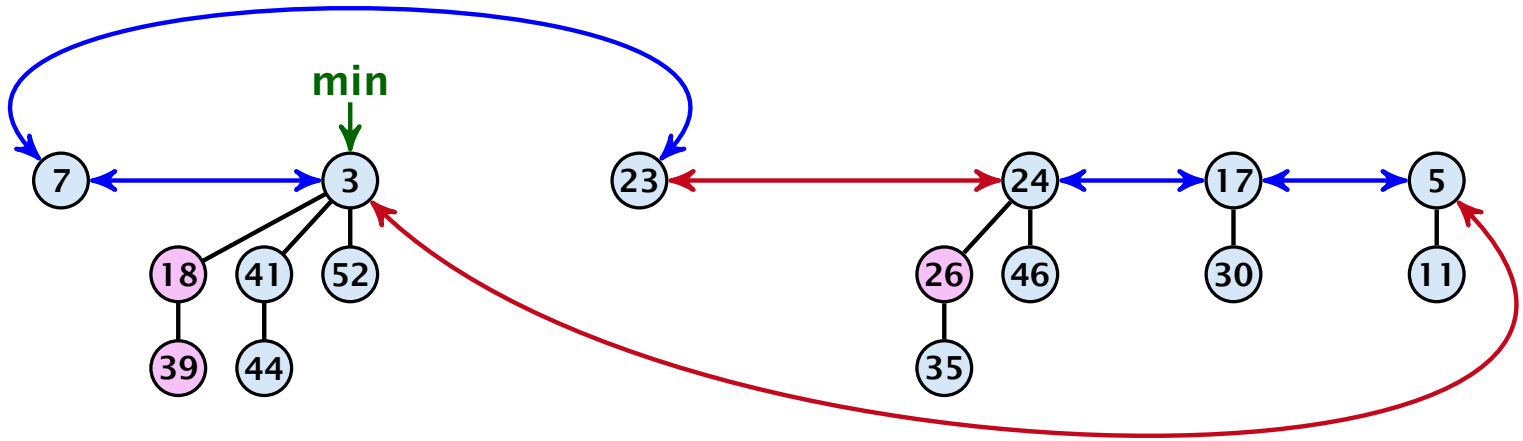


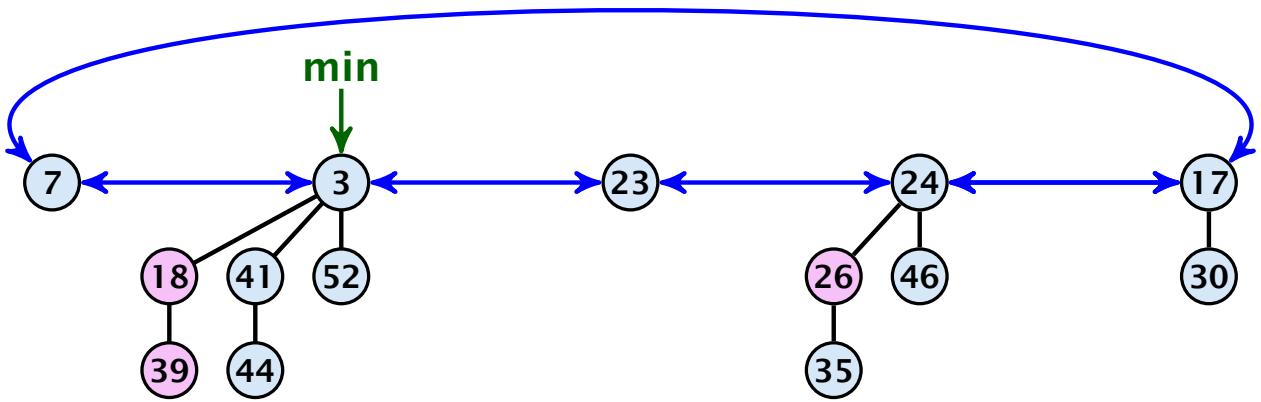
Running time:

- Actual cost $\mathcal{O}(1)$.
- No change in potential.
- Hence, amortized cost is $\mathcal{O}(1)$.

8.3 Fibonacci Heaps

S. insert(x)

- Create a new tree containing x.
- Insert x into the root-list.
- Update min-pointer, if necessary.



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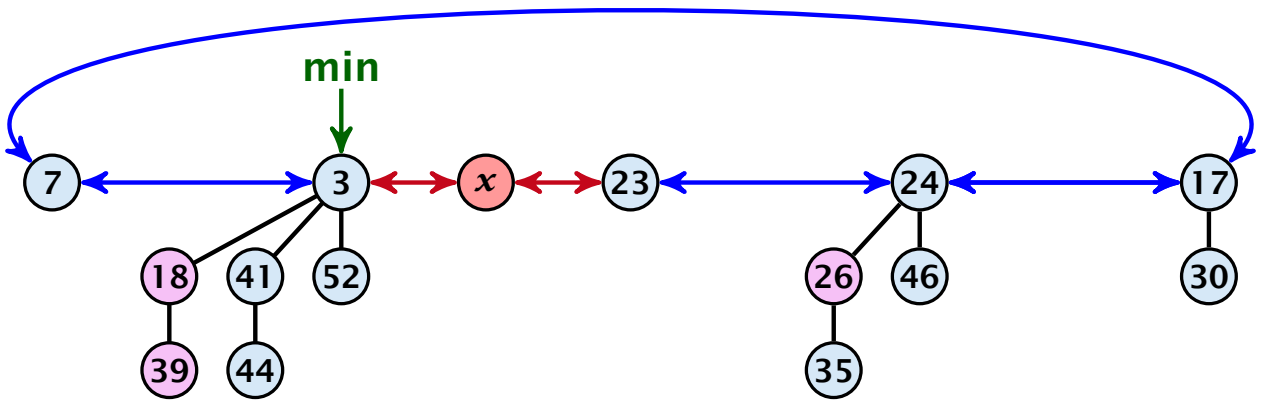

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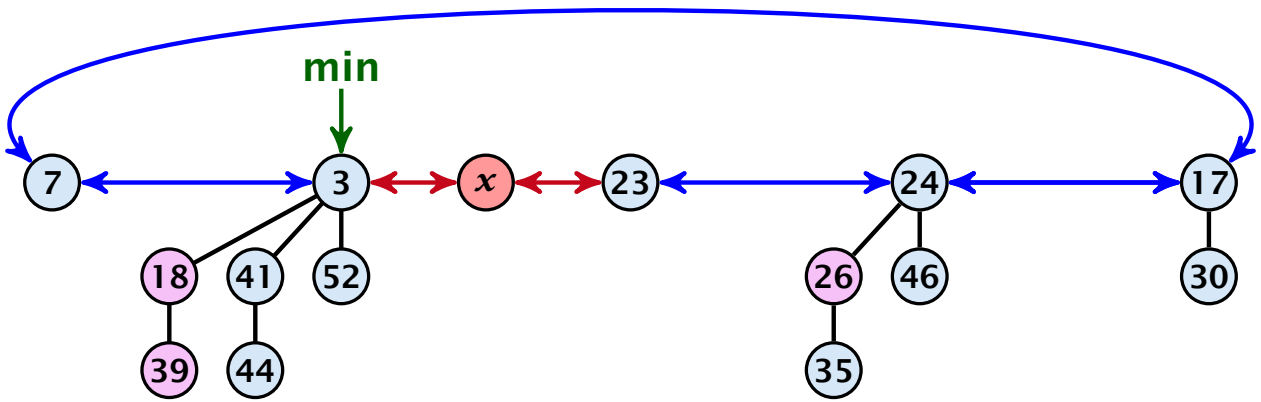
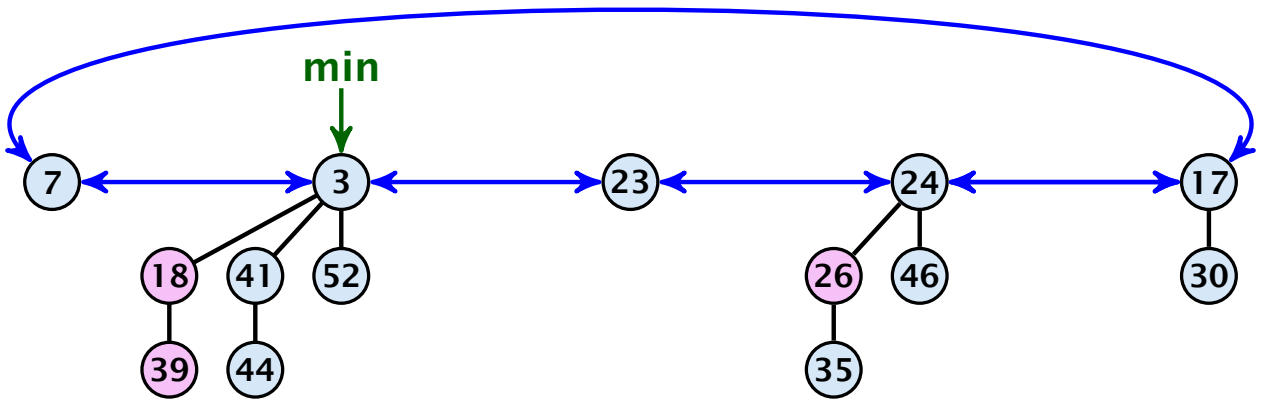


Running time:

- Actual cost $\mathcal{O}(1)$.
- Change in potential is $+1$.
- Amortized cost is $c + \mathcal{O}(1) = \mathcal{O}(1)$.

8.3 Fibonacci Heaps

S. delete-min(x)



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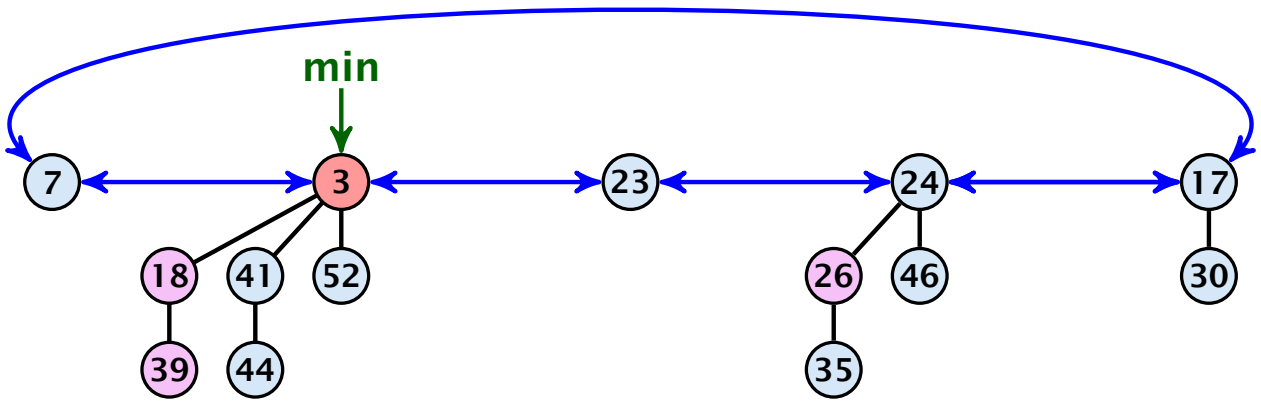
- Delete minimum; add child-trees to heap;
  time: $D(\min) \cdot \mathcal{O}(1)$.



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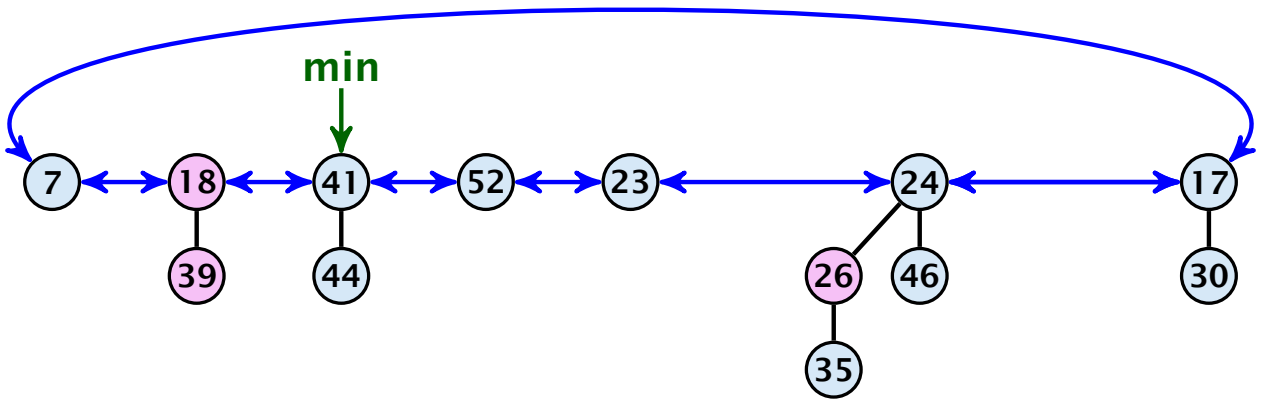
- Update min-pointer; time: $(t + D(\min)) \cdot \mathcal{O}(1)$.



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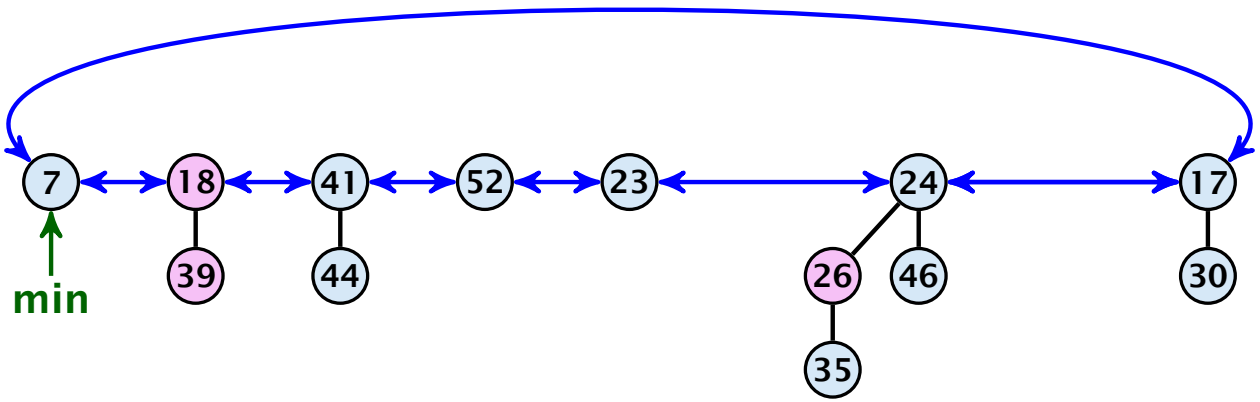

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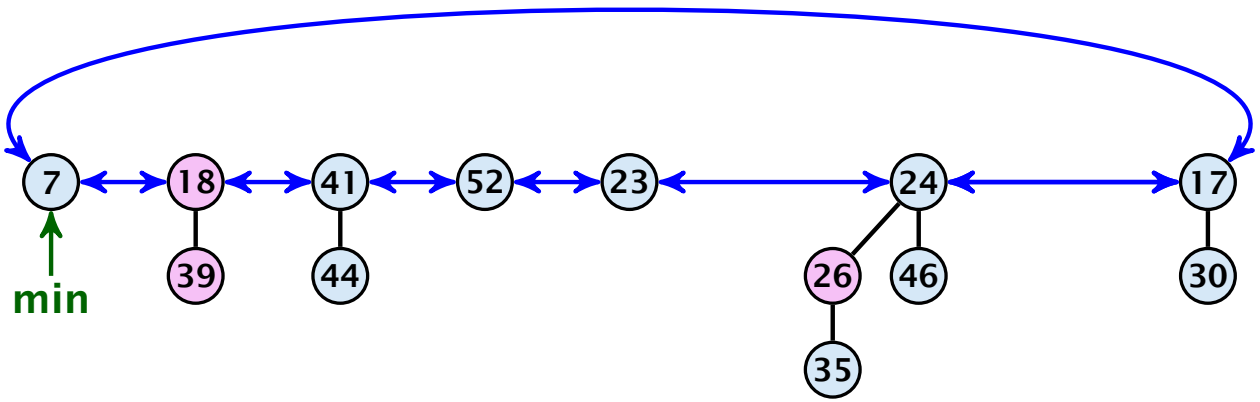
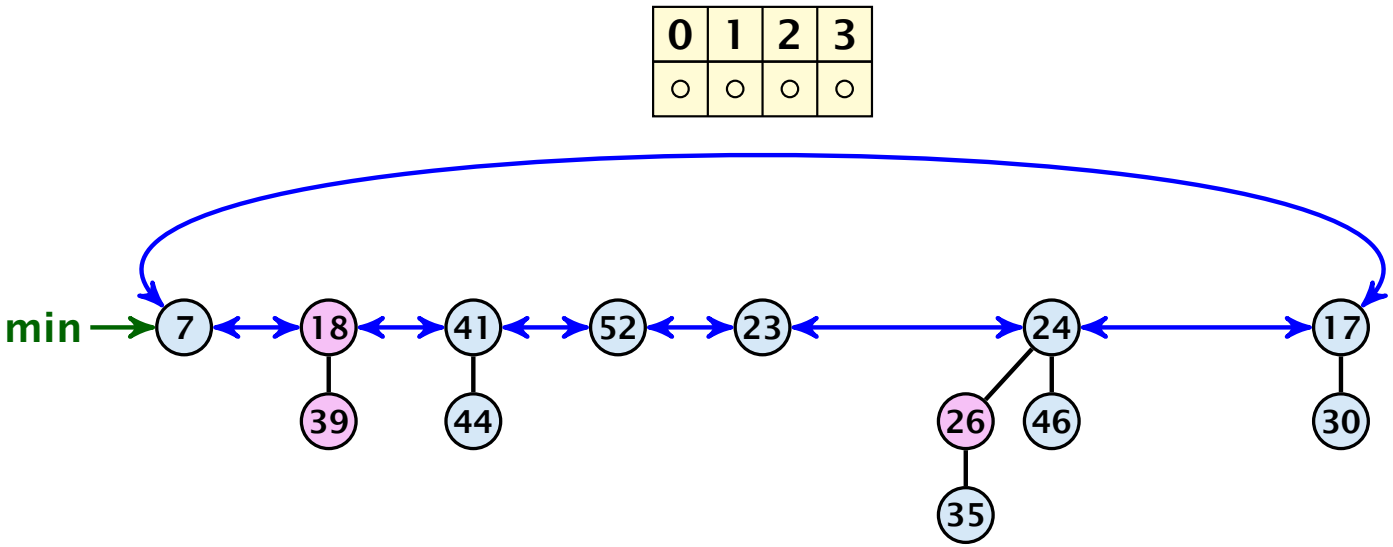


- Consolidate root-list so that no roots have the same degree. Time $t \cdot \mathcal{O}(1)$ (see next slide).

8.3 Fibonacci Heaps

Consolidate:



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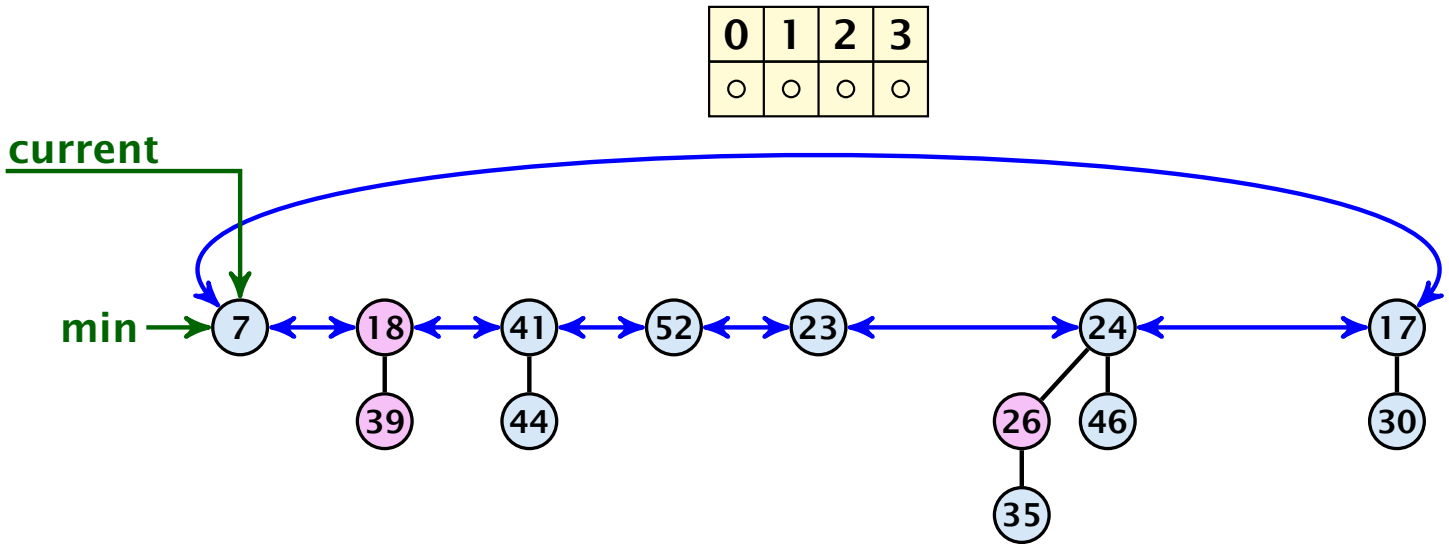

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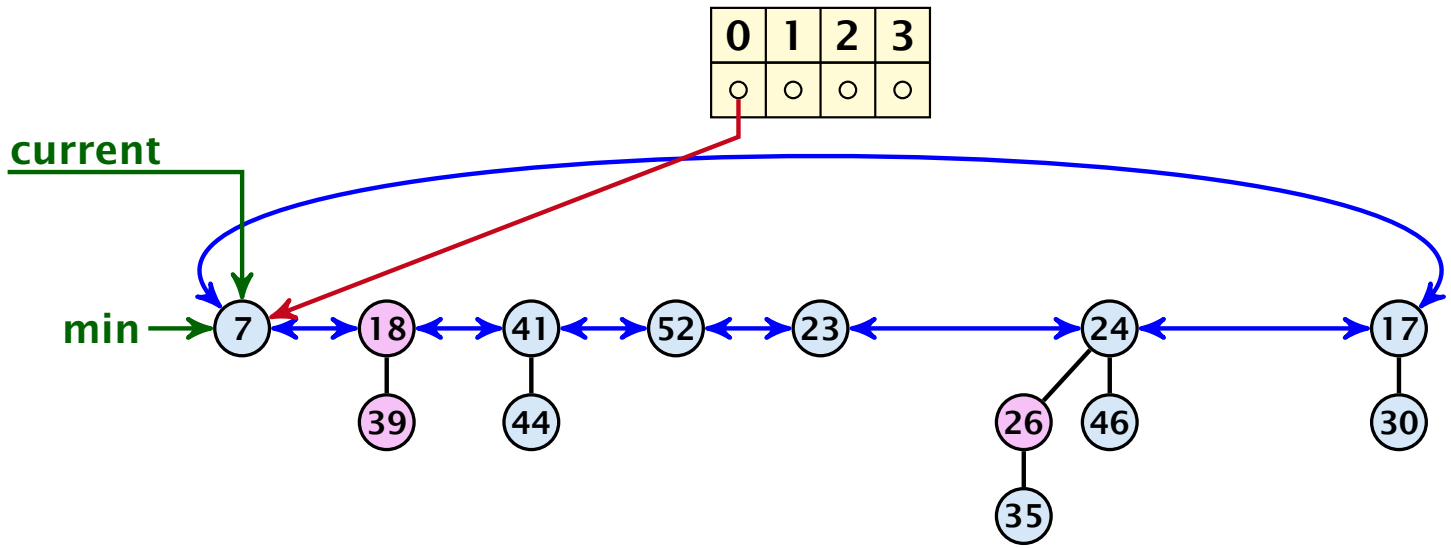

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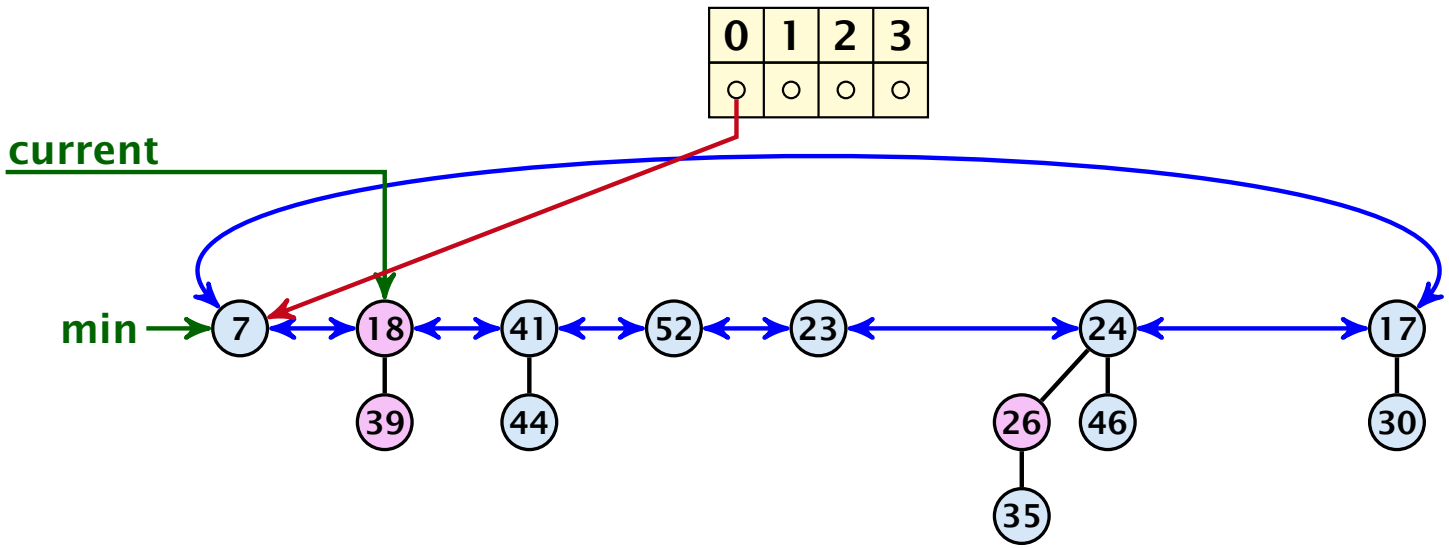

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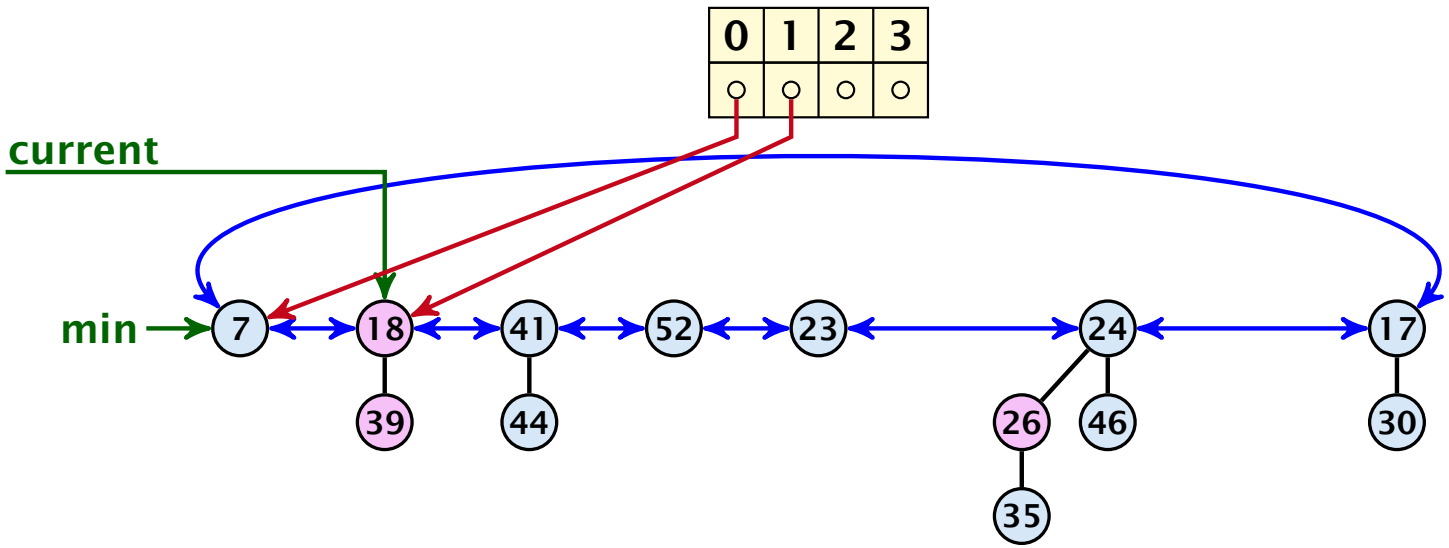

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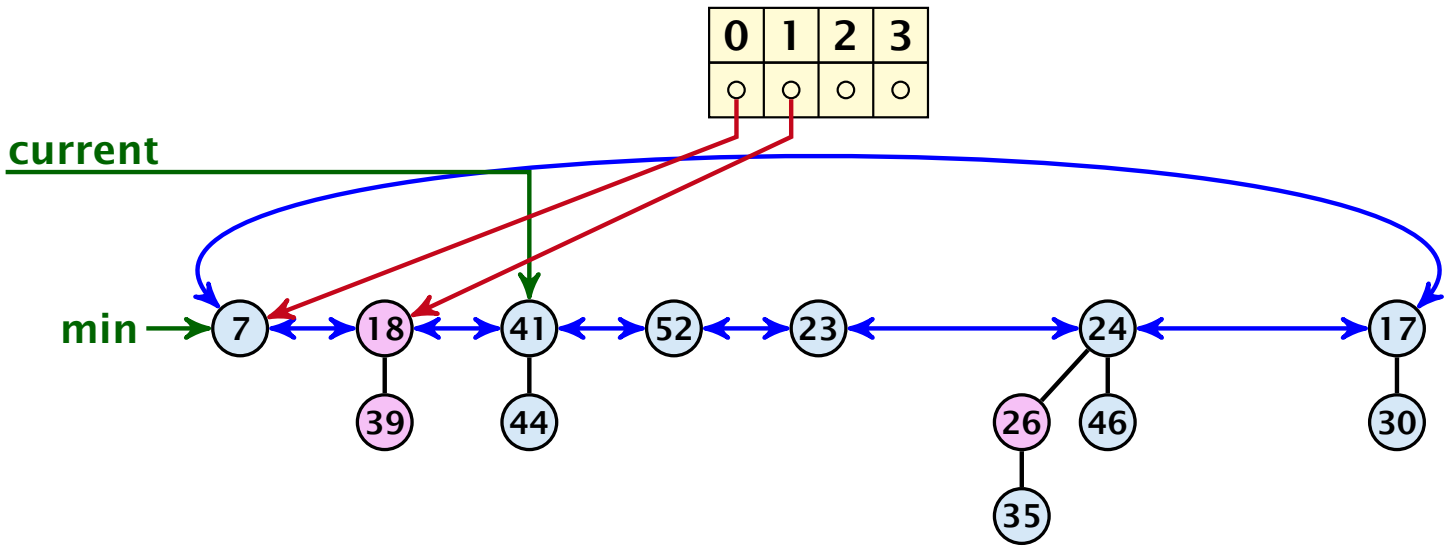

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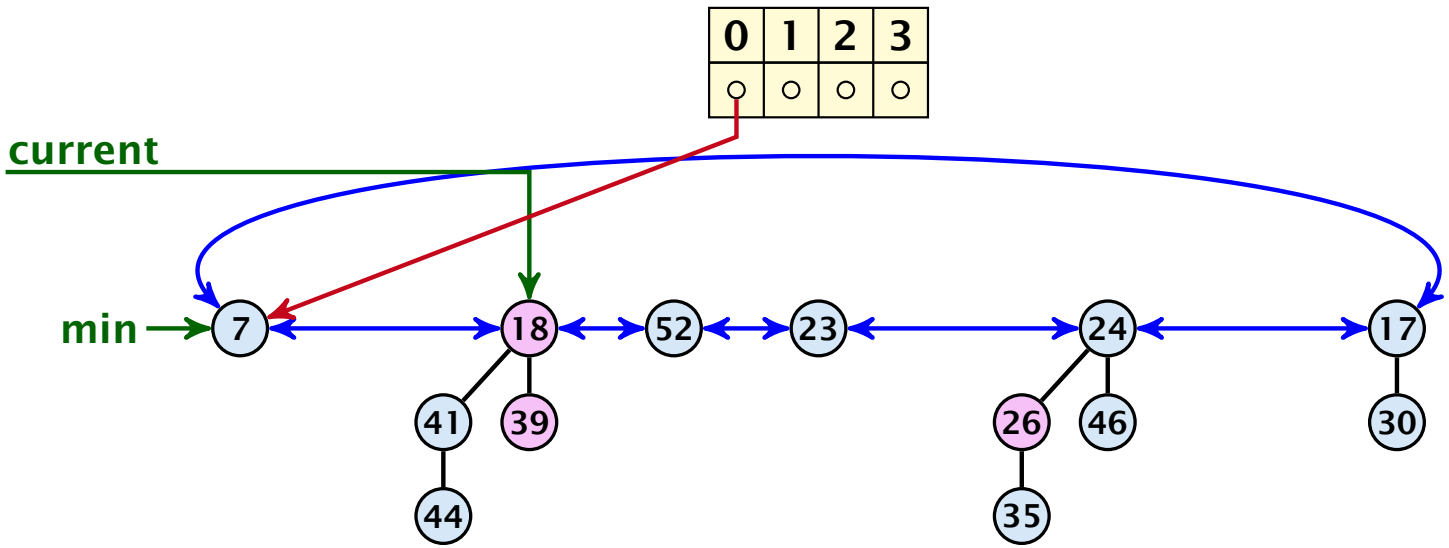

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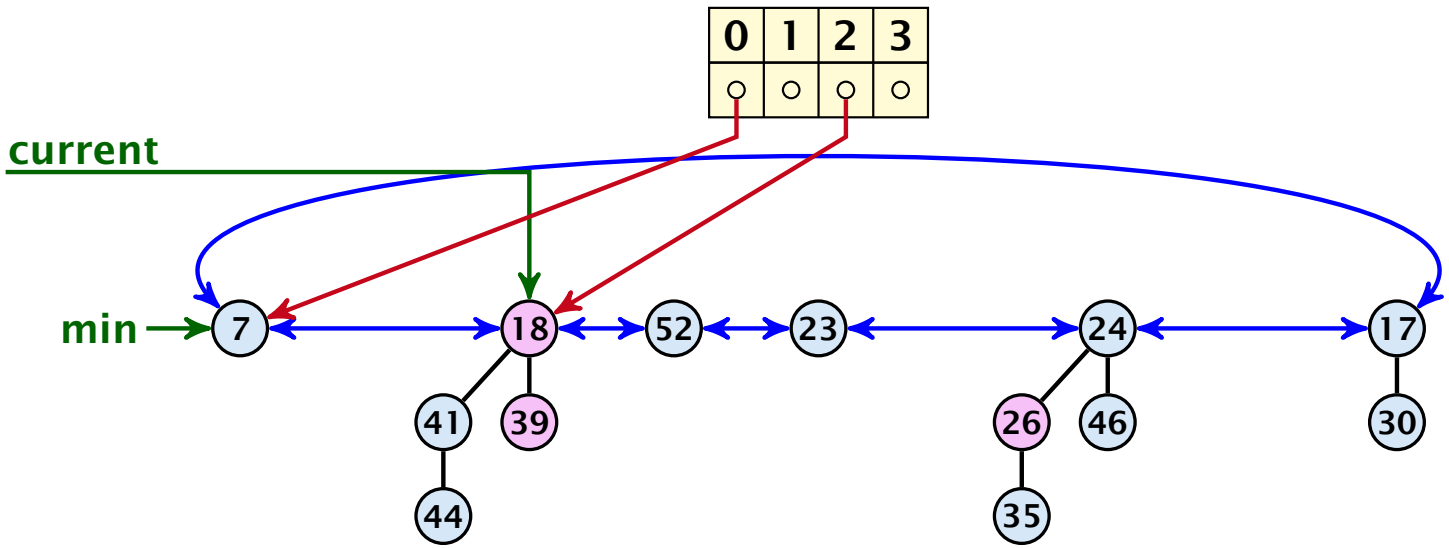

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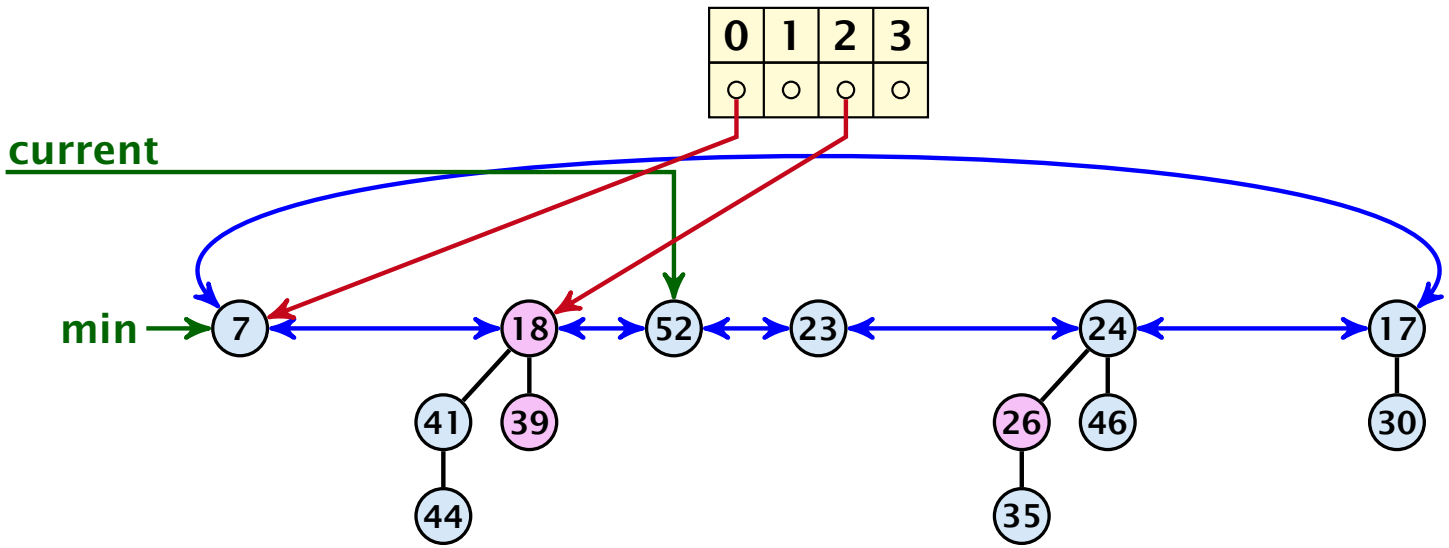

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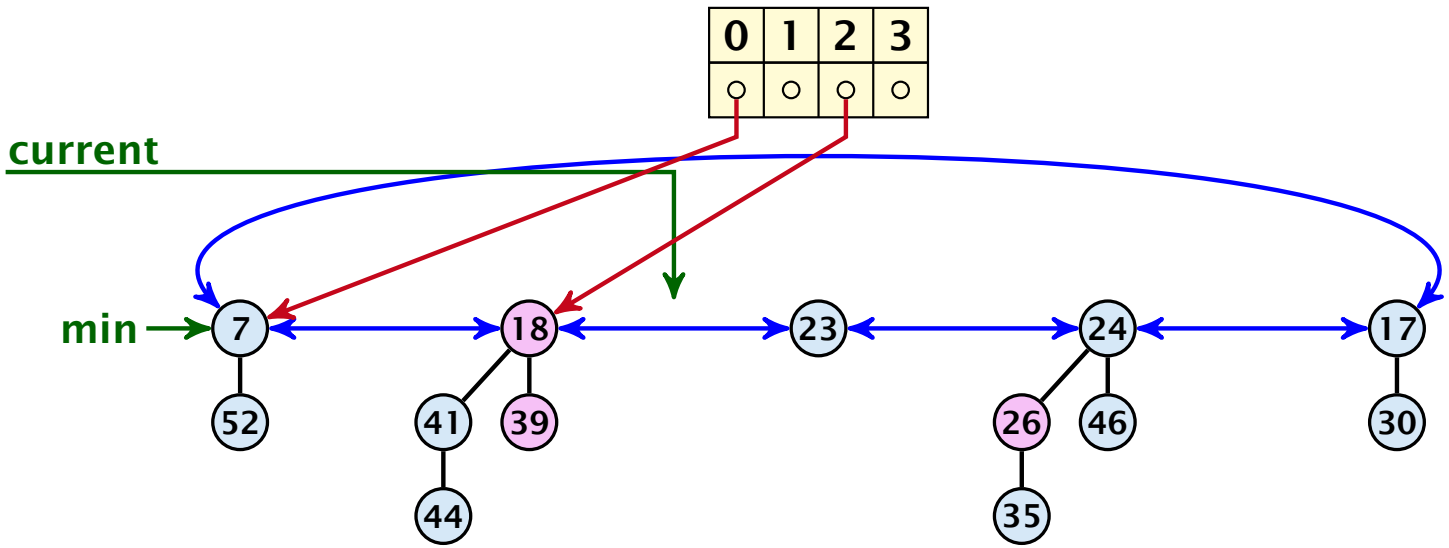

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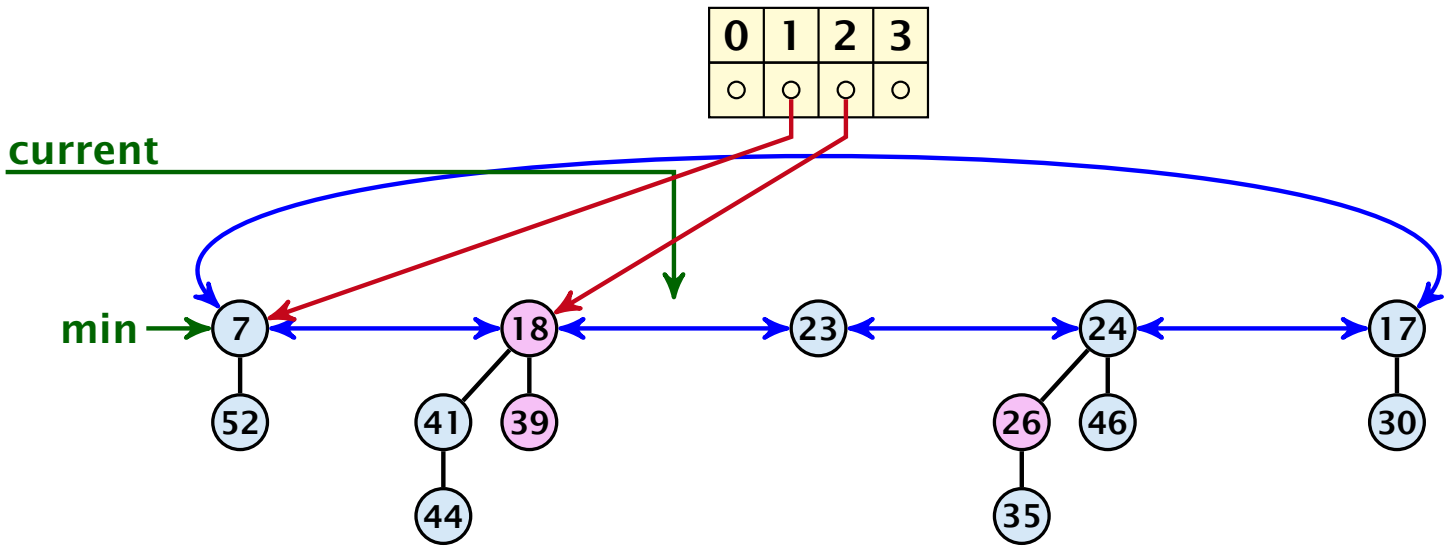

8.3 Fibonacci Heaps

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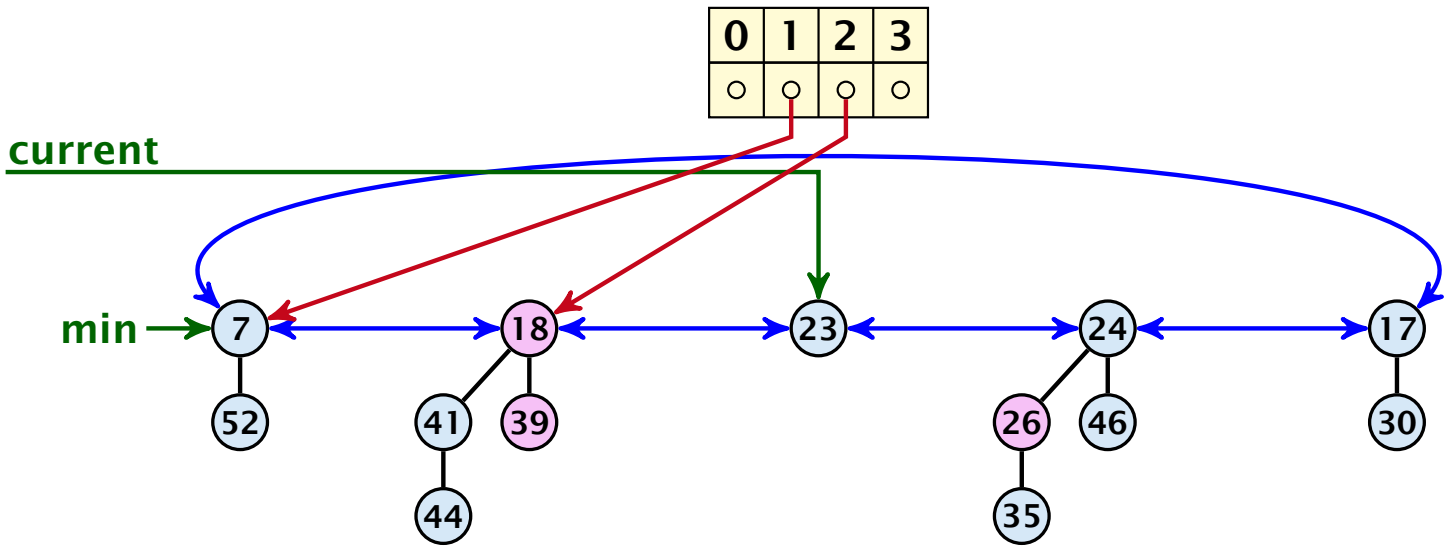

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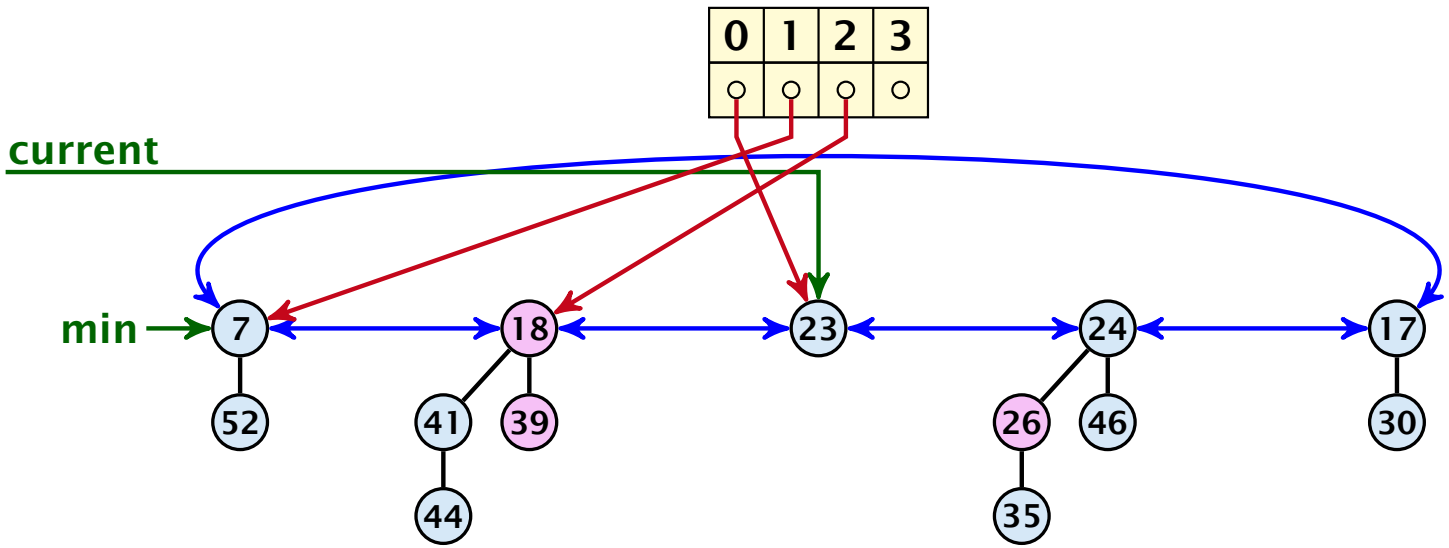

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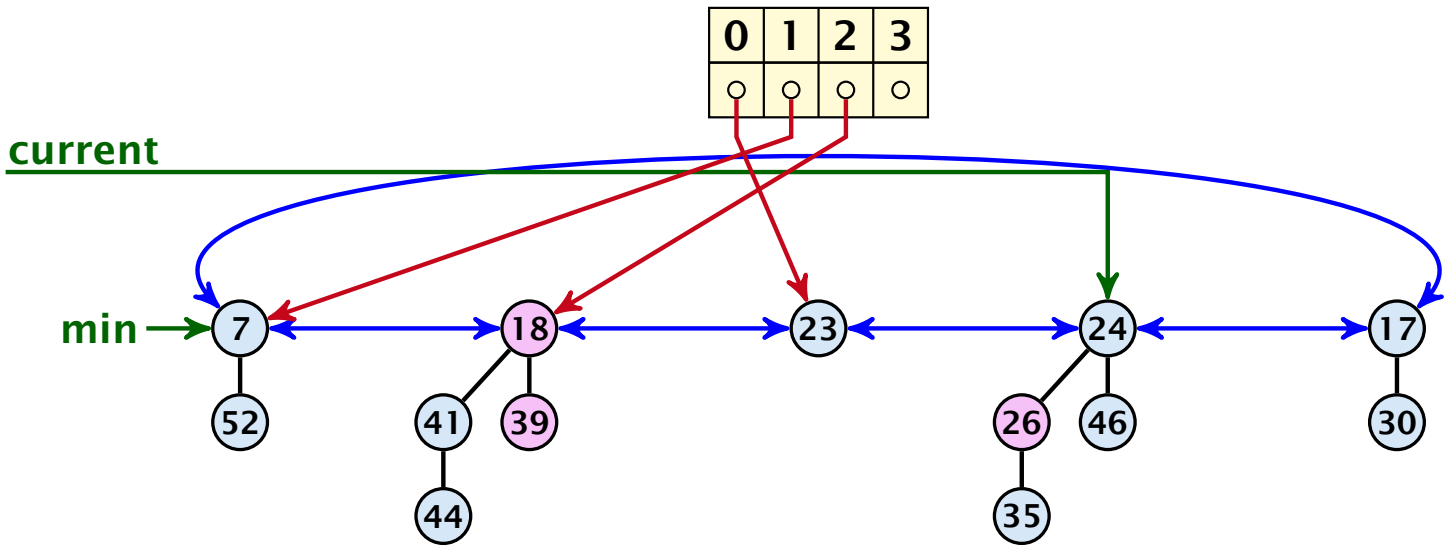

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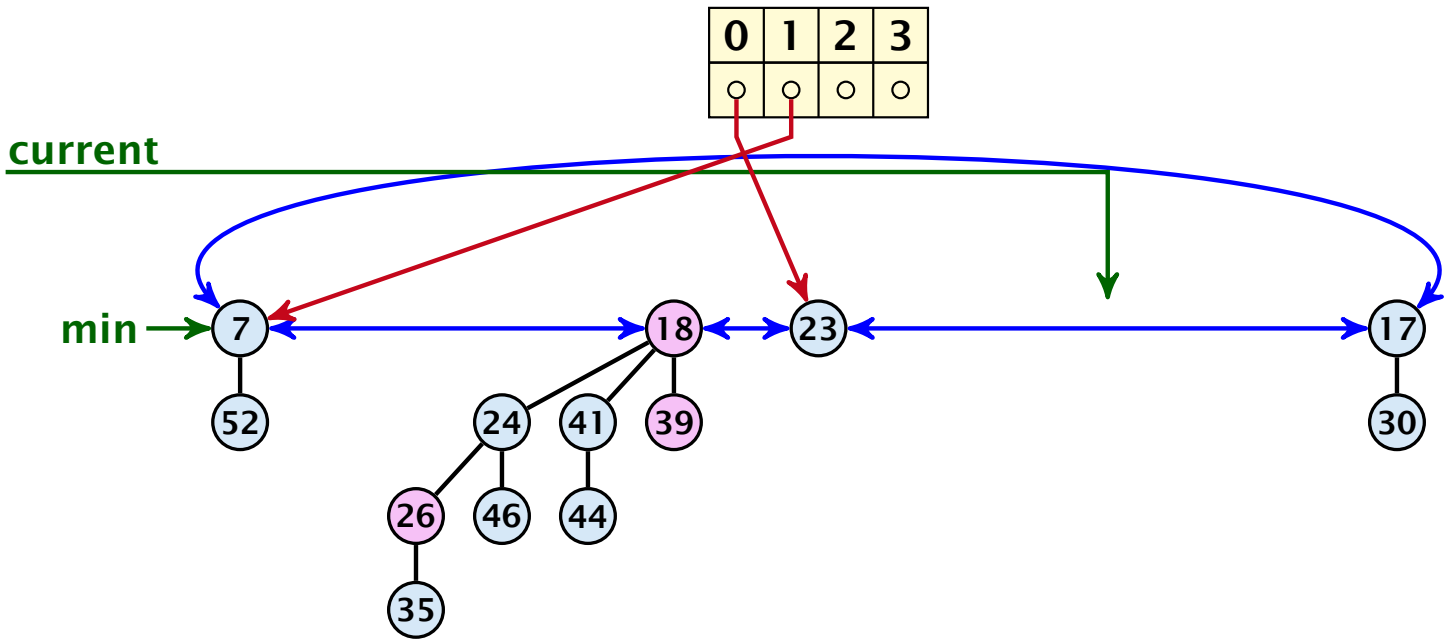

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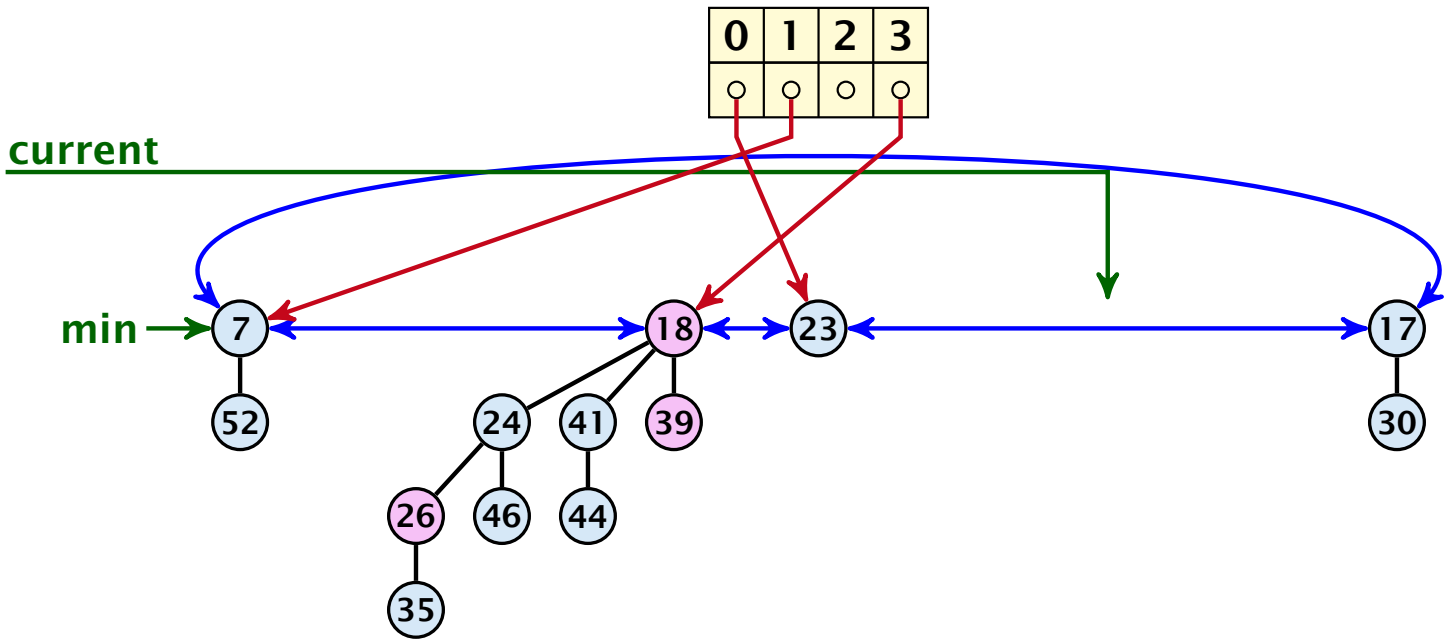

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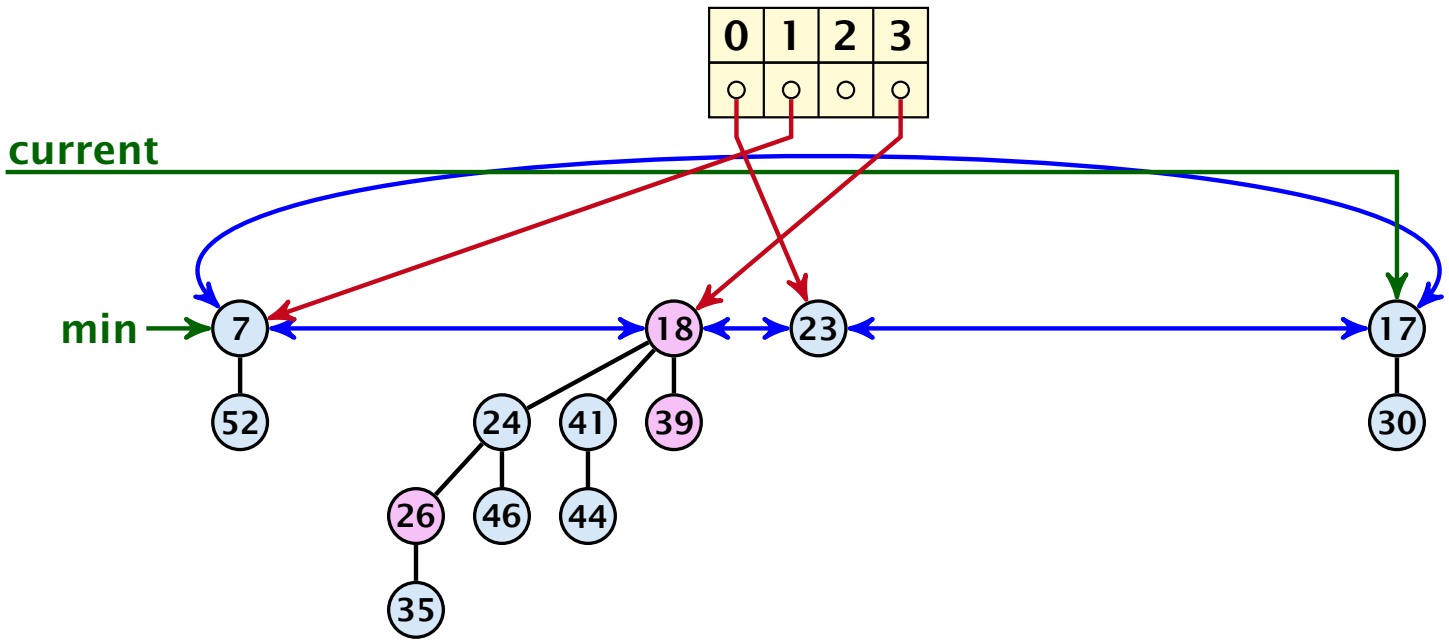

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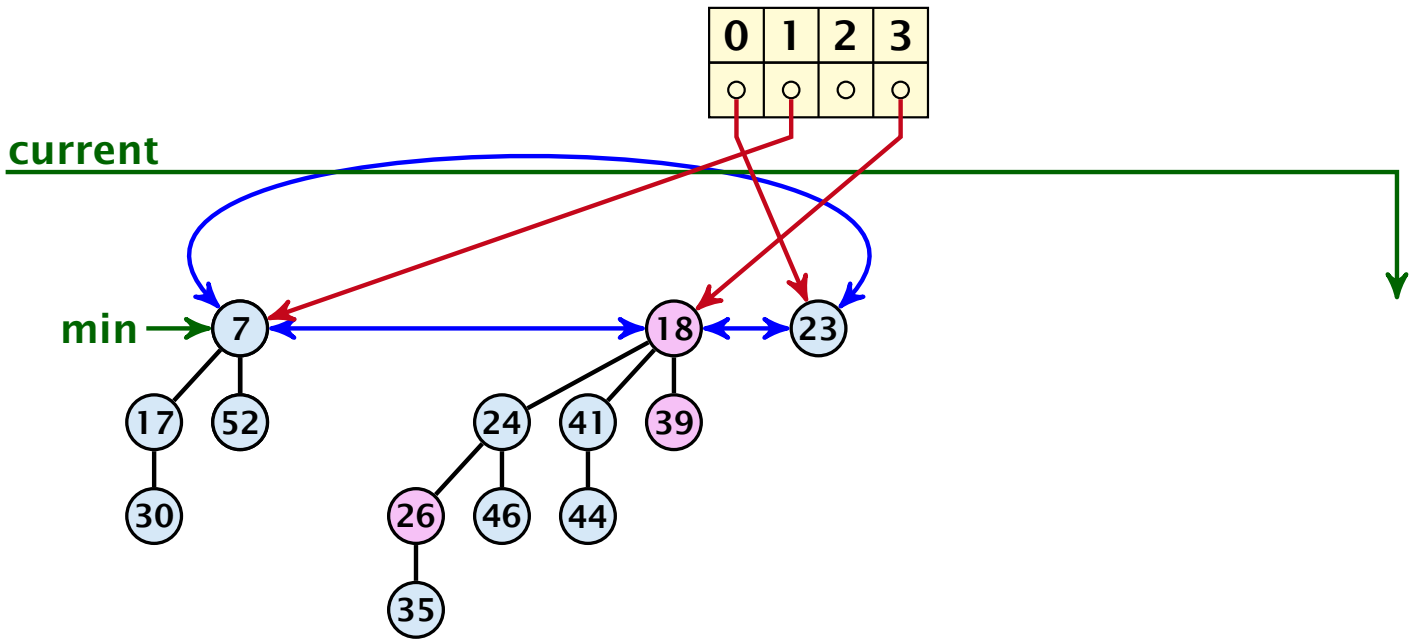

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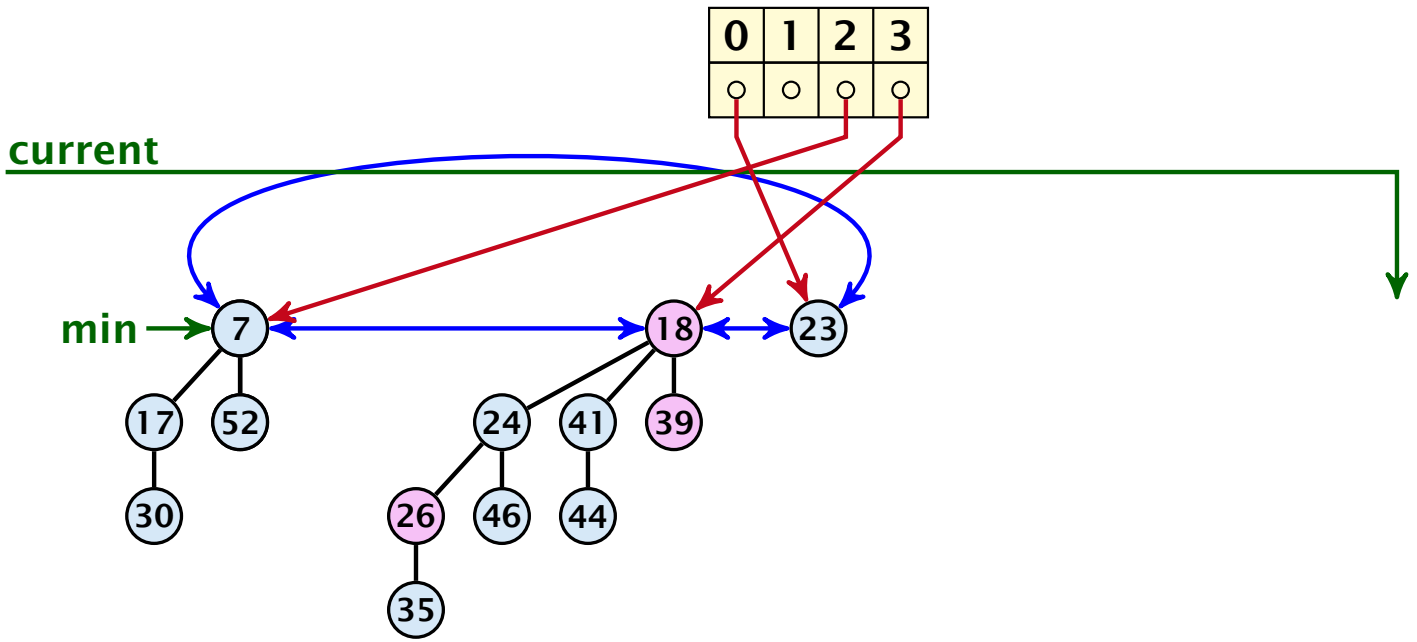

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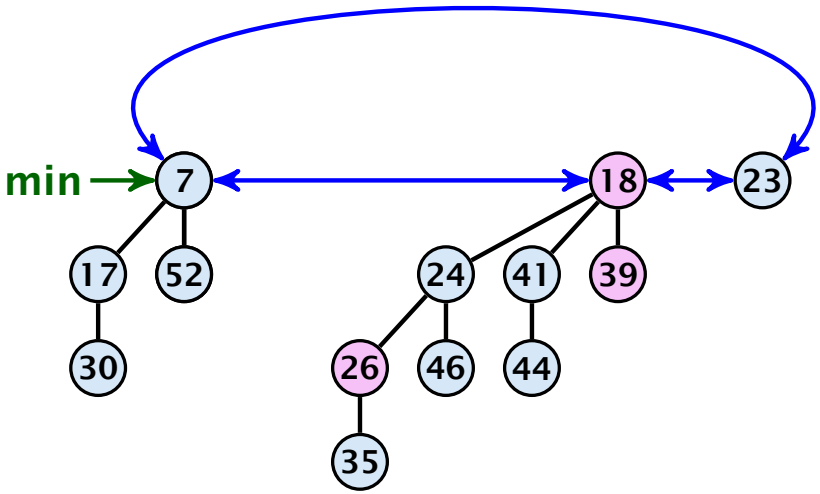

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for $c \geq c_1$.

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If the input trees of the consolidation procedure are binomial trees (for example only singleton vertices) then the output will be a set of distinct binomial trees, and, hence, the Fibonacci heap will be (more or less) a Binomial heap right after the consolidation.

If we do not have delete or decrease-key operations then $D_n \leq \log n$.

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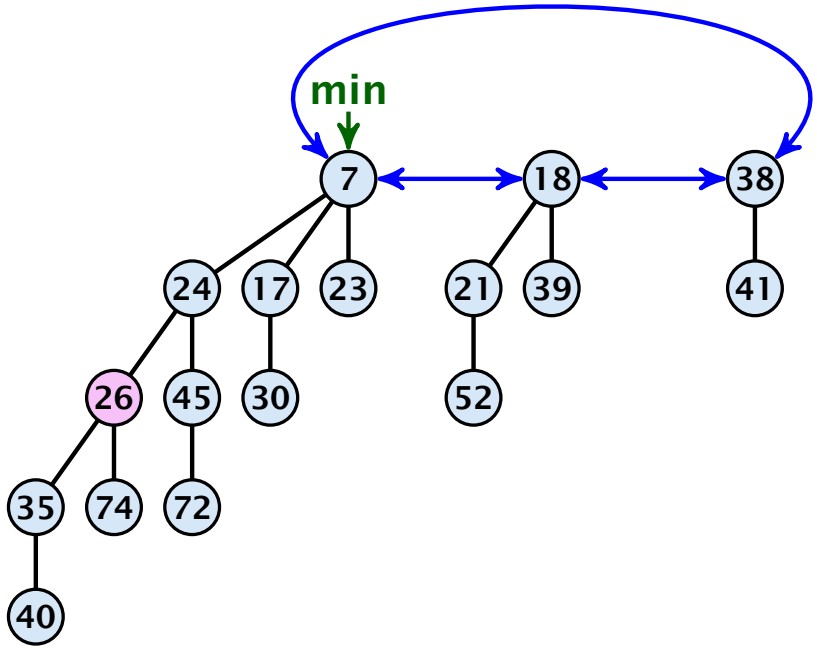

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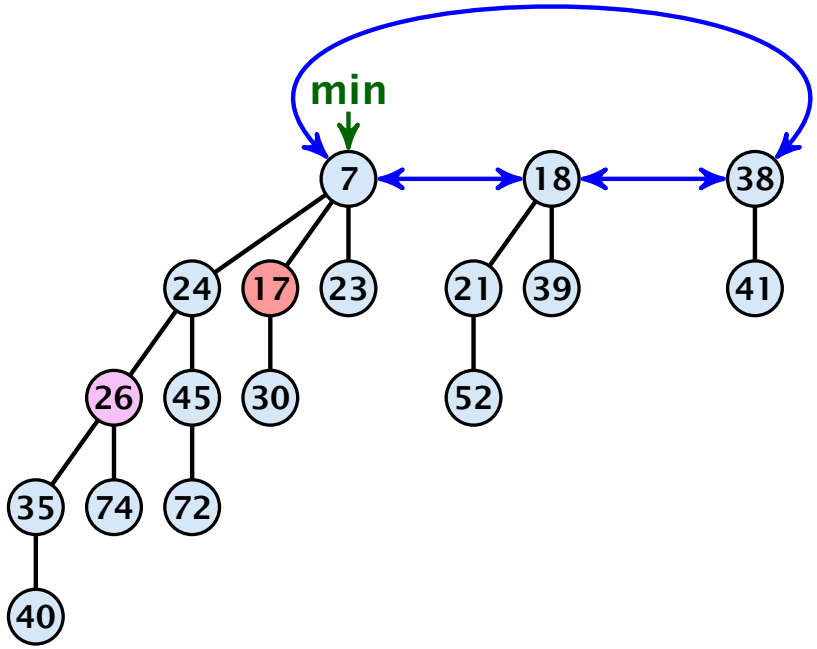

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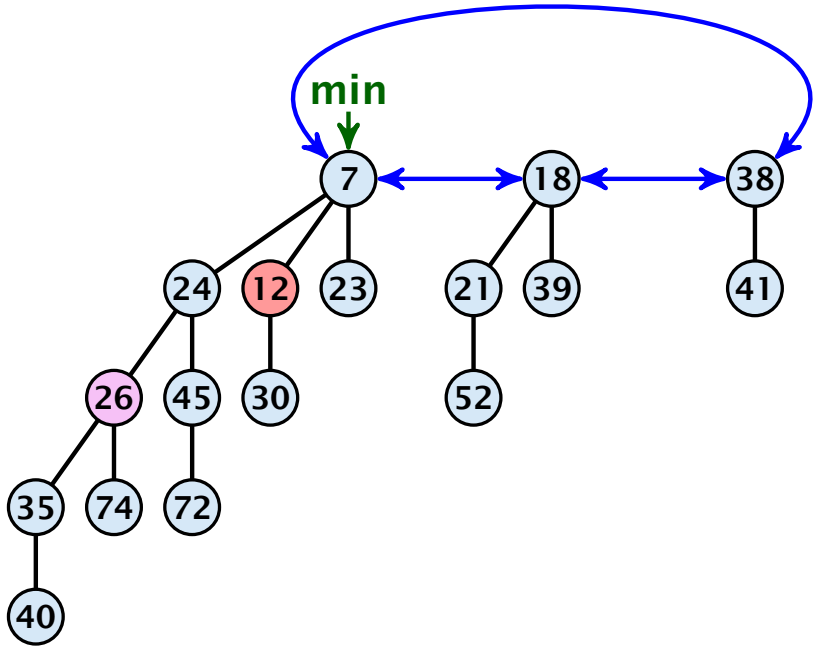

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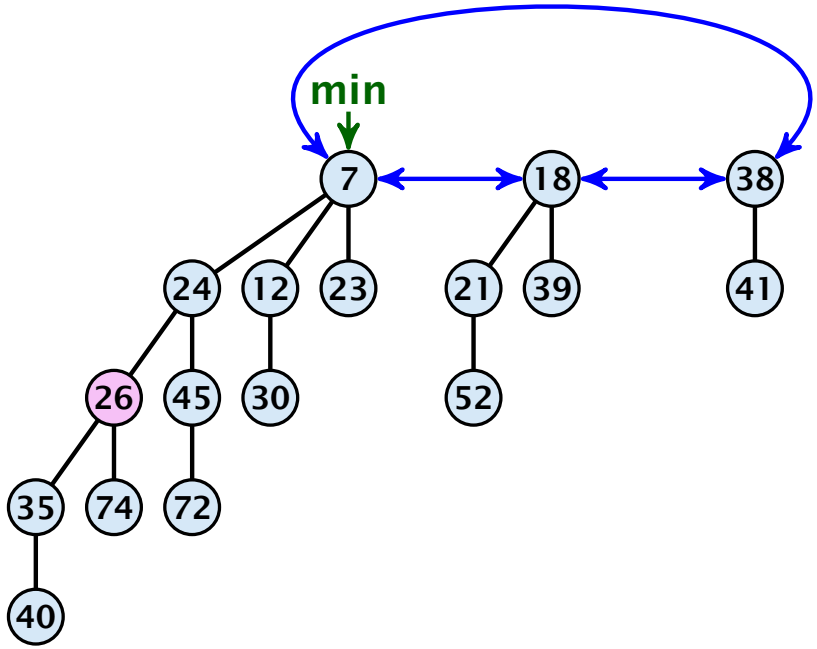

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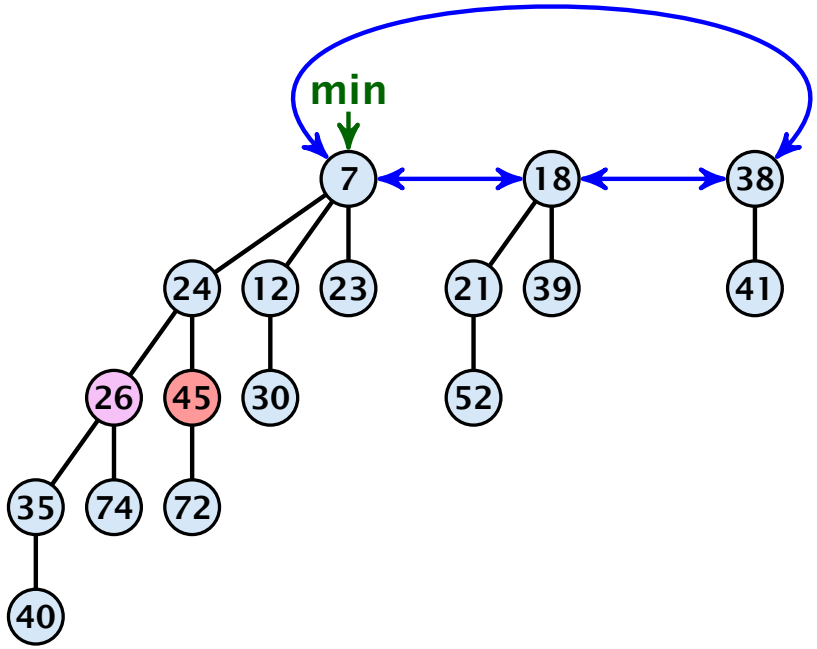

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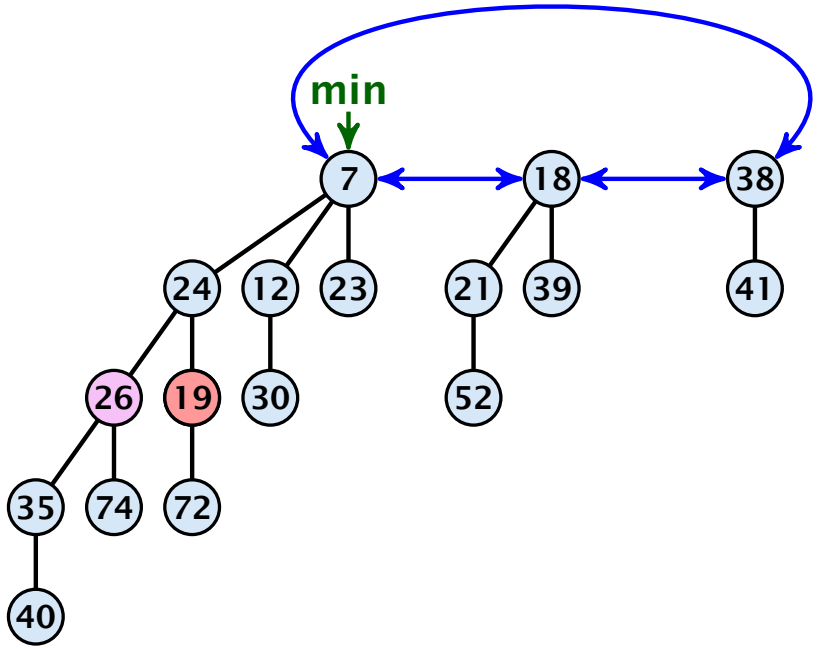

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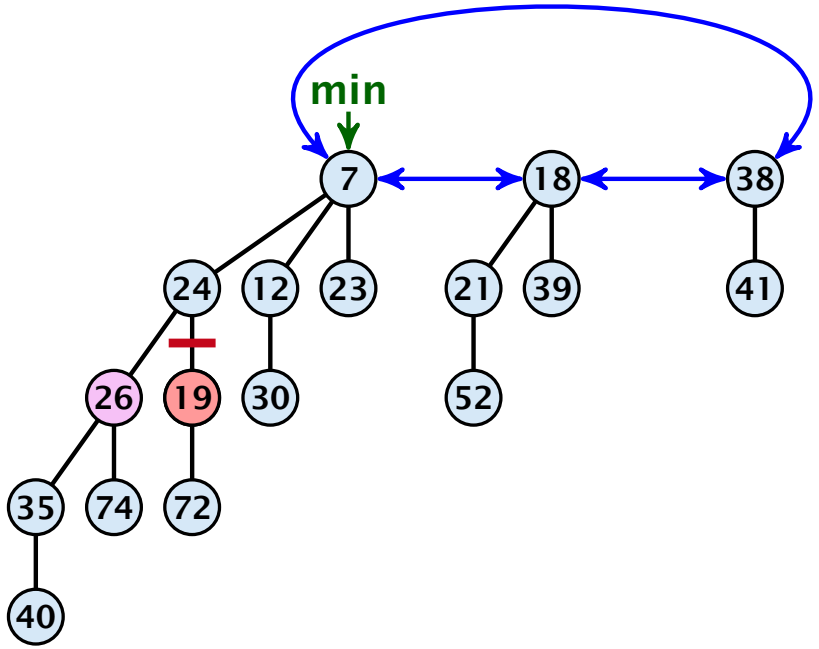

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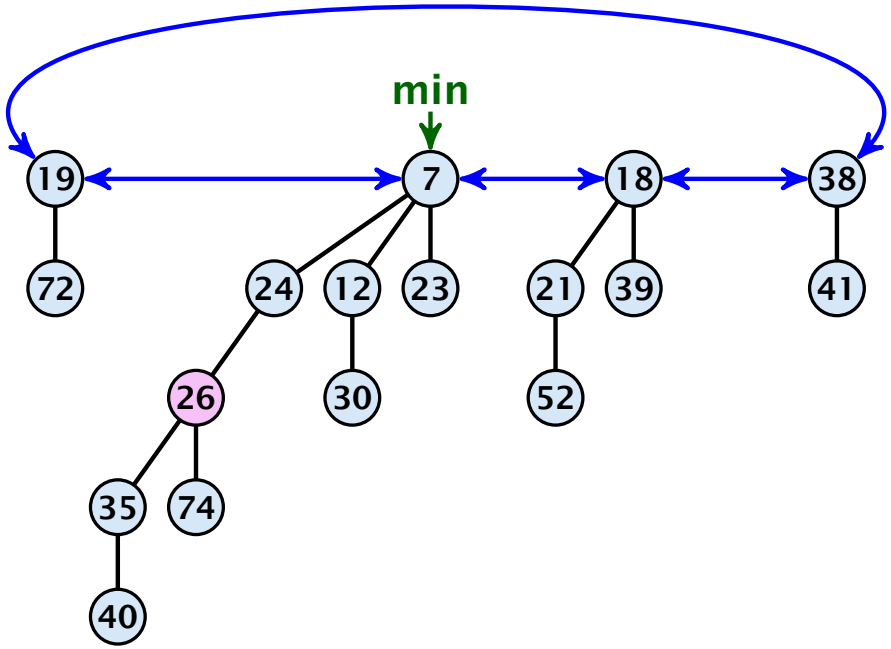

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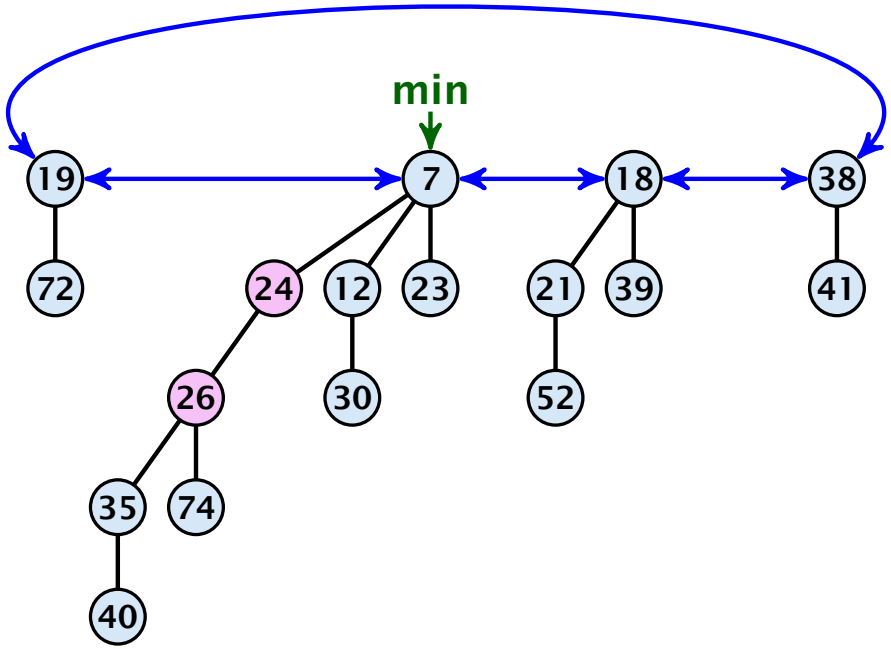

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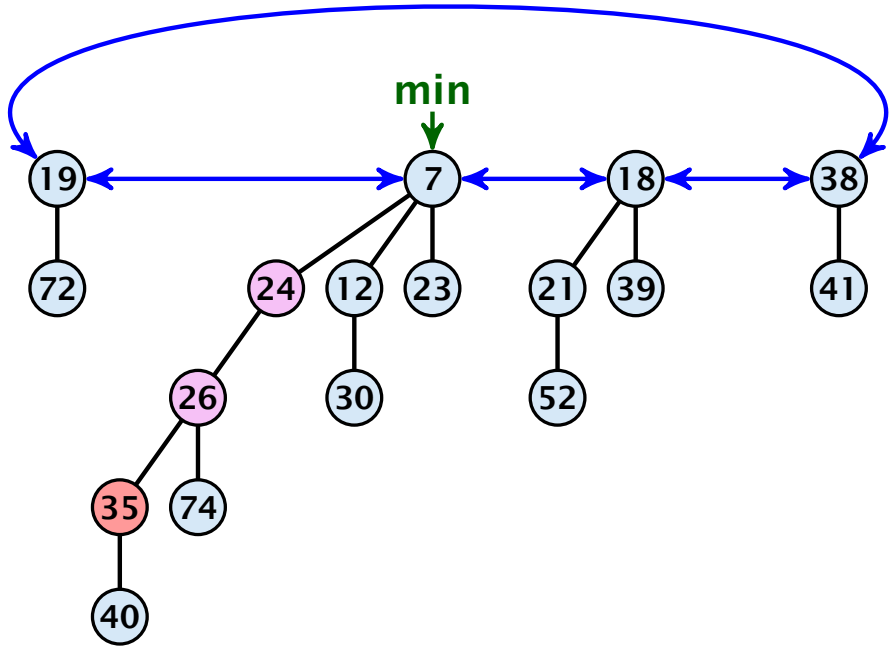

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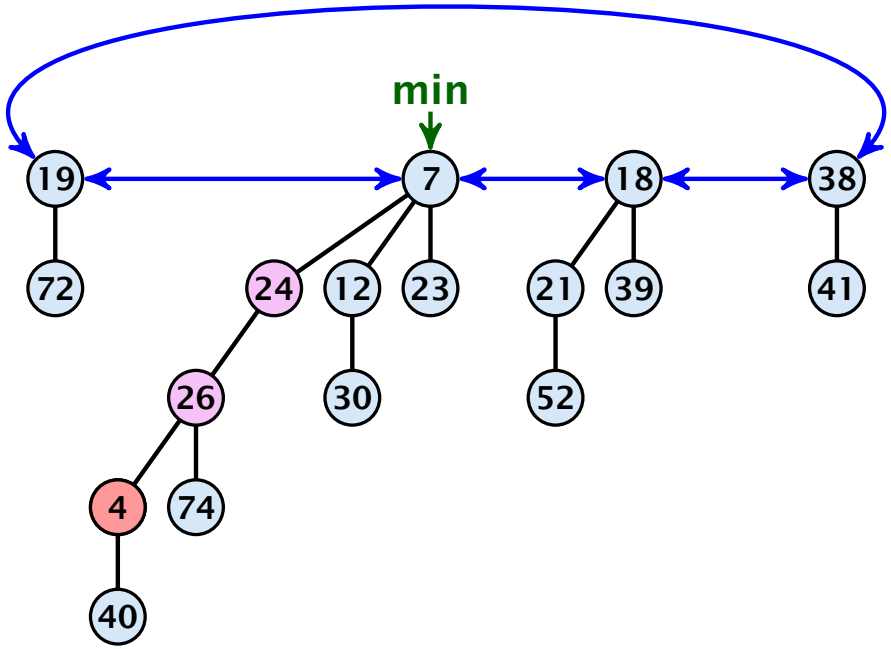

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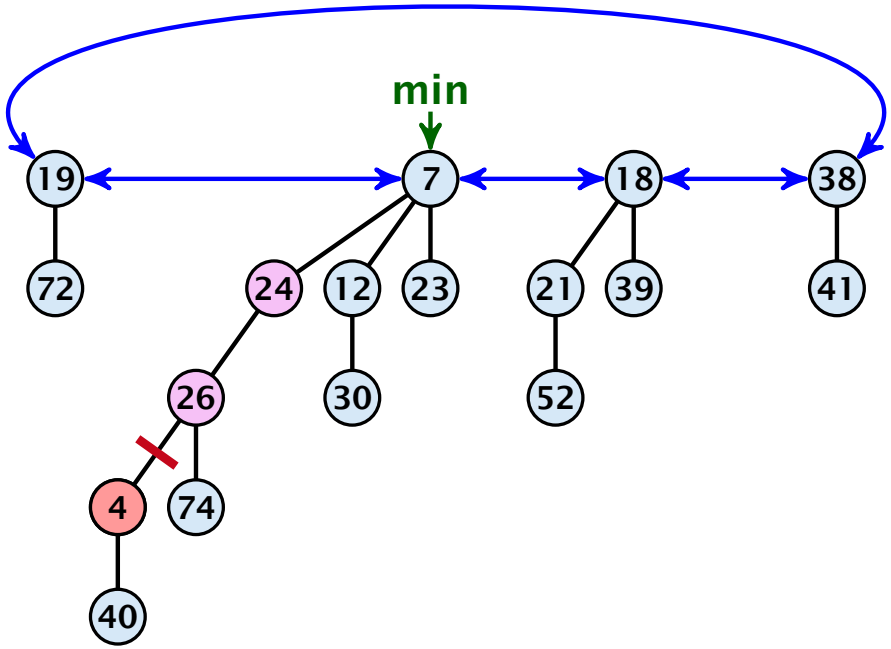

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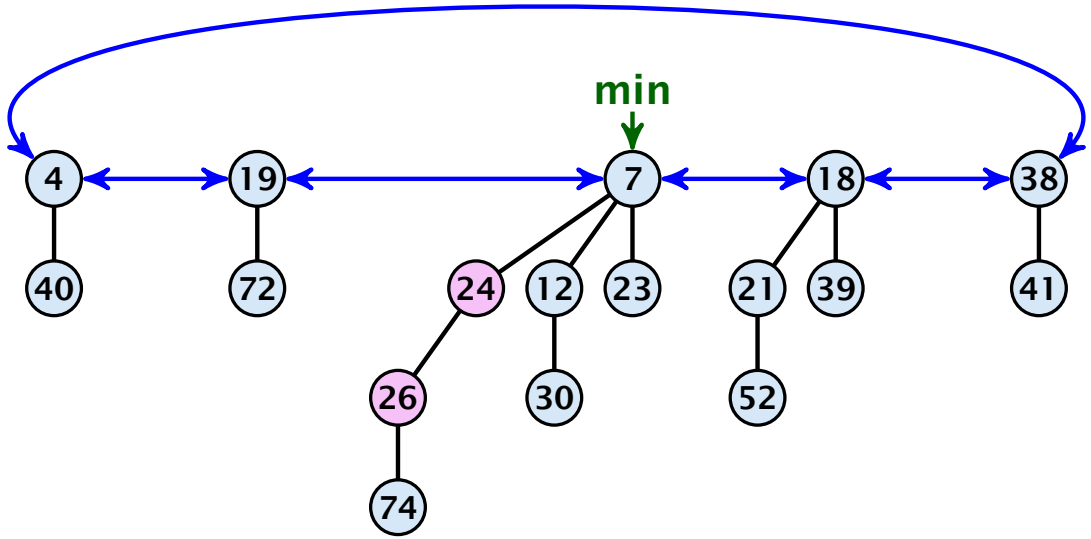

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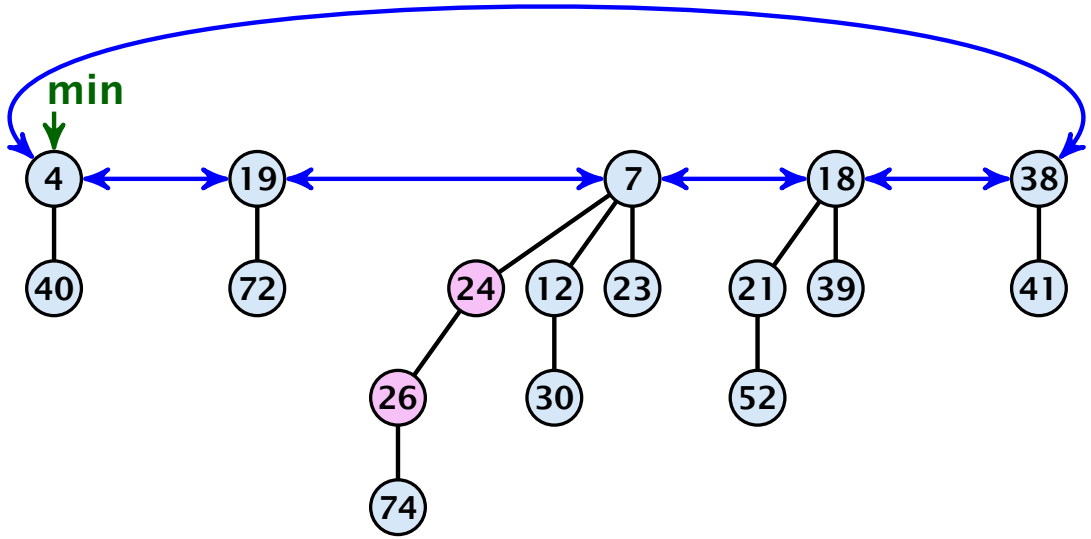

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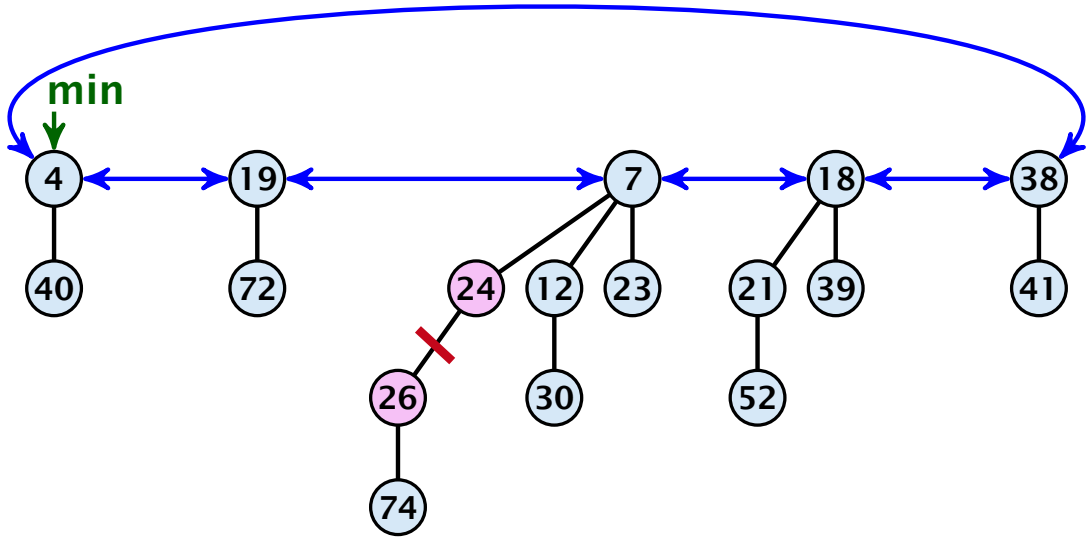

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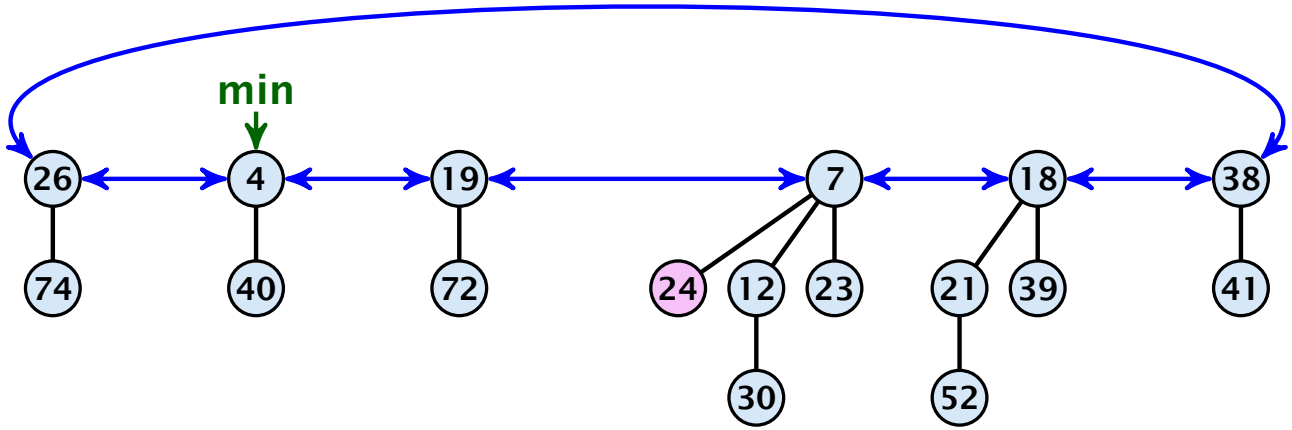

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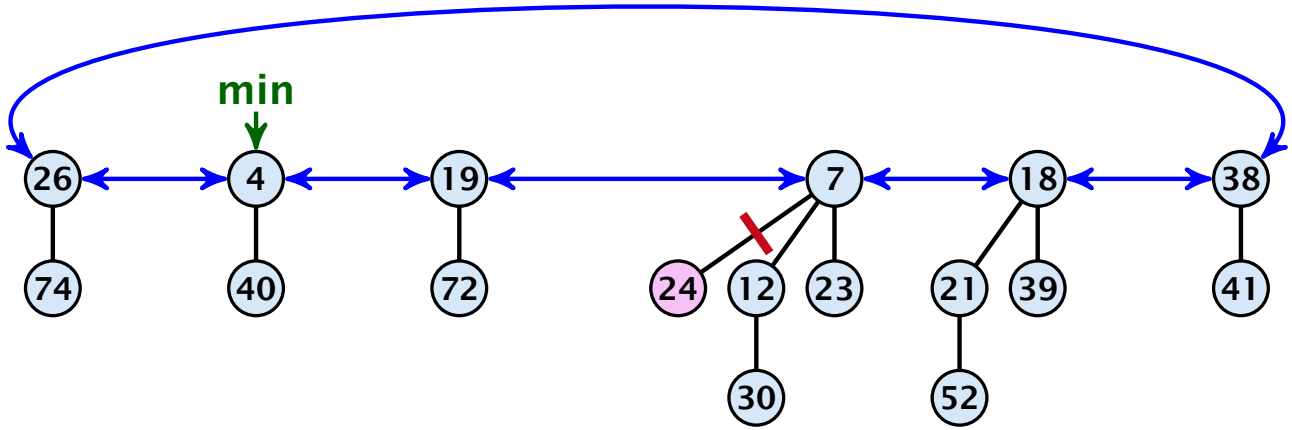

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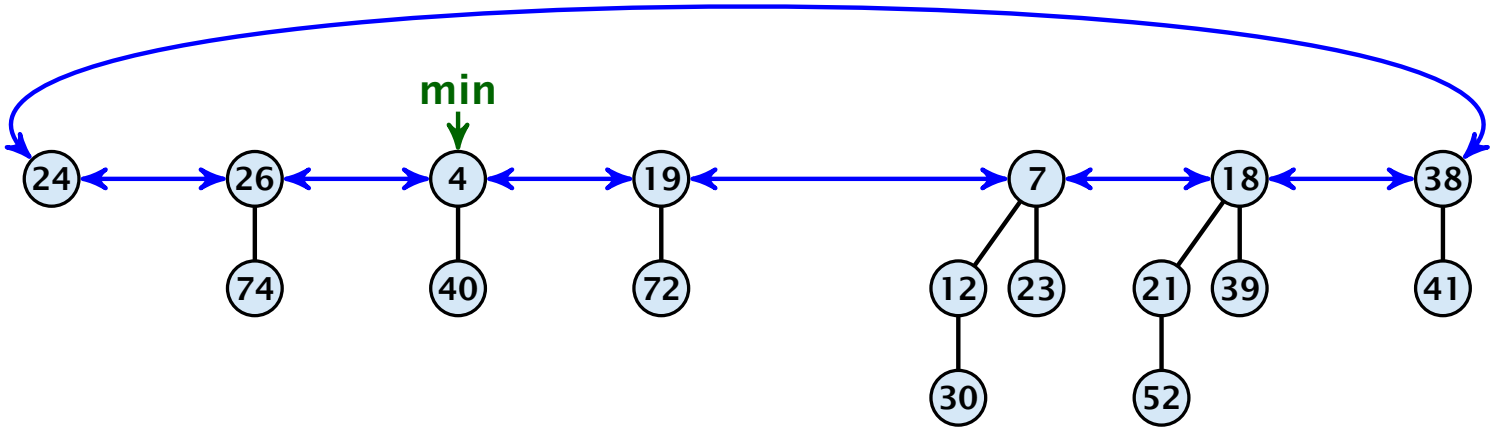

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- Cut the parent edge of x, and make x into a root.
- Adjust min-pointers, if necessary.
- Execute the following:

$p \leftarrow \text{parent}[x]$;
while (p is marked)
    $pp \leftarrow \text{parent}[p]$;
    cut of p; make it into a root; unmark it;
    $p \leftarrow pp$;
if p is unmarked and not a root mark it;

Fibonacci Heaps: decrease-key(handle h, v)

Actual cost:

- Constant cost for decreasing the value.
- Constant cost for each of ℓ cuts.
- Hence, cost is at most $c_2 \cdot (\ell + 1)$, for some constant c_2.

Amortized cost:

- $\Delta\Phi \leq 1 - \ell + 2(\ell - 1) = \ell - 1$, since all but the first cut unmarks a node, the last cut may mark a node.
- Amortized cost is at most $c_2 \cdot (\ell + 1) + c_1(\ell - 1) \leq (c_2 + c_1)\ell + c_2$.

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Amortized cost:

- $\Phi(H') = \Phi(H) + \ell$, as every cut creates one new root.
- Actual cost: $c_2 \cdot (\ell + 1)$, since all but the first cut unmarks a node, the last cut may mark a node.
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Amortized cost:

- $\Phi_{new} - \Phi_{old} \leq 1$, as every cut creates one new root.
- $\Phi_{new} - \Phi_{old} \leq 2 - \ell$, since all but the first cut unmarks a node, the last cut may mark a node.
- Amortized cost is at most

$$c_2 \cdot (\ell + 1) + c_1 \cdot (2 - \ell) \leq c_1 \cdot 2 + c_2 \cdot (\ell + 1)$$

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Amortized cost:

- $t' = t + \ell$, as every cut creates one new root.
- $m' \leq m - (\ell - 1) + 1 = m - \ell + 2$, since all but the first cut unmarks a node; the last cut may mark a node.
- $\Delta\Phi \leq \ell + 2(-\ell + 2) = 4 - \ell$
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$$c_2(\ell+1)+c(4-\ell) \leq (c_2-c)\ell+4c+c_2 = \mathcal{O}(1)\,,$$

if $c \geq c_2$.

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Delete node

H.**delete**(x):

- decrease value of x to $-\infty$.
- delete-min.

Amortized cost: $\mathcal{O}(D_n)$

- $\mathcal{O}(1)$ for decrease-key.
- $\mathcal{O}(D_n)$ for delete-min.

8.3 Fibonacci Heaps

Lemma 35

Let x be a node with degree k and let $y_1, \ldots, y_k$ denote the children of x in the order that they were linked to x. Then

$$\text{degree}(y_i) \geq \begin{cases} 0 & \text{if } i = 1 \\ i-2 & \text{if } i > 1 \end{cases}$$

8.3 Fibonacci Heaps

Proof

- When y_i was linked to x, at least $y_1, \ldots, y_{i-1}$ were already linked to x.
- Hence, at this time $\text{degree}(x) \geq i-1$, and therefore also $\text{degree}(y_i) \geq i-1$ as the algorithm links nodes of equal degree only.
- Since, then y_i has lost at most one child.
- Therefore, $\text{degree}(y_i) \geq i-2$.

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- Let s_k be the minimum possible size of a sub-tree rooted at a node of degree k that can occur in a Fibonacci heap.

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$$s_k = 2 + \sum_{i=2}^{k} \text{size}(y_i)$$

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$$\geq 2 + \sum_{i=2}^{k} s_{i-2}$$

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Let x be a degree k node of size s_k and let $y_1, \ldots, y_k$ be its children.

$$
\begin{aligned}
s_k &= 2 + \sum_{i=2}^{k} \text{size}(y_i) \\
&\geq 2 + \sum_{i=2}^{k} s_{i-2} \\
&= 2 + \sum_{i=0}^{k-2} s_i
\end{aligned}
$$

8.3 Fibonacci Heaps

Definition 36

Consider the following non-standard Fibonacci type sequence:

$$F_k = \begin{cases} 1 & \text{if } k = 0 \\ 2 & \text{if } k = 1 \\ F_{k-1} + F_{k-2} & \text{if } k \geq 2 \end{cases}$$

Facts:

1. $F_k \geq \phi^k$.
2. For $k \geq 2$: $F_k = 2 + \sum_{i=0}^{k-2} F_i$.

The above facts can be easily proved by induction. From this it follows that $s_k \geq F_k \geq \phi^k$, which gives that the maximum degree in a Fibonacci heap is logarithmic.

k=0: $1 = F_0 \geq \Phi^0 = 1$

k=1: $2 = F_1 \geq \Phi^1 \approx 1.61$

k-2,k-1→ k: $F_k = F_{k-1} + F_{k-2} \geq \Phi^{k-1} + \Phi^{k-2} = \Phi^{k-2}\overbrace{(\Phi + 1)}^{\Phi^2} = \Phi^k$

k=2: $3 = F_2 = 2 + 1 = 2 + F_0$

k-1→ k: $F_k = F_{k-1} + F_{k-2} = 2 + \sum_{i=0}^{k-3} F_i + F_{k-2} = 2 + \sum_{i=0}^{k-2} F_i$

9 Union Find

Union Find Data Structure $\mathcal{P}$: Maintains a partition of disjoint sets over elements.

- $\mathcal{P}.\mathbf{makeset}(x)$: Given an element x, adds x to the data-structure and creates a singleton set that contains only this element. Returns a locator/handle for x in the data-structure.
- $\mathcal{P}.\mathbf{find}(x)$: Given a handle for an element x; find the set that contains x. Returns a representative/identifier for this set.
- $\mathcal{P}.\mathbf{union}(x, y)$: Given two elements x, and y that are currently in sets S_x and S_y, respectively, the function replaces S_x and S_y by $S_x \cup S_y$ and returns an identifier for the new set.

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Applications:

- Keep track of the connected components of a dynamic graph that changes due to insertion of nodes and edges.
- Kruskals Minimum Spanning Tree Algorithm

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9 Union Find

Algorithm 20 Kruskal-MST$(G = (V, E), w)$

```
1: A ← ∅;
2: for all v ∈ V do
3:         v.set ← P.makeset(v.label)
4: sort edges in non-decreasing order of weight w
5: for all (u, v) ∈ E in non-decreasing order do
6:         if P.find(u.set) ≠ P.find(v.set) then
7:                 A ← A ∪ {(u, v)}
8:                 P.union(u.set, v.set)
```

List Implementation

- The elements of a set are stored in a list; each node has a backward pointer to the head.
- The head of the list contains the identifier for the set and a field that stores the size of the set.

- makeset(x) can be performed in constant time.
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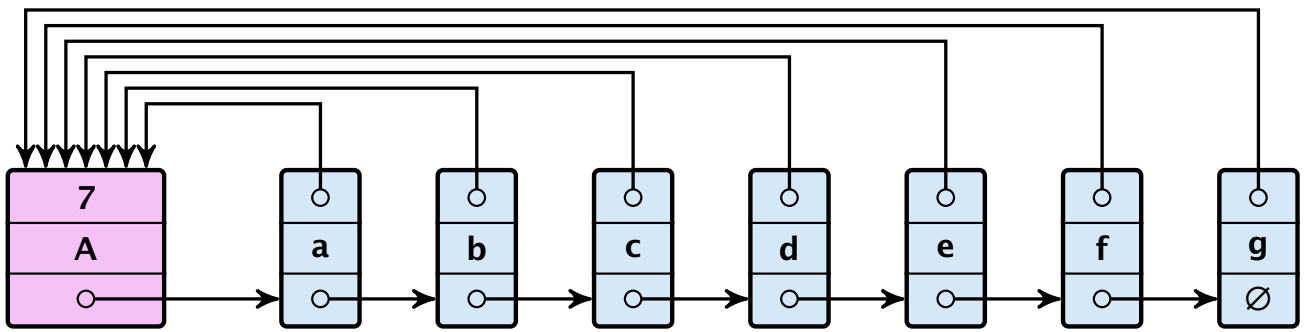

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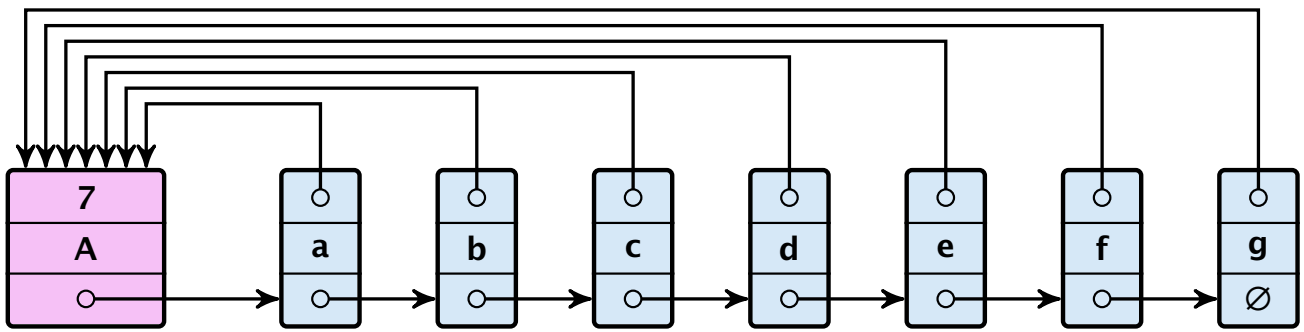

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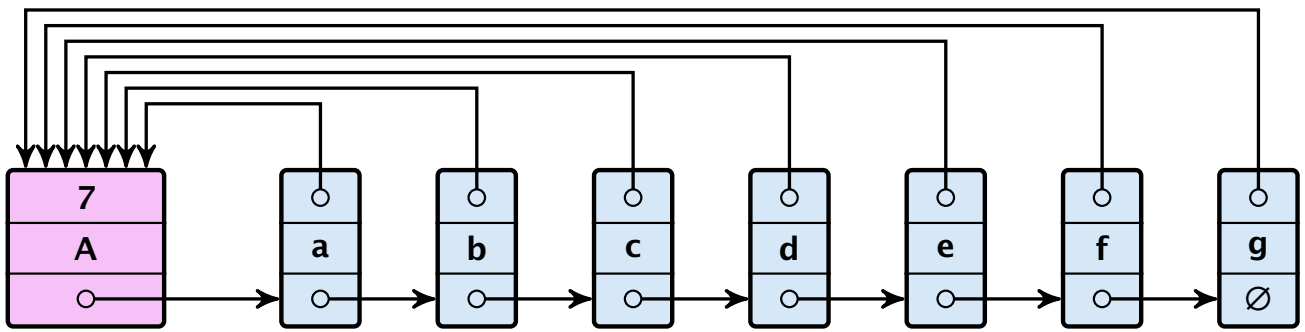

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List Implementation

union(x, y)

- Determine sets S_x and S_y.
- Traverse the smaller list (say S_y), and change all backward pointers to the head of list S_x.
- Insert list S_y at the head of S_x.
- Adjust the size-field of list S_x.
- Time: $\min\{|S_x|, |S_y|\}$.

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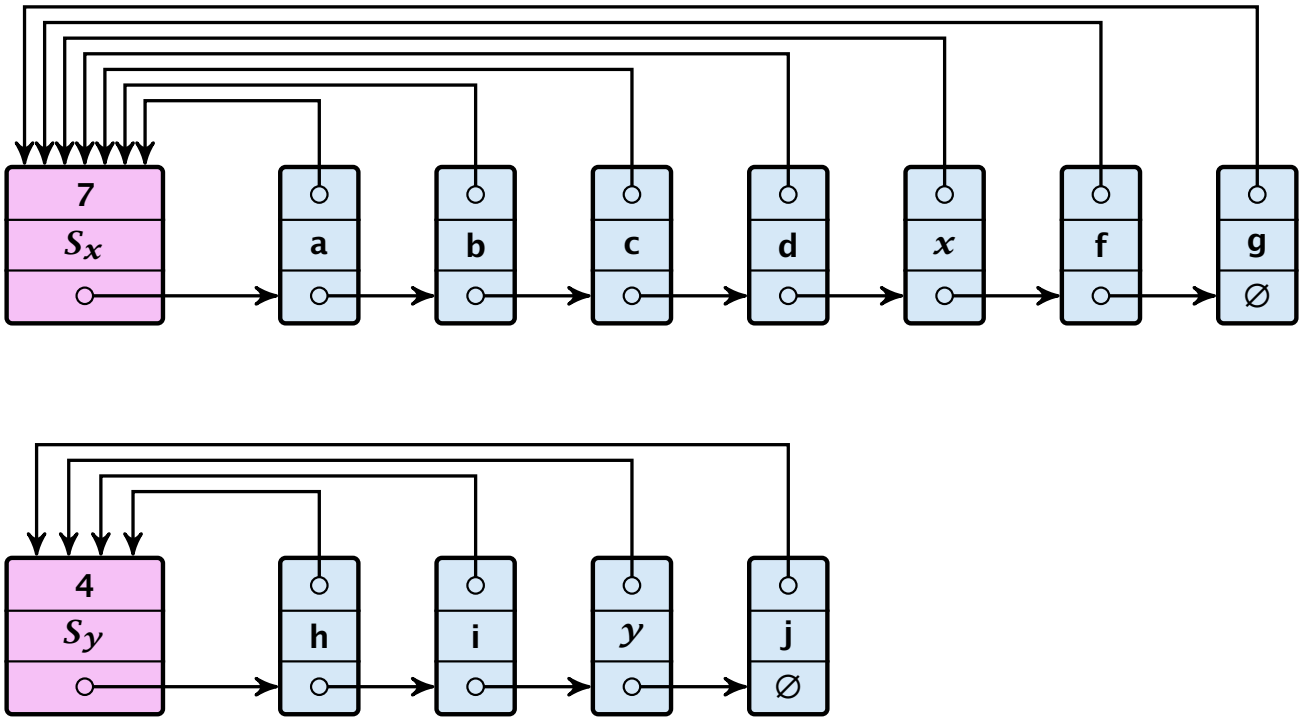

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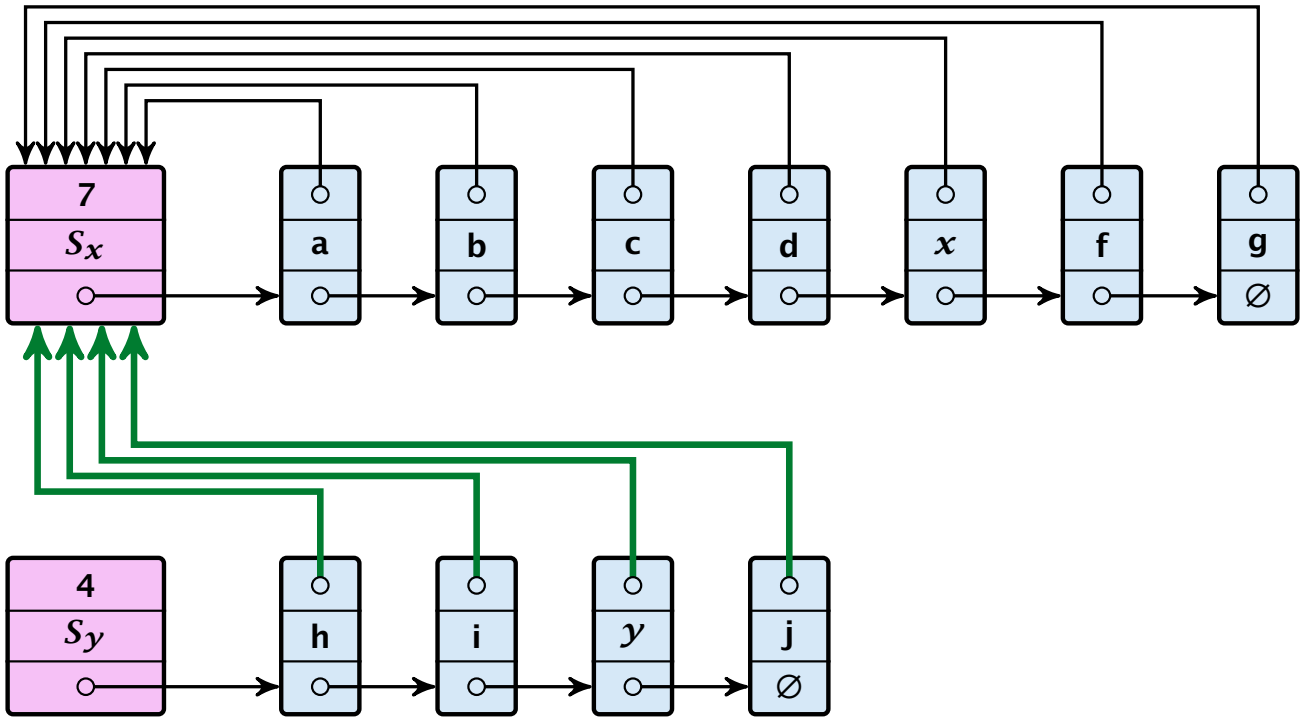

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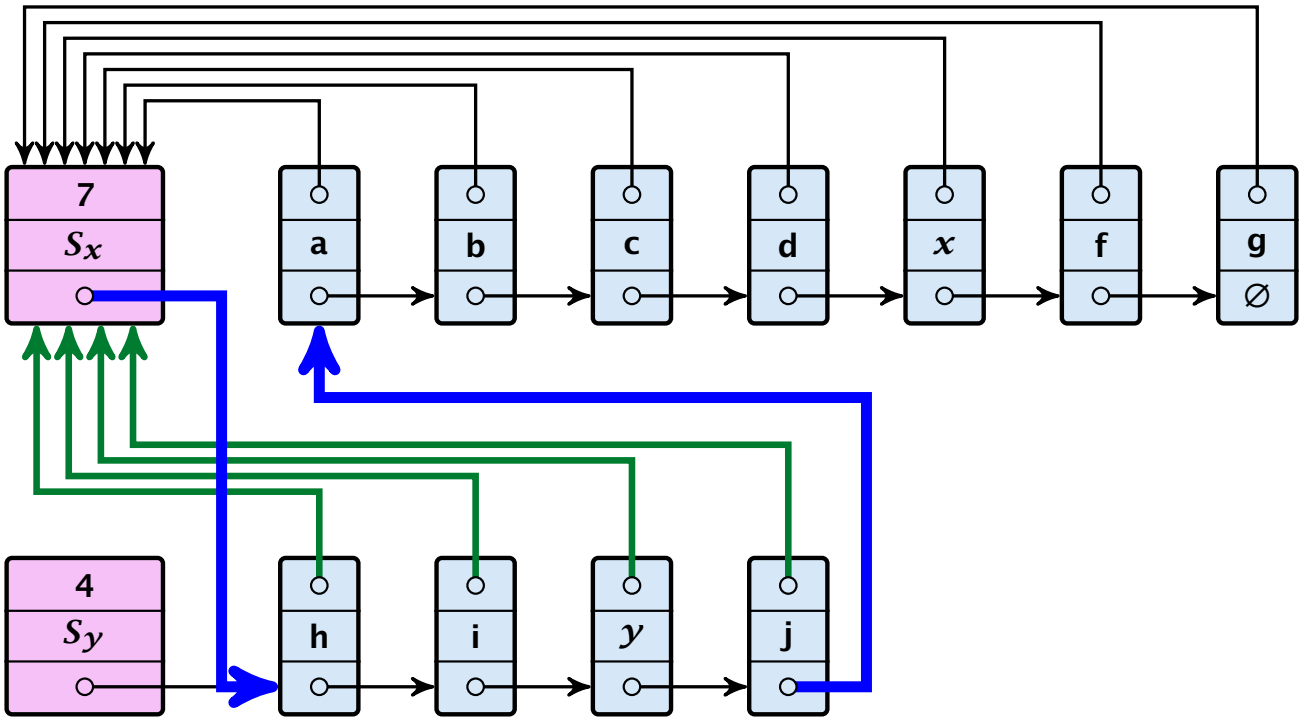

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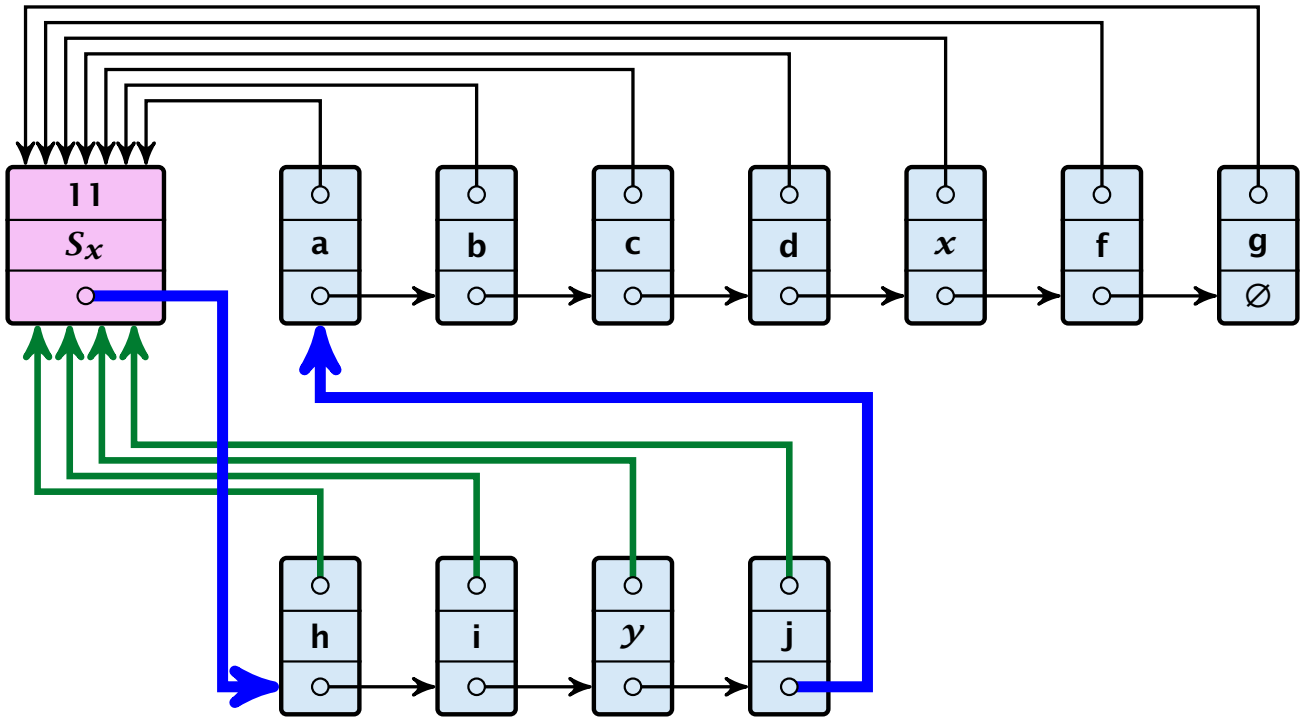

List Implementation

Running times:

- $\text{find}(x)$: constant
- $\text{makeset}(x)$: constant
- $\text{union}(x,y)$: $\mathcal{O}(n)$, where n denotes the number of elements contained in the set system.

List Implementation

Lemma 37

The list implementation for the ADT union find fulfills the following amortized time bounds:

- $\text{find}(x)$: $\mathcal{O}(1)$.
- $\text{makeset}(x)$: $\mathcal{O}(\log n)$.
- $\text{union}(x,y)$: $\mathcal{O}(1)$.

The Accounting Method for Amortized Time Bounds

- There is a bank account for every element in the data structure.
- Initially the balance on all accounts is zero.
- Whenever for an operation the amortized time bound exceeds the actual cost, the difference is credited to some bank accounts of elements involved.
- Whenever for an operation the actual cost exceeds the amortized time bound, the difference is charged to bank accounts of some of the elements involved.
- If we can find a charging scheme that guarantees that balances always stay positive the amortized time bounds are proven.

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List Implementation

- For an operation whose actual cost exceeds the amortized cost we charge the excess to the elements involved.
- In total we will charge at most $\mathcal{O}(\log n)$ to an element (regardless of the request sequence).
- For each element a makeset operation occurs as the first operation involving this element.
- We inflate the amortized cost of the makeset-operation to $\Theta(\log n)$, i.e., at this point we fill the bank account of the element to $\Theta(\log n)$.
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- We inflate the amortized cost of the makeset-operation to $\Theta(\log n)$, i.e., at this point we fill the bank account of the element to $\Theta(\log n)$.
- Later operations charge the account but the balance never drops below zero.

List Implementation

- For an operation whose actual cost exceeds the amortized cost we charge the excess to the elements involved.
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List Implementation

makeset(x): The actual cost is $\mathcal{O}(1)$. Due to the cost inflation the amortized cost is $\mathcal{O}(\log n)$.

find(x): For this operation we define the amortized cost and the actual cost to be the same. Hence, this operation does not change any accounts. Cost: $\mathcal{O}(1)$.

union(x, y):

- If $x = y$, the cost is constant, no bank accounts change.
- Otw. the actual cost is $\mathcal{O}(\log n)$.
- Assume wlog. that x_y is the smaller set, let c denote the hidden constant, i.e., the actual cost is at most $c \log n$.
- Charge c to every element in set x_y.

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union(x, y):

- If $S_x = S_y$ the cost is constant; no bank accounts change.
- Otw. the actual cost is $\mathcal{O}(\min\{|S_x|, |S_y|\})$.
- Assume wlog. that S_x is the smaller set; let c denote the hidden constant, i.e., the actual cost is at most $c \cdot |S_x|$.
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List Implementation

Lemma 38

An element is charged at most $\lfloor \log_2 n \rfloor$ times, where n is the total number of elements in the set system.

Proof.

Whenever an element x is charged the number of elements in x's set doubles. This can happen at most $\lfloor \log n \rfloor$ times. □

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Implementation via Trees

- Maintain nodes of a set in a tree.
- The root of the tree is the label of the set.
- Only pointer to parent exists; we cannot list all elements of a given set.
- Example:

Set system $\{2,5,10,12\}$, $\{3,6,7,8,9,14,17\}$, $\{16,19,23\}$.

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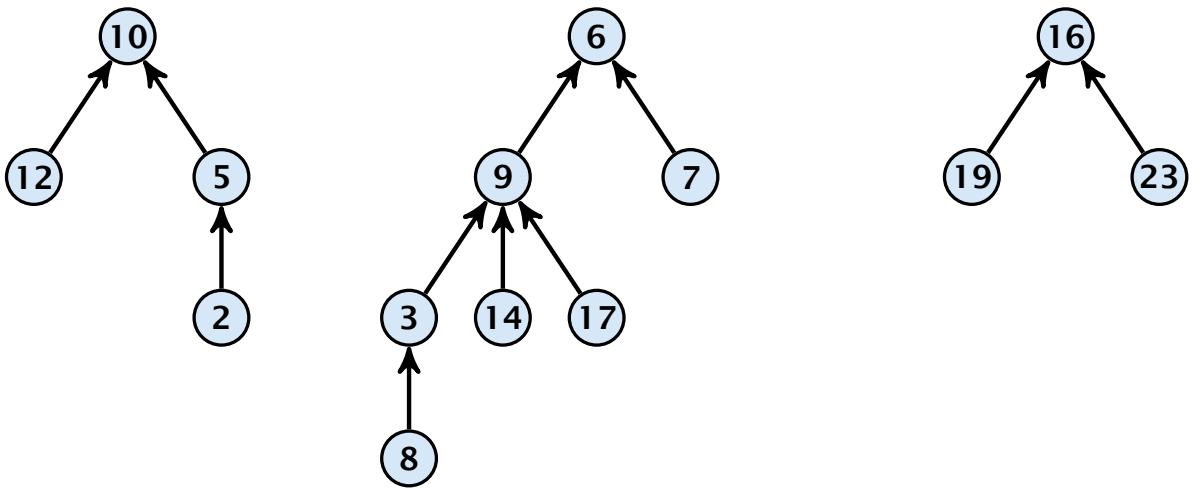

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Implementation via Trees

makeset(x)

- Create a singleton tree. Return pointer to the root.
- Time: $O(1)$.

find(x)

- Start at element x in the tree. Go upwards until you reach the root.
- Time: $O(\text{rank}(x))$, where $\text{rank}(x)$ is the distance of element x to the root in its tree. Note $\text{rank}(x) \leq \log n$.

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makeset(x)

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- Time: $\mathcal{O}(\log n)$ where $\log n$ is the distance of element x to the root in its tree. Worst case is n.

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To support union we store the size of a tree in its root.

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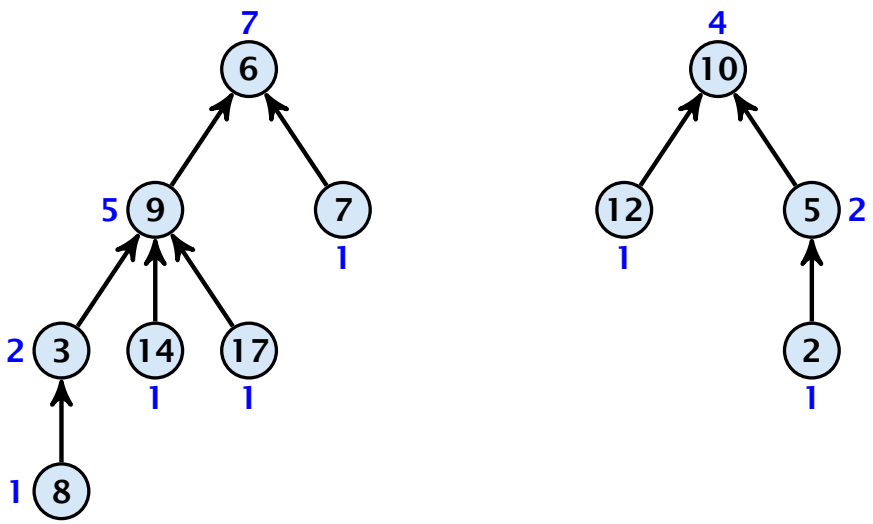

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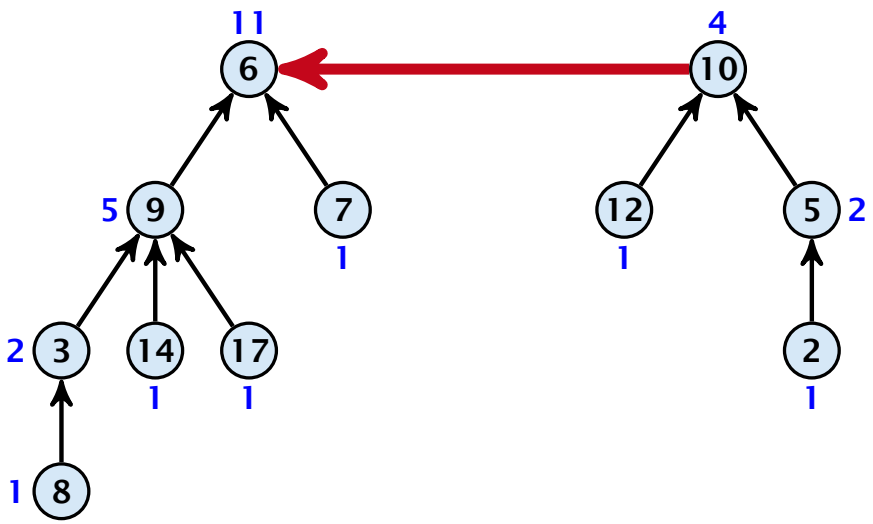

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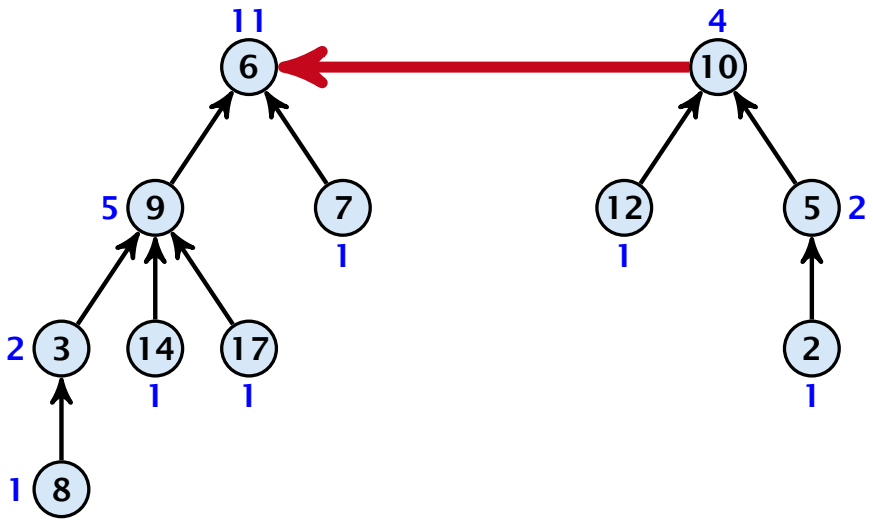

- Time: constant for $\text{link}(a, b)$ plus two find-operations.

Implementation via Trees

Lemma 39

The running time (non-amortized!!!) for $\mathrm{find}(x)$ *is* $\mathcal{O}(\log n)$.

Proof.

- When we attach a tree with root r_1 to become a child of a tree with root r_2 then $\mathrm{size}(r_2)$ increases by $\mathrm{size}(r_1)$, where $\mathrm{size}(\cdot)$ denotes the value of the size field right after the operation.
- After that the value of $\mathrm{size}(r_2)$ stays fixed, while the value of $\mathrm{size}(r_1)$ may still increase.
- Hence, at any point in time a tree fulfills $\mathrm{size}(v) \geq 2^{\mathrm{height}(v)}$ for any pair of nodes (v, r_v), where r_v is a parent of v.

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- When we attach a tree with root c to become a child of a tree with root p, then $\mathrm{size}(p) \geq 2\,\mathrm{size}(c)$, where size denotes the value of the size-field right after the operation.
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Path Compression

find(x):

- Go upward until you find the root.
- Re-attach all visited nodes as children of the root.
- Speeds up successive find-operations.

- Note that the size fields now only give an upper bound on the size of a subtree.

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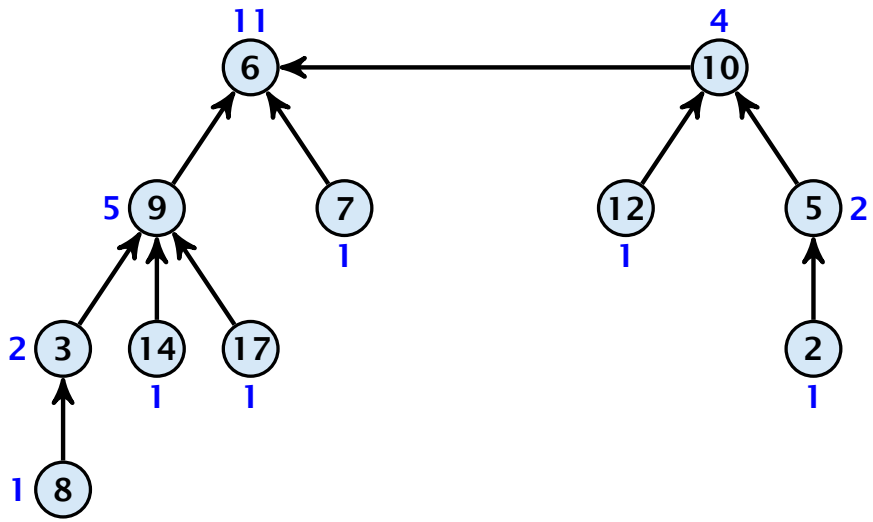

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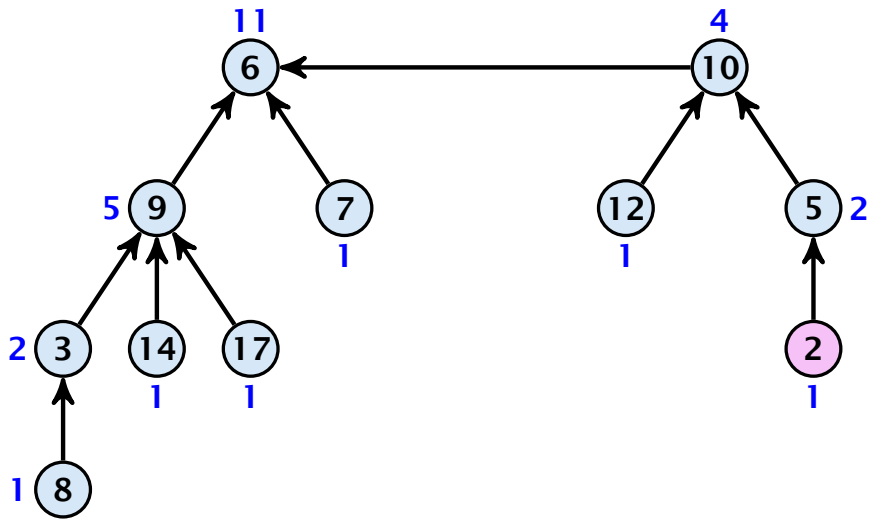

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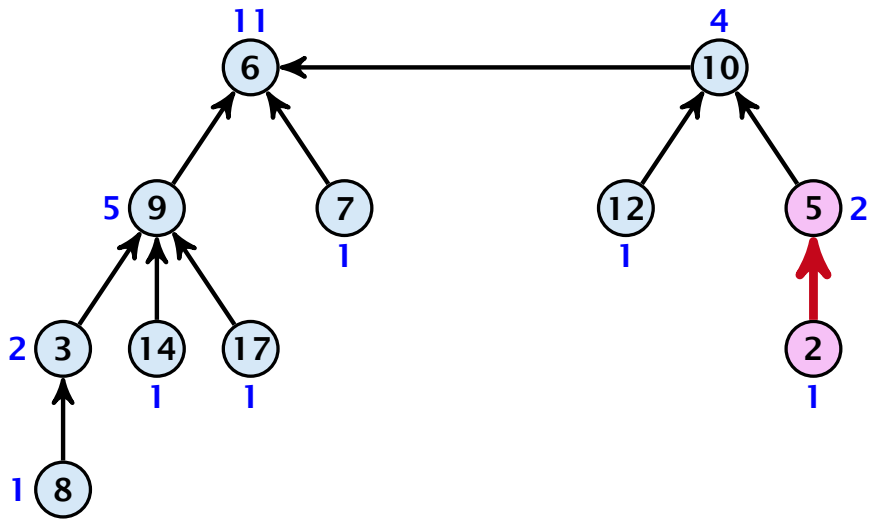

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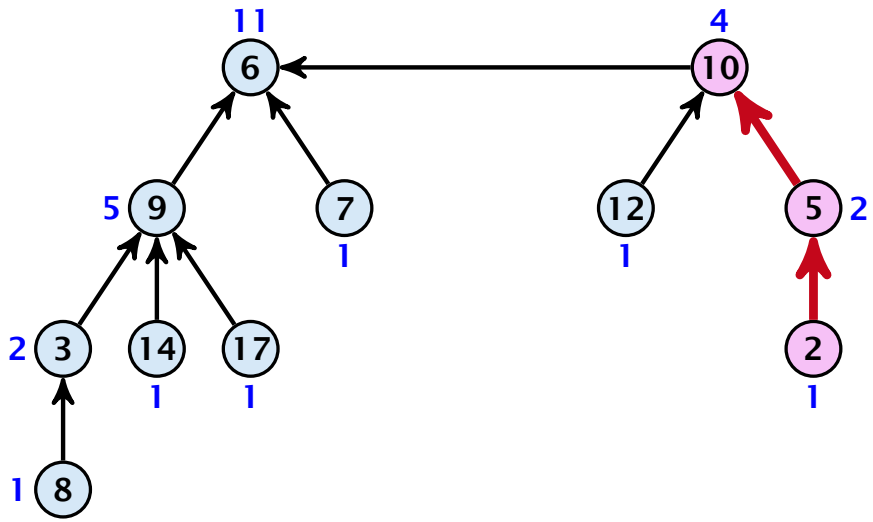

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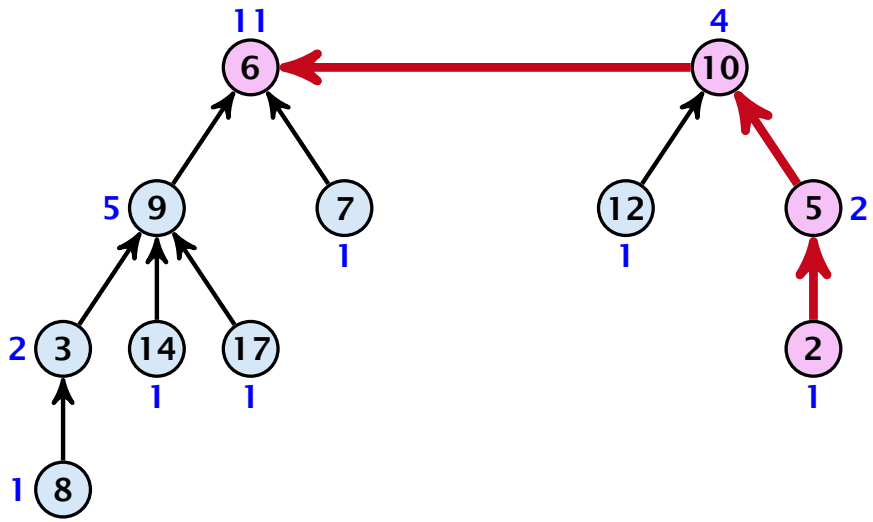

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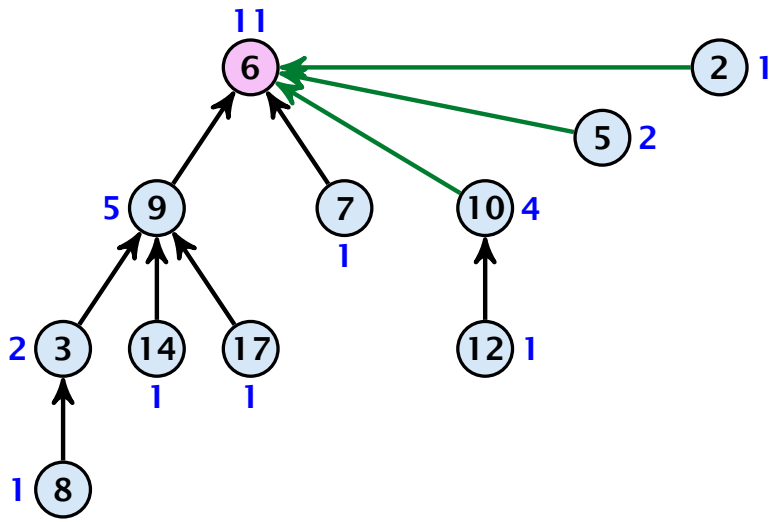

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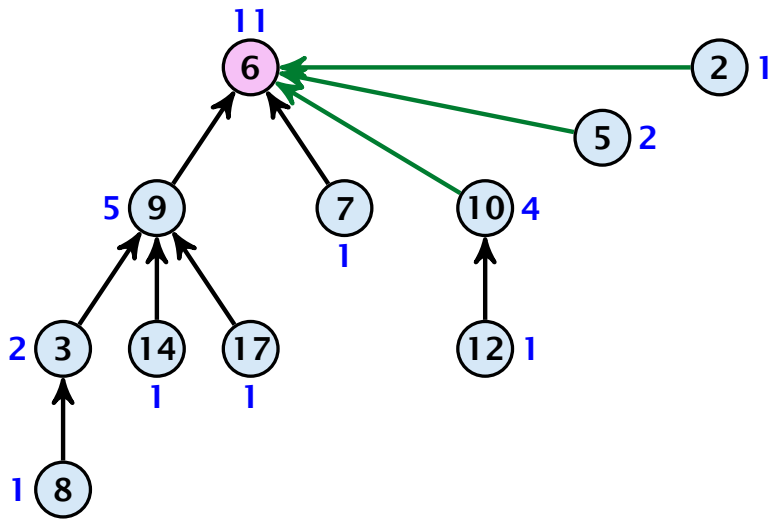

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Asymptotically the cost for a find-operation does not increase due to the path compression heuristic.

However, for a worst-case analysis there is no improvement on the running time. It can still happen that a find-operation takes time $\mathcal{O}(\log n)$.

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Amortized Analysis

Definitions:

- $s(v)$ = the number of nodes that were in the sub-tree rooted at v when v became the child of another node (or the number of nodes if v is the root).

  Note that this is the same as the size of v's subtree in the case that there are no find-operations.

Lemma 40

The rank of a parent must be strictly larger than the rank of a child.

Amortized Analysis

Definitions:

- $\text{size}(v) :=$ the number of nodes that were in the sub-tree rooted at v when v became the child of another node (or the number of nodes if v is the root).

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- $\text{rank}(v) := \lfloor \log(\text{size}(v)) \rfloor$.
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Amortized Analysis

Lemma 41

There are at most $n/2^s$ nodes of rank s.

Proof.

- Let's say a node v sees node w if w is in v's subtree at the time that v becomes a child.
- A node v sees at most one node of rank s during the running time of the algorithm.
- This holds because the rank-sequence of the roots of the different trees that contain v during the running time of the algorithm is a strictly increasing sequence.
- Hence, every node sees at most one rank s node, but every rank s node is seen by at least 2^s different nodes. □

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$$\text{tow}(i) := \begin{cases} 1 & \text{if } i = 0 \\ 2^{\text{tow}(i-1)} & \text{otw.} \end{cases} \qquad \text{tow}(i) = \underbrace{2^{2^{2^{2^{2}}}}}_{i \text{ times}}$$

and

$$\log^*(n) := \min\{i \mid \text{tow}(i) \geq n\} \ .$$

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Theorem 42

Union find with path compression fulfills the following amortized running times:

- $\text{makeset}(x) : \mathcal{O}(\log^*(n))$
- $\text{find}(x) : \mathcal{O}(\log^*(n))$
- $\text{union}(x, y) : \mathcal{O}(\log^*(n))$

Amortized Analysis

In the following we assume $n \geq 2$.

rank-group:

- A node with rank $r(v)$ is in rank group $\log r(v)$.
- The rank-group g contains only nodes with rank r or rank $\ldots$
- A rank group g contains ranks $\ldots$
- The maximum non-empty rank group is $\ldots$ which is the case for $\ldots$ (which holds for $\ldots$)
- Hence, the total number of rank-groups is $\ldots$

Amortized Analysis

In the following we assume $n \geq 2$.

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- A node with rank $\mathrm{rank}(v)$ is in rank group $\log^*(\mathrm{rank}(v))$.
- The rank-group $g = 0$ contains only nodes with rank 0 or rank 1.
- A rank group $g \geq 1$ contains ranks $\mathrm{tow}(g-1)+1, \ldots, \mathrm{tow}(g)$.
- The maximum non-empty rank group is $\log^*(\lfloor \log n \rfloor) \leq \log^*(n) - 1$ (which holds for $n \geq 2$).
- Hence, the total number of rank-groups is at most $\log^* n$.

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Amortized Analysis

Accounting Scheme:

- create an account for every find-operation
- create an account for every node v

The cost for a find-operation is equal to the length of the path traversed. We charge the cost for going from v to $\text{parent}[v]$ as follows:

- If $\text{parent}[v]$ is the root we charge the cost to the find-account.
- If the group-number of v and $\text{parent}[v]$ is the same as that of v (before starting path compression) we charge the cost to the node-account of v.
- Otherwise we charge the cost to the find-account.

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Amortized Analysis

Observations:

- A find-account is charged at most $\log^* n + 1$ times (once for the root and at most $\log^* n$ times when increasing the rank-group).
- After a node v is charged its parent-edge is reassigned. The rank of the parent strictly increases.
- After some charges to v the parent will be in a larger rank-group. Then v will never be charged again.
- The total charge made to a node in rank-group k is at most $\mathrm{tower}(k) - \mathrm{tower}(k-1)$.

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- The total charge is at most

$$\sum_g n(g) \cdot \mathrm{tow}(g) \ ,$$

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Hence,

$$\sum_g n(g)\,\mathrm{tow}(g) \leq n(0)\,\mathrm{tow}(0) + \sum_{g\geq 1} n(g)\,\mathrm{tow}(g) \leq n \log^*(n)$$

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Without loss of generality we can assume that all makeset-operations occur at the start.

This means if we inflate the cost of makeset to $\log^* n$ and add this to the node account of v then the balances of all node accounts will sum up to a positive value (this is sufficient to obtain an amortized bound).

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The analysis is not tight. In fact it has been shown that the amortized time for the union-find data structure with path compression is $\mathcal{O}(\alpha(m,n))$, where $\alpha(m,n)$ is the inverse Ackermann function which grows a lot lot slower than $\log^* n$. (Here, we consider the average running time of m operations on at most n elements).

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Amortized Analysis

$$A(x,y) = \begin{cases} y+1 & \text{if } x = 0 \\ A(x-1,1) & \text{if } y = 0 \\ A(x-1,A(x,y-1)) & \text{otw.} \end{cases}$$

$$\alpha(m,n) = \min\{i \geq 1 : A(i,\lfloor m/n \rfloor) \geq \log n\}$$

- $A(0,y) = y+1$
- $A(1,y) = y+2$
- $A(2,y) = 2y+3$
- $A(3,y) = 2^{y+3} - 3$
- $A(4,y) = \underbrace{2^{2^{2^{2}}} - 3}_{y+3 \text{ times}}$

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10 van Emde Boas Trees

Dynamic Set Data Structure S:

- $S.\text{insert}(x)$
- $S.\text{delete}(x)$
- $S.\text{search}(x)$
- $S.\min()$
- $S.\max()$
- $S.\text{succ}(x)$
- $S.\text{pred}(x)$

10 van Emde Boas Trees

For this chapter we ignore the problem of storing satellite data:

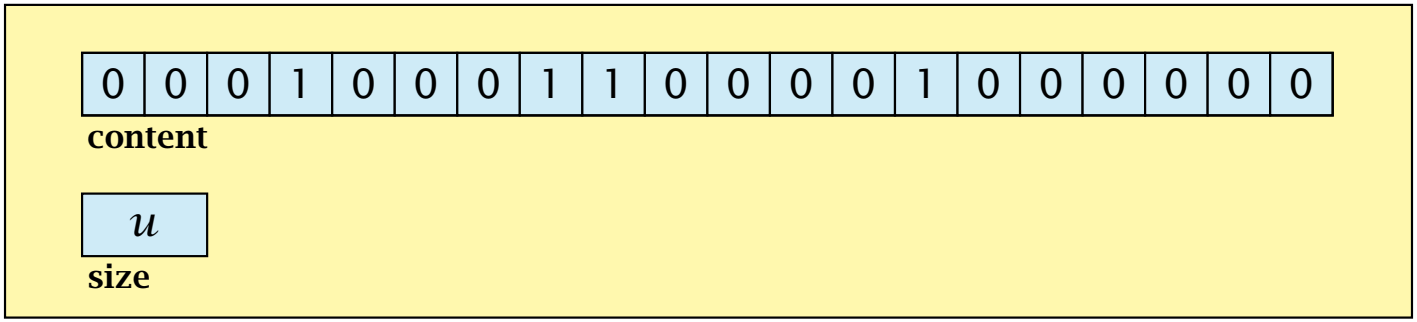
- $S.\mathbf{insert}(x)$: Inserts x into S.
- $S.\mathbf{delete}(x)$: Deletes x from S. Usually assumes that $x \in S$.
- $S.\mathbf{member}(x)$: Returns 1 if $x \in S$ and 0 otw.
- $S.\mathbf{min}()$: Returns the value of the minimum element in S.
- $S.\mathbf{max}()$: Returns the value of the maximum element in S.
- $S.\mathbf{succ}(x)$: Returns successor of x in S. Returns null if x is maximum or larger than any element in S. Note that x needs not to be in S.
- $S.\mathbf{pred}(x)$: Returns the predecessor of x in S. Returns null if x is minimum or smaller than any element in S. Note that x needs not to be in S.

10 van Emde Boas Trees

Can we improve the existing algorithms when the keys are from a restricted set?

In the following we assume that the keys are from $\{0, 1, \ldots, u-1\}$, where u denotes the size of the universe.

Implementation 1: Array



one array of u bits

Use an array that encodes the indicator function of the dynamic set.

Implementation 1: Array

Algorithm 1 array.insert(x)

1: content[x] $\leftarrow$ 1;

Algorithm 2 array.delete(x)

1: content[x] $\leftarrow$ 0;

Algorithm 3 array.member(x)

1: **return** content[x];

- Note that we assume that x is valid, i.e., it falls within the array boundaries.
- Obviously(?) the running time is constant.

Implementation 1: Array

Algorithm 4 array.max()

```
1: for (i = size −1; i ≥ 0; i−−) do
2:         if content[i] = 1 then return i;
3: return null;
```

Algorithm 5 array.min()

```
1: for (i = 0; i < size; i++) do
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- Running time is $\mathcal{O}(u)$ in the worst case.

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Implementation 1: Array

Algorithm 6 array.succ(x)

```
1: for (i = x + 1; i < size; i++) do
2:         if content[i] = 1 then return i;
3: return null;
```

Algorithm 7 array.pred(x)

```
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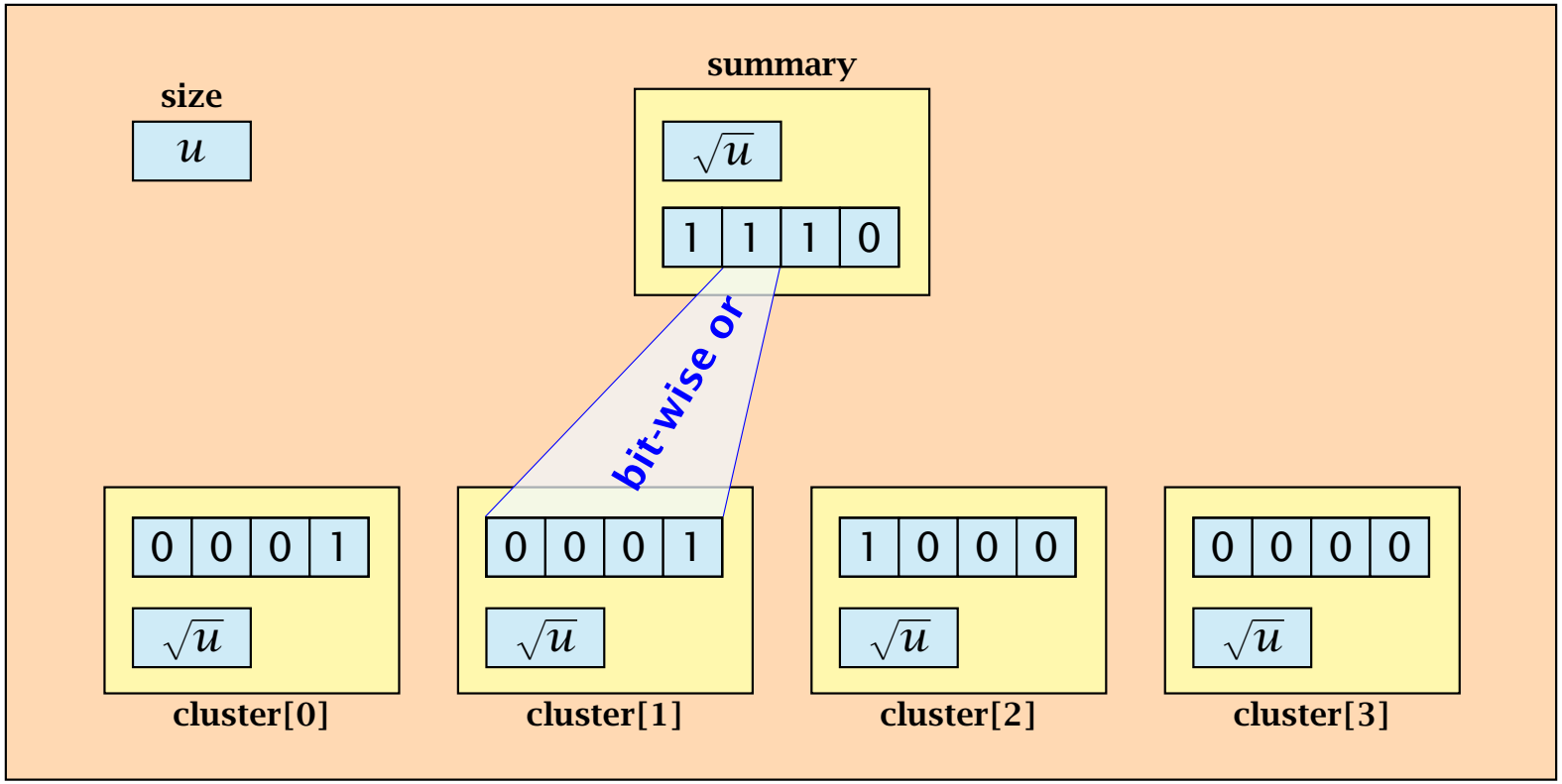
Implementation 2: Summary Array



- $\sqrt{u}$ cluster-arrays of $\sqrt{u}$ bits.
- One summary-array of $\sqrt{u}$ bits. The i-th bit in the summary array stores the bit-wise or of the bits in the i-th cluster.

Implementation 2: Summary Array

The bit for a key x is contained in cluster number $\left\lfloor \frac{x}{\sqrt{u}} \right\rfloor$.

Within the cluster-array the bit is at position $x \bmod \sqrt{u}$.

For simplicity we assume that $u = 2^{2k}$ for some $k \geq 1$. Then we can compute the cluster-number for an entry x as $\text{high}(x)$ (the upper half of the dual representation of x) and the position of x within its cluster as $\text{low}(x)$ (the lower half of the dual representation).

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1: return cluster[high(x)].member(low(x));
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Algorithm 9 insert(x)

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1: cluster[high(x)].insert(low(x));
2: summary.insert(high(x));
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- The running times are constant, because the corresponding array-functions have constant running times.

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Implementation 2: Summary Array

Algorithm 10 delete(x)

```
1: cluster[high(x)].delete(low(x));
2: if cluster[high(x)].min() = null then
3:         summary.delete(high(x));
```

- The running time is dominated by the cost of a minimum computation on an array of size $\sqrt{u}$. Hence, $\Theta(\sqrt{u})$.

Implementation 2: Summary Array

Algorithm 10 delete(x)

```
1: cluster[high(x)].delete(low(x));
2: if cluster[high(x)].min() = null then
3:         summary.delete(high(x));
```

- The running time is dominated by the cost of a minimum computation on an array of size $\sqrt{u}$. Hence, $\mathcal{O}(\sqrt{u})$.

Implementation 2: Summary Array

Algorithm 11 max()

```
1: maxcluster ← summary.max();
2: if maxcluster = null return null;
3: offs ← cluster[maxcluster].max()
4: return maxcluster ∘ offs;
```

Algorithm 12 min()

```
1: mincluster ← summary.min();
2: if mincluster = null return null;
3: offs ← cluster[mincluster].min();
4: return mincluster ∘ offs;
```

- Running time is roughly $2\sqrt{u} = \mathcal{O}(\sqrt{u})$ in the worst case.

Implementation 2: Summary Array

Algorithm 11 max()

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1: maxcluster ← summary.max();
2: if maxcluster = null return null;
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4: return mincluster ∘ offs;
```

- Running time is roughly $2\sqrt{u} = \mathcal{O}(\sqrt{u})$ in the worst case.

Implementation 2: Summary Array

Algorithm 13 succ(x)

```
1: m ← cluster[high(x)].succ(low(x))
2: if m ≠ null then return high(x) ∘ m;
3: succcluster ← summary.succ(high(x));
4: if succcluster ≠ null then
5:         offs ← cluster[succcluster].min();
6:         return succcluster ∘ offs;
7: return null;
```

- Running time is roughly $3\sqrt{u} = \mathcal{O}(\sqrt{u})$ in the worst case.

Implementation 2: Summary Array

Algorithm 13 succ(x)

```
1: m ← cluster[high(x)].succ(low(x))
2: if m ≠ null then return high(x) ∘ m;
3: succcluster ← summary.succ(high(x));
4: if succcluster ≠ null then
5:         offs ← cluster[succcluster].min();
6:         return succcluster ∘ offs;
7: return null;
```

- Running time is roughly $3\sqrt{u} = \mathcal{O}(\sqrt{u})$ in the worst case.

Implementation 2: Summary Array

Algorithm 14 pred(x)

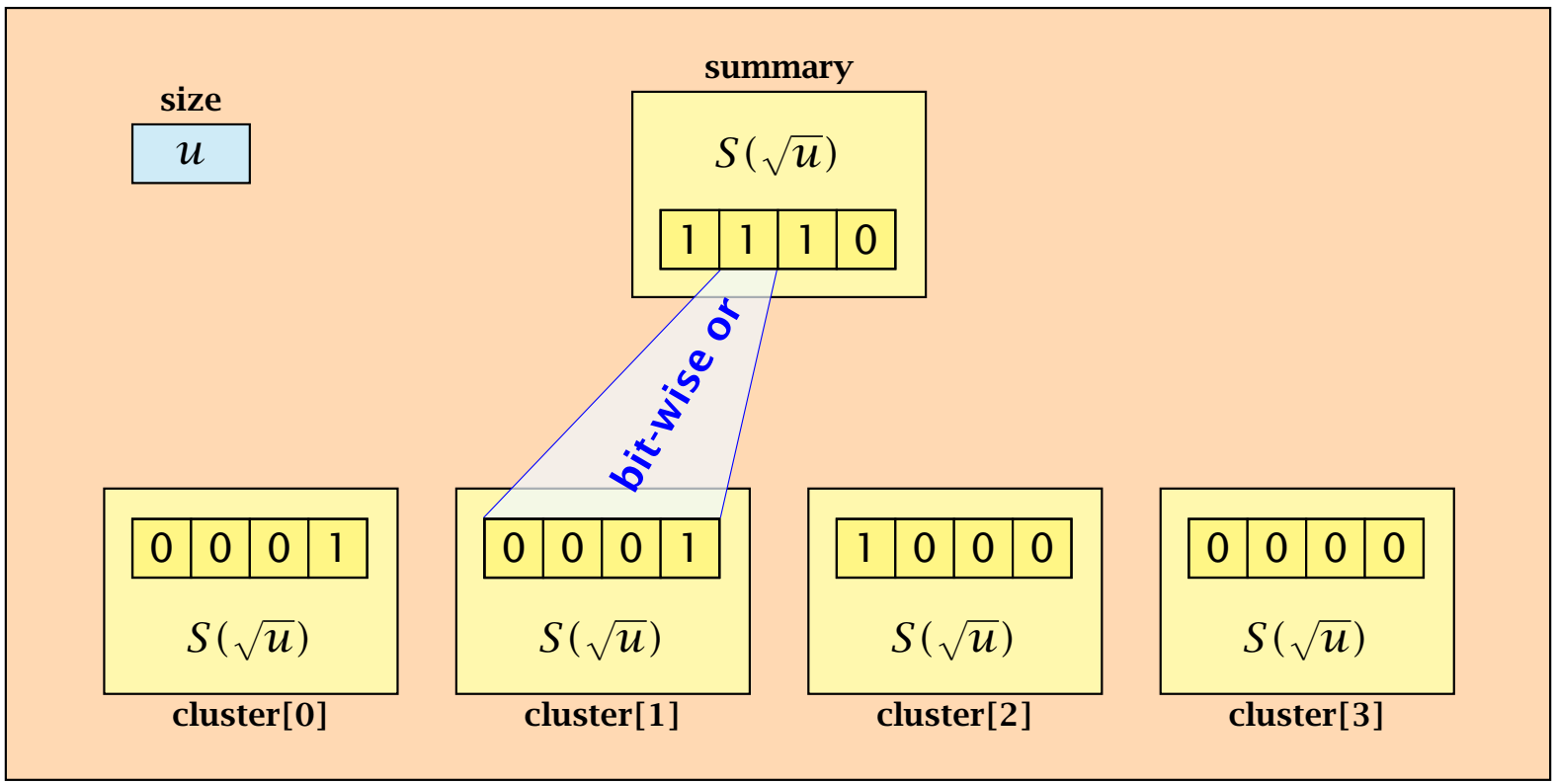
```
1: m ← cluster[high(x)].pred(low(x))
2: if m ≠ null then return high(x) ∘ m;
3: predcluster ← summary.pred(high(x));
4: if predcluster ≠ null then
5:         offs ← cluster[predcluster].max();
6:         return predcluster ∘ offs;
7: return null;
```

- Running time is roughly $3\sqrt{u} = \mathcal{O}(\sqrt{u})$ in the worst case.

Implementation 3: Recursion

Instead of using sub-arrays, we build a recursive data-structure.

$S(u)$ is a dynamic set data-structure representing u bits:



Implementation 3: Recursion

We assume that $u = 2^{2^k}$ for some k.

The data-structure $S(2)$ is defined as an array of 2-bits (end of the recursion).

Implementation 3: Recursion

The code from Implementation 2 can be used unchanged. We only need to redo the analysis of the running time.

Note that in the code we do not need to specifically address the non-recursive case. This is achieved by the fact that an $S(4)$ will contain $S(2)$'s as sub-datastructures, which are arrays. Hence, a call like $\text{cluster}[1].\min()$ from within the data-structure $S(4)$ is not a recursive call as it will call the function $\text{array}.\min()$.

This means that the non-recursive case is been dealt with while initializing the data-structure.

Implementation 3: Recursion

The code from Implementation 2 can be used unchanged. We only need to redo the analysis of the running time.

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This means that the non-recursive case is been dealt with while initializing the data-structure.

Implementation 3: Recursion

Algorithm 15 member(x)

1: **return** cluster[high(x)].member(low(x));

- $T_{\text{mem}}(u) = T_{\text{mem}}(\sqrt{u}) + 1$.

Implementation 3: Recursion

Algorithm 16 insert(x)

```
1: cluster[high(x)].insert(low(x));
2: summary.insert(high(x));
```

- $T_{\text{ins}}(u) = 2T_{\text{ins}}(\sqrt{u}) + 1$.

Implementation 3: Recursion

Algorithm 17 delete(x)

```
1: cluster[high(x)].delete(low(x));
2: if cluster[high(x)].min() = null then
3:         summary.delete(high(x));
```

- $T_{\text{del}}(u) = 2T_{\text{del}}(\sqrt{u}) + T_{\min}(\sqrt{u}) + 1$.

Implementation 3: Recursion

Algorithm 18 min()

```
1: mincluster ← summary.min();
2: if mincluster = null return null;
3: offs ← cluster[mincluster].min();
4: return mincluster ∘ offs;
```

- $T_{\min}(u) = 2T_{\min}(\sqrt{u}) + 1$.

Implementation 3: Recursion

Algorithm 19 succ(x)

```
1: m ← cluster[high(x)].succ(low(x))
2: if m ≠ null then return high(x) ∘ m;
3: succcluster ← summary.succ(high(x));
4: if succcluster ≠ null then
5:         offs ← cluster[succcluster].min();
6:         return succcluster ∘ offs;
7: return null;
```

- $T_{\text{succ}}(u) = 2T_{\text{succ}}(\sqrt{u}) + T_{\min}(\sqrt{u}) + 1$.

Implementation 3: Recursion

$$T_{\text{mem}}(u) = T_{\text{mem}}(\sqrt{u}) + 1:$$

Implementation 3: Recursion

$$T_{\text{mem}}(u) = T_{\text{mem}}(\sqrt{u}) + 1:$$

Set $\ell := \log u$ and $X(\ell) := T_{\text{mem}}(2^\ell)$.

Implementation 3: Recursion

$$T_{\text{mem}}(u) = T_{\text{mem}}(\sqrt{u}) + 1:$$

Set $\ell := \log u$ and $X(\ell) := T_{\text{mem}}(2^\ell)$.Then

Implementation 3: Recursion

$$T_{\text{mem}}(u) = T_{\text{mem}}(\sqrt{u}) + 1:$$

Set $\ell := \log u$ and $X(\ell) := T_{\text{mem}}(2^\ell)$.Then

$$X(\ell)$$

Implementation 3: Recursion

$$T_{\text{mem}}(u) = T_{\text{mem}}(\sqrt{u}) + 1:$$

Set $\ell := \log u$ and $X(\ell) := T_{\text{mem}}(2^\ell)$.Then

$$X(\ell) = T_{\text{mem}}(2^\ell)$$

Implementation 3: Recursion

$$T_{\text{mem}}(u) = T_{\text{mem}}(\sqrt{u}) + 1:$$

Set $\ell := \log u$ and $X(\ell) := T_{\text{mem}}(2^\ell)$.Then

$$X(\ell) = T_{\text{mem}}(2^\ell) = T_{\text{mem}}(u)$$

Implementation 3: Recursion

$$T_{\mathrm{mem}}(u) = T_{\mathrm{mem}}(\sqrt{u}) + 1:$$

Set $\ell := \log u$ and $X(\ell) := T_{\mathrm{mem}}(2^{\ell})$.Then

$$X(\ell) = T_{\mathrm{mem}}(2^{\ell}) = T_{\mathrm{mem}}(u) = T_{\mathrm{mem}}(\sqrt{u}) + 1$$

Implementation 3: Recursion

$$T_{\text{mem}}(u) = T_{\text{mem}}(\sqrt{u}) + 1:$$

Set $\ell := \log u$ and $X(\ell) := T_{\text{mem}}(2^\ell)$.Then

$$X(\ell) = T_{\text{mem}}(2^\ell) = T_{\text{mem}}(u) = T_{\text{mem}}(\sqrt{u}) + 1$$
$$= T_{\text{mem}}(2^{\frac{\ell}{2}}) + 1$$

Implementation 3: Recursion

$$T_{\mathrm{mem}}(u) = T_{\mathrm{mem}}(\sqrt{u}) + 1:$$

Set $\ell := \log u$ and $X(\ell) := T_{\mathrm{mem}}(2^\ell)$.Then

$$\begin{aligned} X(\ell) = T_{\mathrm{mem}}(2^\ell) = T_{\mathrm{mem}}(u) &= T_{\mathrm{mem}}(\sqrt{u}) + 1 \\ &= T_{\mathrm{mem}}(2^{\frac{\ell}{2}}) + 1 = X(\tfrac{\ell}{2}) + 1 \ . \end{aligned}$$

Implementation 3: Recursion

$T_{\mathrm{mem}}(u) = T_{\mathrm{mem}}(\sqrt{u}) + 1$:

Set $\ell := \log u$ and $X(\ell) := T_{\mathrm{mem}}(2^\ell)$.Then

$$X(\ell) = T_{\mathrm{mem}}(2^\ell) = T_{\mathrm{mem}}(u) = T_{\mathrm{mem}}(\sqrt{u}) + 1$$
$$= T_{\mathrm{mem}}(2^{\frac{\ell}{2}}) + 1 = X(\tfrac{\ell}{2}) + 1 \ .$$

Using Master theorem gives $X(\ell) = \mathcal{O}(\log \ell)$, and hence $T_{\mathrm{mem}}(u) = \mathcal{O}(\log \log u)$.

Implementation 3: Recursion

$$T_{\text{ins}}(u) = 2T_{\text{ins}}(\sqrt{u}) + 1.$$

Implementation 3: Recursion

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Implementation 3: Recursion

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$$X(\ell)$$

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$$X(\ell) = T_{\text{ins}}(2^\ell) = T_{\text{ins}}(u)$$

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Implementation 3: Recursion

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Using Master theorem gives $X(\ell) = \mathcal{O}(\ell)$, and hence $T_{\text{ins}}(u) = \mathcal{O}(\log u)$.

Implementation 3: Recursion

$$T_{\text{ins}}(u) = 2T_{\text{ins}}(\sqrt{u}) + 1.$$

Set $\ell := \log u$ and $X(\ell) := T_{\text{ins}}(2^\ell)$. Then

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Using Master theorem gives $X(\ell) = \mathcal{O}(\ell)$, and hence $T_{\text{ins}}(u) = \mathcal{O}(\log u)$.

The same holds for $T_{\max}(u)$ and $T_{\min}(u)$.

Implementation 3: Recursion

$$T_{\mathrm{del}}(u) = 2T_{\mathrm{del}}(\sqrt{u}) + T_{\mathrm{min}}(\sqrt{u}) + 1 \leq 2T_{\mathrm{del}}(\sqrt{u}) + c\log(u).$$

Implementation 3: Recursion

$$T_{\mathrm{del}}(u) = 2T_{\mathrm{del}}(\sqrt{u}) + T_{\mathrm{min}}(\sqrt{u}) + 1 \leq 2T_{\mathrm{del}}(\sqrt{u}) + c\log(u).$$

Set $\ell := \log u$ and $X(\ell) := T_{\mathrm{del}}(2^{\ell})$.

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$$X(\ell)$$

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Set $\ell := \log u$ and $X(\ell) := T_{\mathrm{del}}(2^\ell)$. Then

$$X(\ell) = T_{\mathrm{del}}(2^\ell) = T_{\mathrm{del}}(u)$$

Implementation 3: Recursion

$$T_{\mathbf{del}}(u) = 2T_{\mathbf{del}}(\sqrt{u}) + T_{\mathbf{min}}(\sqrt{u}) + 1 \leq 2T_{\mathbf{del}}(\sqrt{u}) + \boldsymbol{c}\log(u).$$

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$$X(\ell) = T_{\mathrm{del}}(2^\ell) = T_{\mathrm{del}}(u) = 2T_{\mathrm{del}}(\sqrt{u}) + c\log u$$

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Set $\ell := \log u$ and $X(\ell) := T_{\mathrm{del}}(2^{\ell})$. Then

$$\begin{aligned} X(\ell) = T_{\mathrm{del}}(2^{\ell}) = T_{\mathrm{del}}(u) &= 2T_{\mathrm{del}}(\sqrt{u}) + c\log u \\ &= 2T_{\mathrm{del}}(2^{\frac{\ell}{2}}) + c\ell \end{aligned}$$

Implementation 3: Recursion

$$T_{\mathrm{del}}(u) = 2T_{\mathrm{del}}(\sqrt{u}) + T_{\mathrm{min}}(\sqrt{u}) + 1 \leq 2T_{\mathrm{del}}(\sqrt{u}) + c\log(u).$$

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Implementation 3: Recursion

$$T_{\mathrm{del}}(u) = 2T_{\mathrm{del}}(\sqrt{u}) + T_{\mathrm{min}}(\sqrt{u}) + 1 \leq 2T_{\mathrm{del}}(\sqrt{u}) + c\log(u).$$

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Using Master theorem gives $X(\ell) = \Theta(\ell \log \ell)$, and hence $T_{\mathrm{del}}(u) = \mathcal{O}(\log u \log\log u)$.

Implementation 3: Recursion

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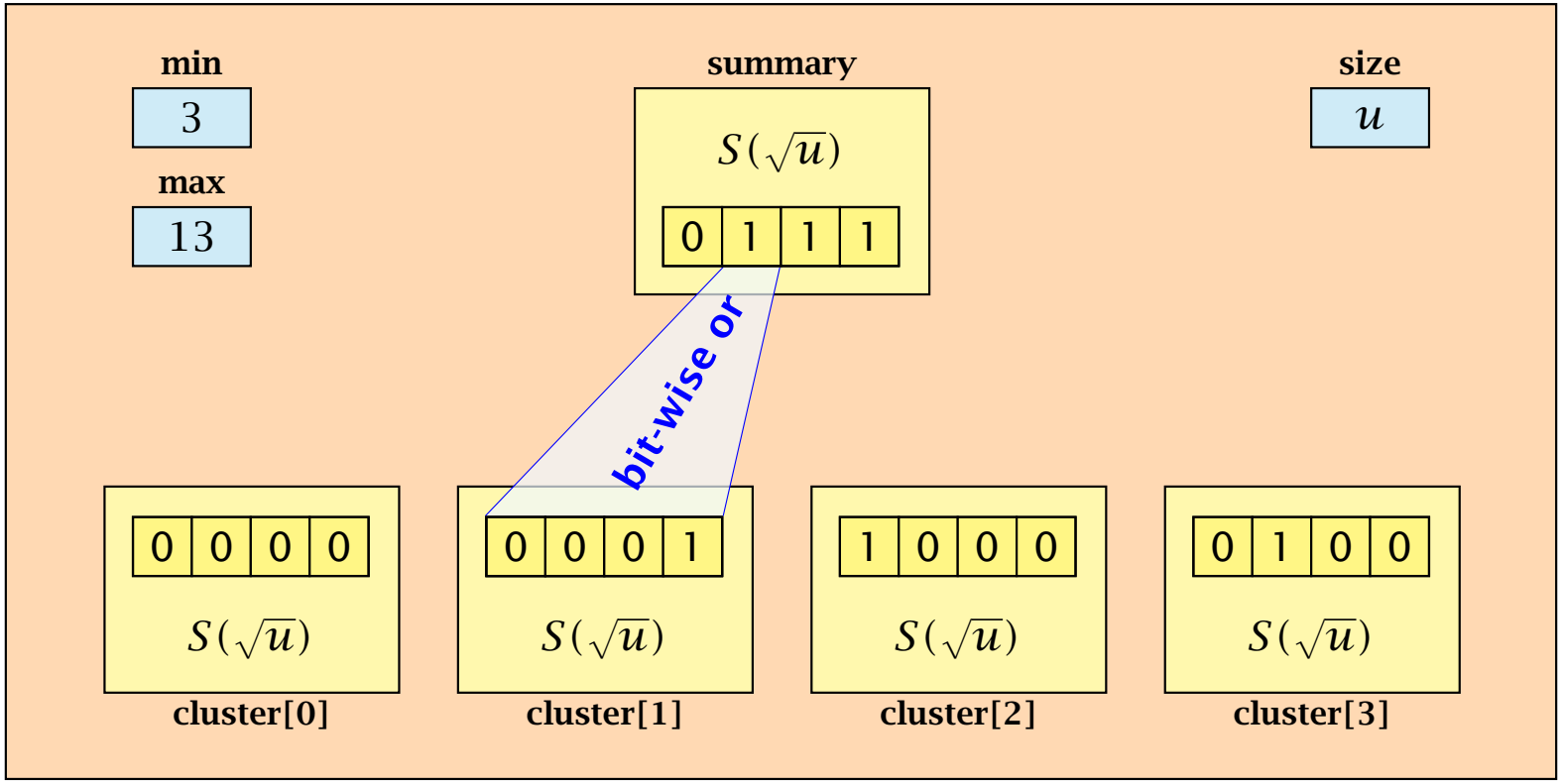
Using Master theorem gives $X(\ell) = \Theta(\ell \log \ell)$, and hence $T_{\mathrm{del}}(u) = \mathcal{O}(\log u \log\log u)$.

The same holds for $T_{\mathrm{pred}}(u)$ and $T_{\mathrm{succ}}(u)$.

Implementation 4: van Emde Boas Trees



- The bit referenced by min is not set within sub-datastructures.
- The bit referenced by max is set within sub-datastructures (if max $\neq$ min).

Implementation 4: van Emde Boas Trees

Advantages of having max/min pointers:

- Recursive calls for $\min$ and $\max$ are constant time.
- $\min = \text{null}$ means that the data-structure is empty.
- $\min = \max \neq \text{null}$ means that the data-structure contains exactly one element.
- We can insert into an empty datastructure in constant time by only setting $\min = \max = x$.
- We can delete from a data-structure that just contains one element in constant time by setting $\min = \max = \text{null}$.

Implementation 4: van Emde Boas Trees

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Implementation 4: van Emde Boas Trees

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Implementation 4: van Emde Boas Trees

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- We can delete from a data-structure that just contains one element in constant time by setting $\min = \max = \text{null}$.

Implementation 4: van Emde Boas Trees

Algorithm 20 max()

```
1: return max;
```

Algorithm 21 min()

```
1: return min;
```

- Constant time.

Implementation 4: van Emde Boas Trees

Algorithm 22 member(x)

1: **if** $x = \min$ **then return** 1; // TRUE
2: **return** cluster[high(x)].member(low(x));

- $T_{\text{mem}}(u) = T_{\text{mem}}(\sqrt{u}) + 1 \Longrightarrow T(u) = \mathcal{O}(\log\log u)$.

Implementation 4: van Emde Boas Trees

Algorithm 23 succ(x)

```
 1: if min ≠ null ∧ x < min then return min;
 2: maxincluster ← cluster[high(x)].max();
 3: if maxincluster ≠ null ∧ low(x) < maxincluster then
 4:         offs ← cluster[high(x)].succ(low(x));
 5:         return high(x) ∘ offs;
 6: else
 7:         succcluster ← summary.succ(high(x));
 8:         if succcluster = null then return null;
 9:         offs ← cluster[succcluster].min();
10:         return succcluster ∘ offs;
```

- $T_{\text{succ}}(u) = T_{\text{succ}}(\sqrt{u}) + 1 \Longrightarrow T_{\text{succ}}(u) = \mathcal{O}(\log\log u)$.

Implementation 4: van Emde Boas Trees

Algorithm 44 insert(x)

```
 1: if min = null then
 2:         min = x; max = x;
 3: else
 4:         if x < min then exchange x and min;
 5:         if cluster[high(x)].min = null; then
 6:                 summary.insert(high(x));
 7:                 cluster[high(x)].insert(low(x));
 8:         else
 9:                 cluster[high(x)].insert(low(x));
10:         if x > max then max = x;
```

- $T_{\text{ins}}(u) = T_{\text{ins}}(\sqrt{u}) + 1 \Longrightarrow T_{\text{ins}}(u) = \mathcal{O}(\log\log u)$.

Implementation 4: van Emde Boas Trees

Note that the recusive call in Line 7 takes constant time as the if-condition in Line 5 ensures that we are inserting in an empty sub-tree.

The only non-constant recursive calls are the call in Line 6 and in Line 9. These are mutually exclusive, i.e., only one of these calls will actually occur.

From this we get that $T_{\text{ins}}(u) = T_{\text{ins}}(\sqrt{u}) + 1$.

Implementation 4: van Emde Boas Trees

- **Assumes that x is contained in the structure.**

Algorithm 45 delete(x)

```
1: if min = max then
2:         min = null; max = null;
3: else
4:         if x = min then
5:                 firstcluster ← summary.min();
6:                 offs ← cluster[firstcluster].min();
7:                 x ← firstcluster ∘ offs;
8:                 min ← x;
9:         cluster[high(x)].delete(low(x));
```

continued...

Implementation 4: van Emde Boas Trees

- **Assumes that x is contained in the structure.**

Algorithm 45 delete(x)

```
1: if min = max then
2:         min = null; max = null;
3: else
4:         if x = min then                          find new minimum
5:                 firstcluster ← summary.min();
6:                 offs ← cluster[firstcluster].min();
7:                 x ← firstcluster ∘ offs;
8:                 min ← x;
9:         cluster[high(x)].delete(low(x));
                                  continued...
```

Implementation 4: van Emde Boas Trees

- **Assumes that x is contained in the structure.**

Algorithm 45 delete(x)

```
1: if min = max then
2:         min = null; max = null;
3: else
4:         if x = min then
5:                 firstcluster ← summary.min();
6:                 offs ← cluster[firstcluster].min();
7:                 x ← firstcluster ∘ offs;
8:                 min ← x;
9:         cluster[high(x)].delete(low(x));                    delete
```

continued...

Implementation 4: van Emde Boas Trees

Algorithm 45 delete(x)

...continued

```
10:     if cluster[high(x)].min() = null then
11:         summary.delete(high(x));
12:         if x = max then
13:             summax ← summary.max();
14:             if summax = null then max ← min;
15:             else
16:                 offs ← cluster[summax].max();
17:                 max ← summax ∘ offs
18:     else
19:         if x = max then
20:             offs ← cluster[high(x)].max();
21:             max ← high(x) ∘ offs;
```

Implementation 4: van Emde Boas Trees

Algorithm 45 delete(x)

...continued — fix maximum

```
10:     if cluster[high(x)].min() = null then
11:         summary.delete(high(x));
12:         if x = max then
13:             summax ← summary.max();
14:             if summax = null then max ← min;
15:             else
16:                 offs ← cluster[summax].max();
17:                 max ← summax ∘ offs
18:     else
19:         if x = max then
20:             offs ← cluster[high(x)].max();
21:             max ← high(x) ∘ offs;
```

Implementation 4: van Emde Boas Trees

Note that only one of the possible recusive calls in Line 9 and Line 11 in the deletion-algorithm may take non-constant time.

To see this observe that the call in Line 11 only occurs if the cluster where x was deleted is now empty. But this means that the call in Line 9 deleted the last element in $\text{cluster}[\text{high}(x)]$. Such a call only takes constant time.

Hence, we get a recurrence of the form

$$T_{\text{del}}(u) = T_{\text{del}}(\sqrt{u}) + c \ .$$

This gives $T_{\text{del}}(u) = \mathcal{O}(\log \log u)$.

10 van Emde Boas Trees

Space requirements:

- The space requirement fulfills the recurrence

$$S(u) = (\sqrt{u}+1)S(\sqrt{u}) + \mathcal{O}(\sqrt{u}) \ .$$

- Note that we cannot solve this recurrence by the Master theorem as the branching factor is not constant.
- One can show by induction that the space requirement is $S(u) = \mathcal{O}(u)$. Exercise.

- Let the "real" recurrence relation be

$$S(k^2) = (k+1)S(k) + c_1 \cdot k;\ S(4) = c_2$$

- Replacing $S(k)$ by $R(k) := S(k)/c_2$ gives the recurrence

$$R(k^2) = (k+1)R(k) + ck;\ R(4) = 1$$

  where $c = c_1/c_2 < 1$.
- Now, we show $R(k) \leq k - 2$ for squares $k \geq 4$.
  - Obviously, this holds for $k = 4$.
  - For $k = \ell^2 > 4$ with ℓ integral we have

$$\begin{aligned} R(k) &= (1+\ell)R(\ell) + c\ell \\ &\leq (1+\ell)(\ell-2) + \ell \leq k - 2 \end{aligned}$$

  - This shows that $R(k)$ and, hence, $S(k)$ grows linearly.

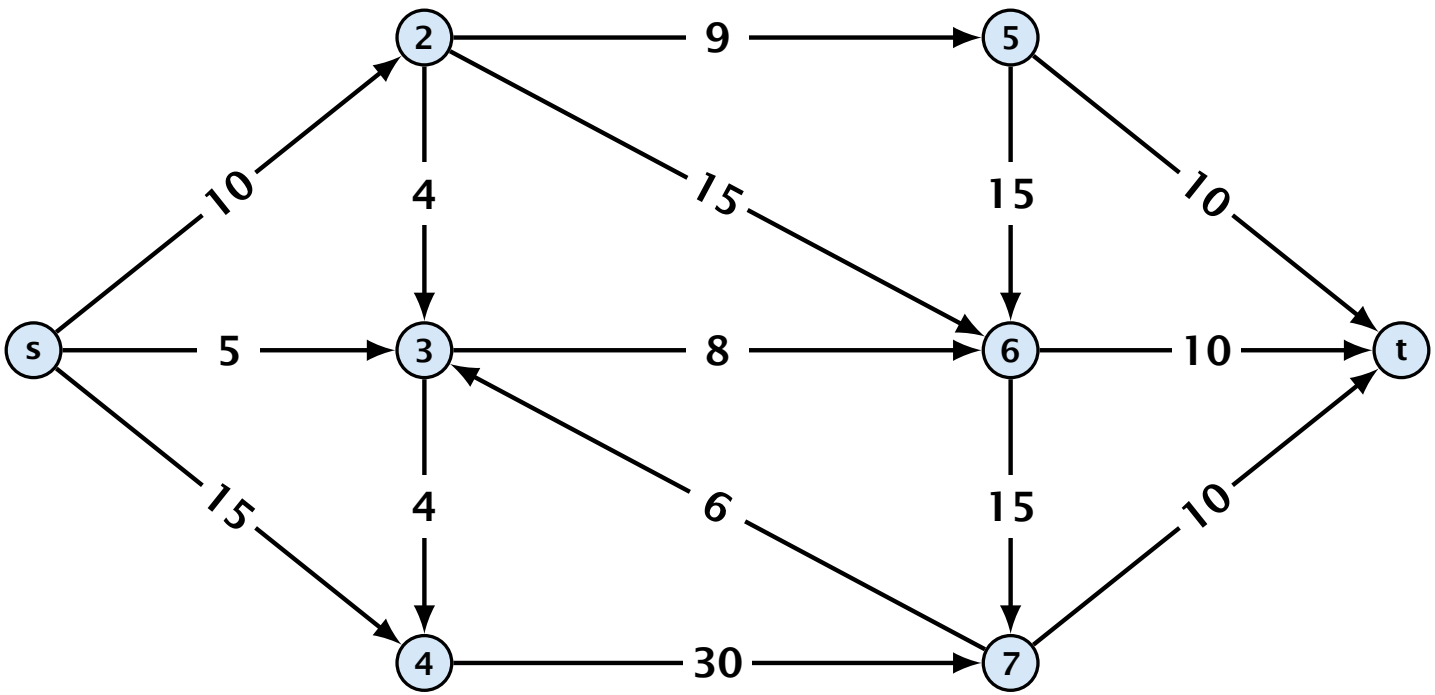
Part IV

Flows and Cuts

11 Introduction

Flow Network

- directed graph $G = (V, E)$; edge capacities $c(e)$
- two special nodes: source s; target t;
- no edges entering s or leaving t;
- at least for now: no parallel edges;



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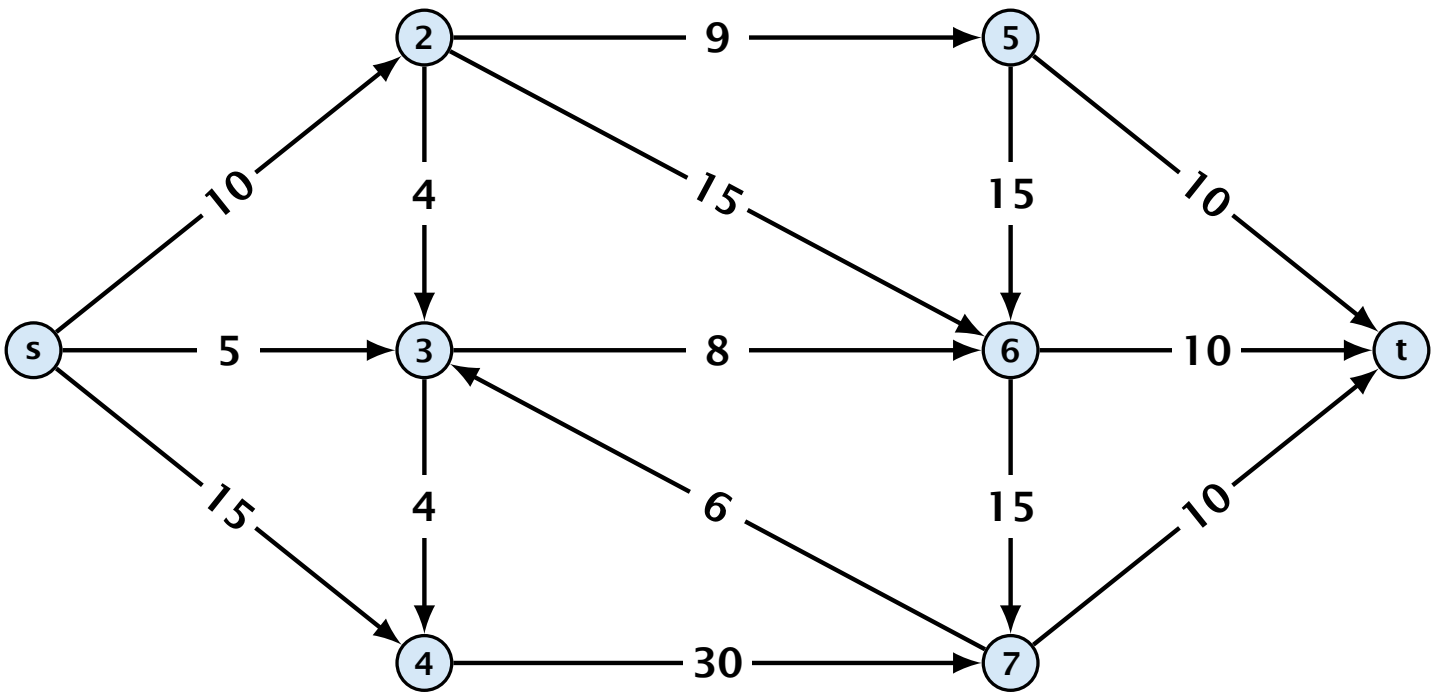

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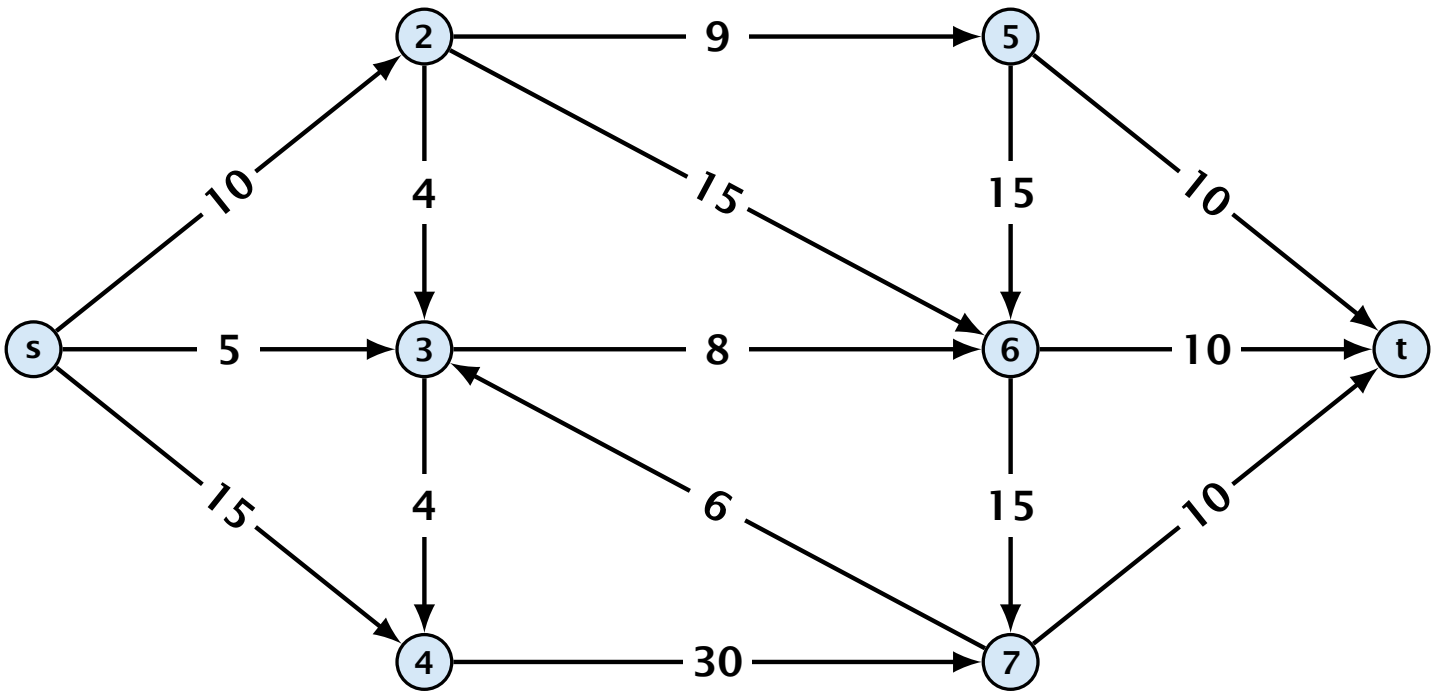

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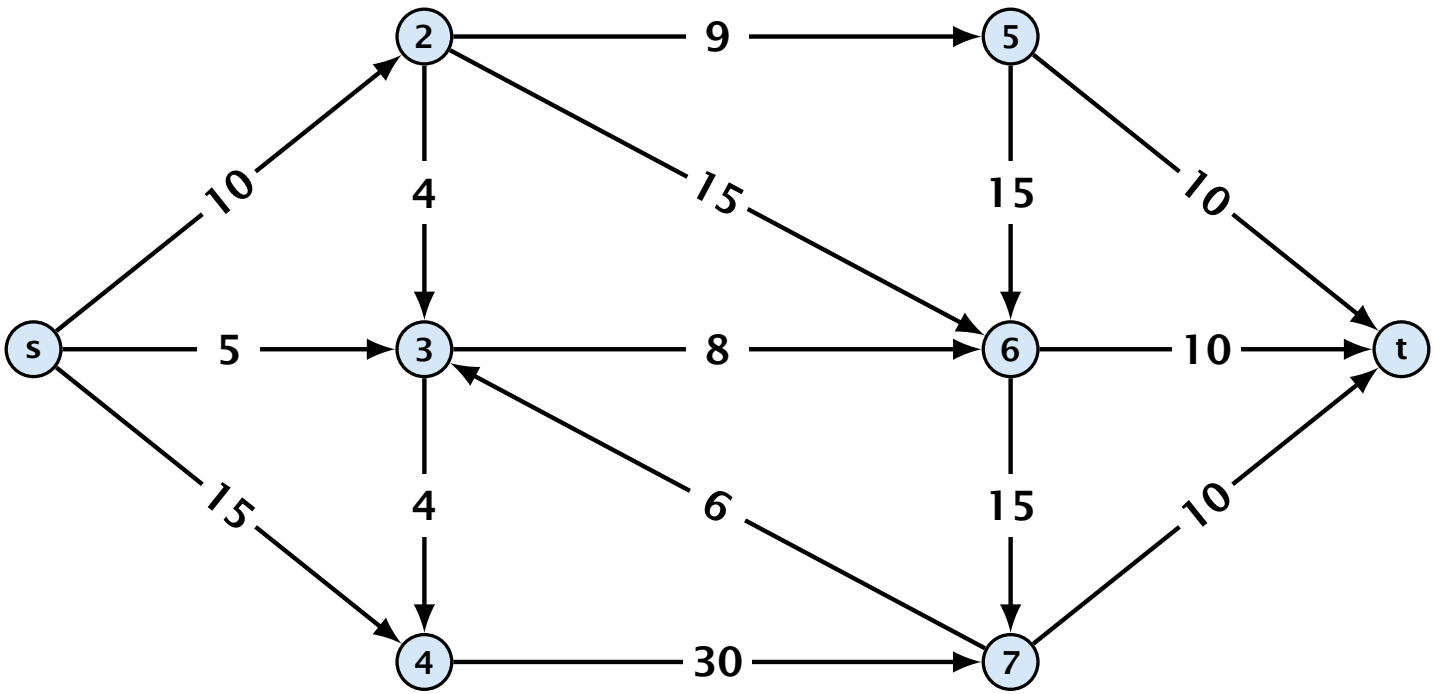

Cuts

Definition 43

An (s,t)-cut in the graph G is given by a set $A \subset V$ with $s \in A$ and $t \in V \setminus A$.

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Definition 44

The capacity of a cut A is defined as

$$\mathrm{cap}(A, V \setminus A) := \sum_{e \in \mathrm{out}(A)} c(e) \ ,$$

where $\mathrm{out}(A)$ denotes the set of edges of the form $A \times V \setminus A$ (i.e. edges leaving A).

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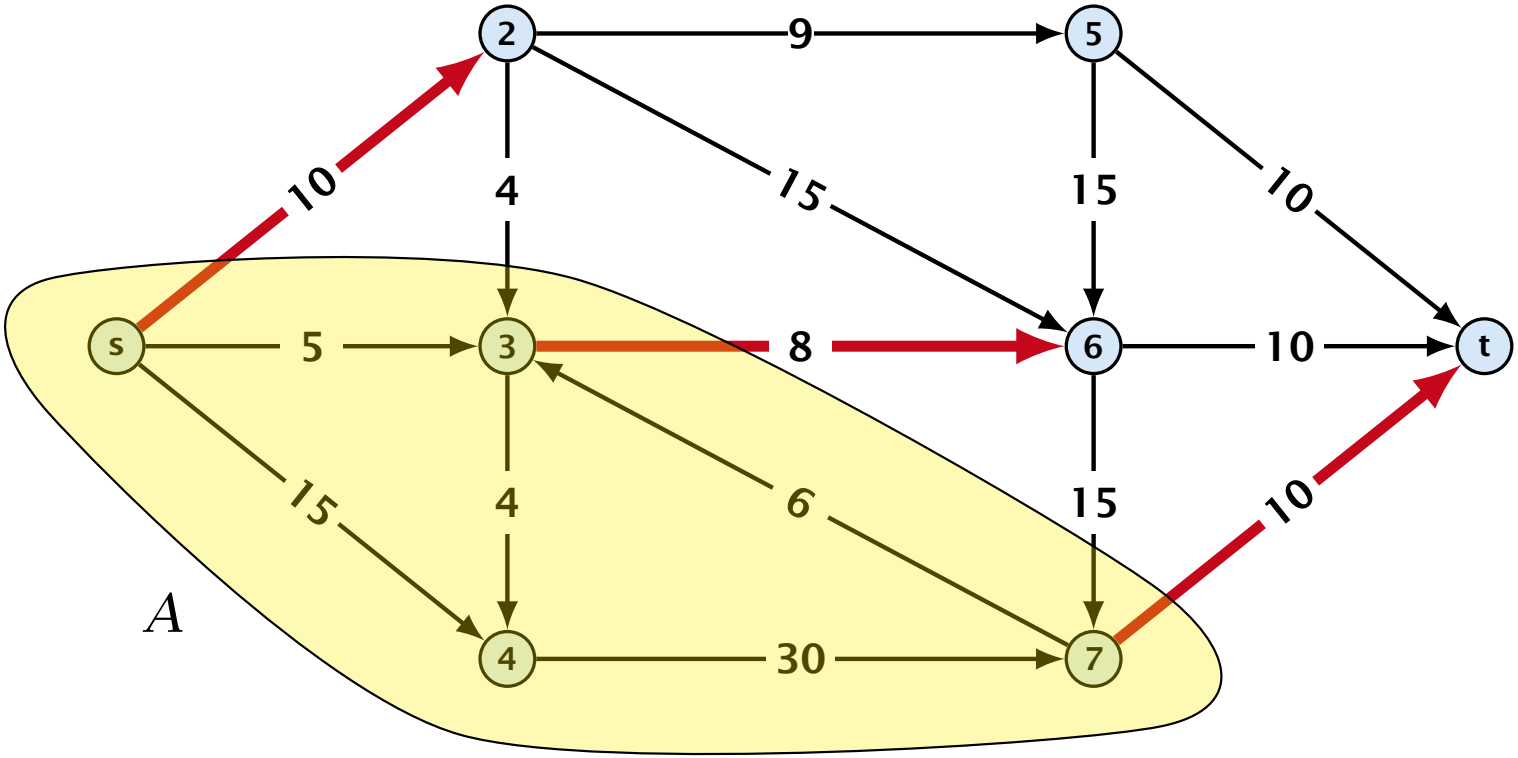
Minimum Cut Problem: Find an (s,t)-cut with minimum capacity.

Cuts

Example 45



The capacity of the cut is $\mathrm{cap}(A, V \setminus A) = 28$.

Flows

Definition 46

An (s,t)-flow is a function $f : E \mapsto \mathbb{R}^+$ that satisfies

1. For each edge e

$$0 \leq f(e) \leq c(e) \ .$$

(capacity constraints)

2. For each $v \in V \setminus \{s,t\}$

$$\sum_{e \in \text{out}(v)} f(e) = \sum_{e \in \text{into}(v)} f(e) \ .$$

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Flows

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$$\sum_{e \in \text{out}(v)} f(e) = \sum_{e \in \text{into}(v)} f(e) \ .$$

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Flows

Definition 47

The value of an (s,t)-flow f is defined as

$$\mathrm{val}(f) = \sum_{e \in \mathrm{out}(s)} f(e) \ .$$

Maximum Flow Problem: Find an (s,t)-flow with maximum value.

Flows

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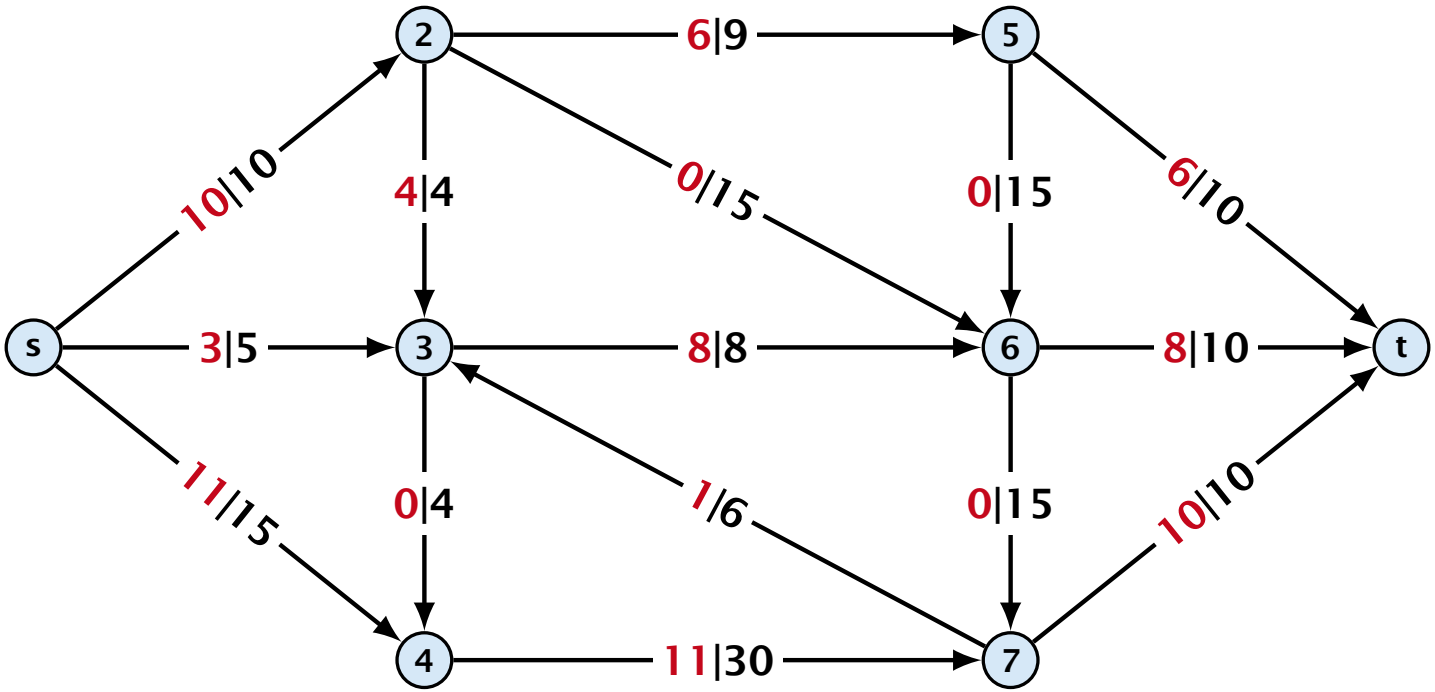
Maximum Flow Problem: Find an (s,t)-flow with maximum value.

Flows

Example 48



The value of the flow is $\mathrm{val}(f) = 24$.

Flows

Lemma 49 (Flow value lemma)

Let f be a flow, and let $A \subseteq V$ be an (s,t)-cut. Then the net-flow across the cut is equal to the amount of flow leaving s, i.e.,

$$\mathrm{val}(f) = \sum_{e \in \mathrm{out}(A)} f(e) - \sum_{e \in \mathrm{into}(A)} f(e) \ .$$

Proof.

$\mathrm{val}(f)$

Proof.

$$\mathrm{val}(f) = \sum_{e \in \mathrm{out}(s)} f(e)$$

Proof.

$$\begin{aligned}\mathrm{val}(f) &= \sum_{e\in \mathrm{out}(s)} f(e) \\ &= \sum_{e\in \mathrm{out}(s)} f(e) + \sum_{v\in A\setminus\{s\}} \left(\sum_{e\in \mathrm{out}(v)} f(e) - \sum_{e\in \mathrm{in}(v)} f(e) \right)\end{aligned}$$

Proof.

$$\mathrm{val}(f) = \sum_{e \in \mathrm{out}(s)} f(e)$$

$$= \sum_{e \in \mathrm{out}(s)} f(e) + \sum_{v \in A \setminus \{s\}} \underbrace{\left(\sum_{e \in \mathrm{out}(v)} f(e) - \sum_{e \in \mathrm{in}(v)} f(e) \right)}_{= 0}$$

Proof.

$$\begin{aligned}
\mathrm{val}(f) &= \sum_{e \in \mathrm{out}(s)} f(e) \\
&= \sum_{e \in \mathrm{out}(s)} f(e) + \sum_{v \in A \setminus \{s\}} \left(\sum_{e \in \mathrm{out}(v)} f(e) - \sum_{e \in \mathrm{in}(v)} f(e) \right) \\
&= \sum_{e \in \mathrm{out}(A)} f(e) - \sum_{e \in \mathrm{into}(A)} f(e)
\end{aligned}$$

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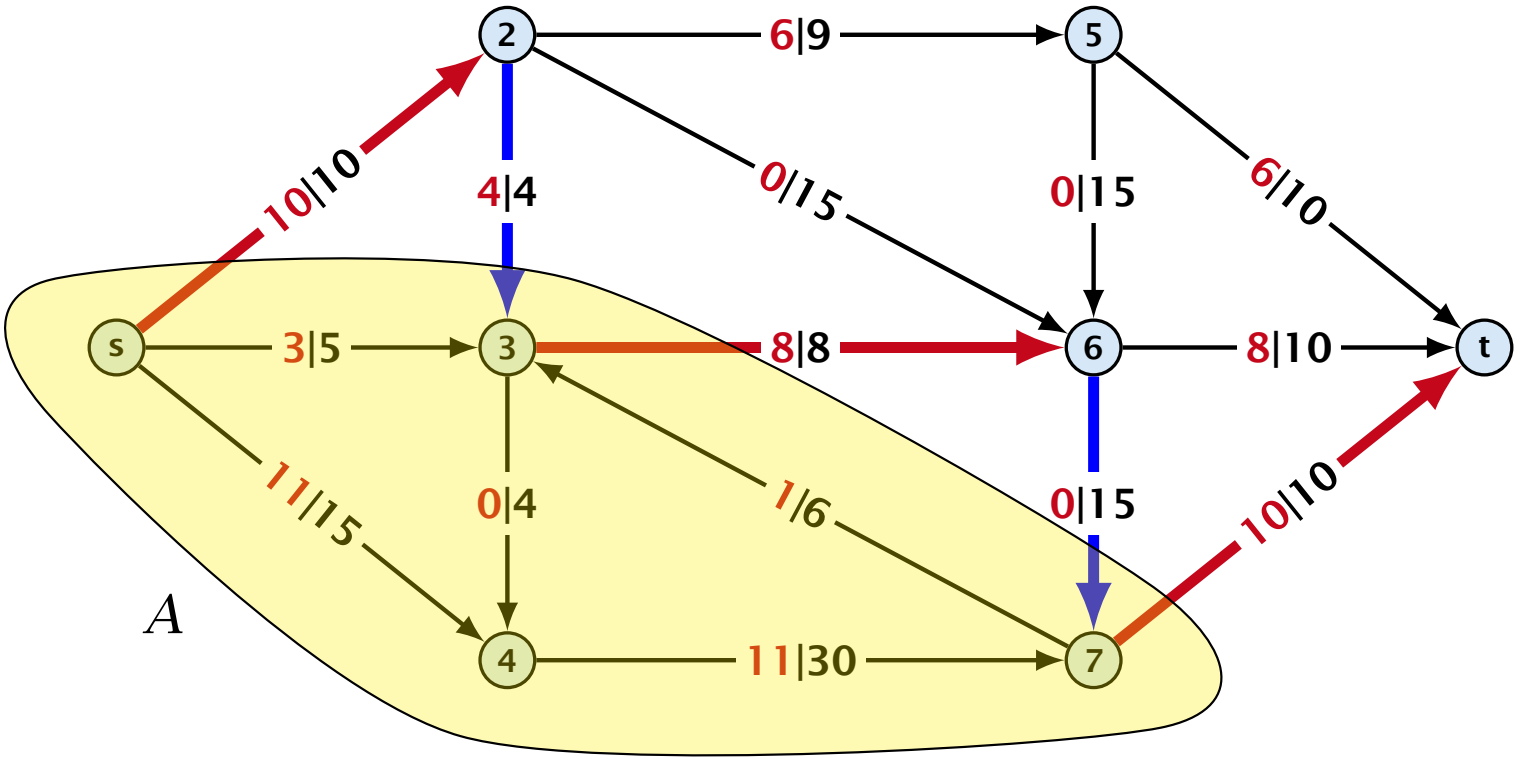
&= \sum_{e\in \mathrm{out}(A)} f(e) - \sum_{e\in \mathrm{into}(A)} f(e)
\end{aligned}$$

The last equality holds since every edge with both end-points in A contributes negatively as well as positively to the sum in Line 2. The only edges whose contribution doesn't cancel out are edges leaving or entering A. □

Example 50


2
6|9
5
10|10
4|4
0|15
0|15
6|10
s
3|5
3
8|8
6
8|10
t
11|15
0|4
1|6
0|15
10|10
A
4
11|30
7

Corollary 51

Let f be an (s,t)-flow and let A be an (s,t)-cut, such that

$$\mathrm{val}(f) = \mathrm{cap}(A, V \setminus A).$$

Then f is a maximum flow.

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Proof.

Suppose that there is a flow f' with larger value. Then

$$\mathrm{cap}(A, V \setminus A) < \mathrm{val}(f')$$

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Then f is a maximum flow.

Proof.

Suppose that there is a flow f' with larger value. Then

$$\begin{aligned}\mathrm{cap}(A, V \setminus A) &< \mathrm{val}(f') \\ &= \sum_{e \in \mathrm{out}(A)} f'(e) - \sum_{e \in \mathrm{into}(A)} f'(e)\end{aligned}$$

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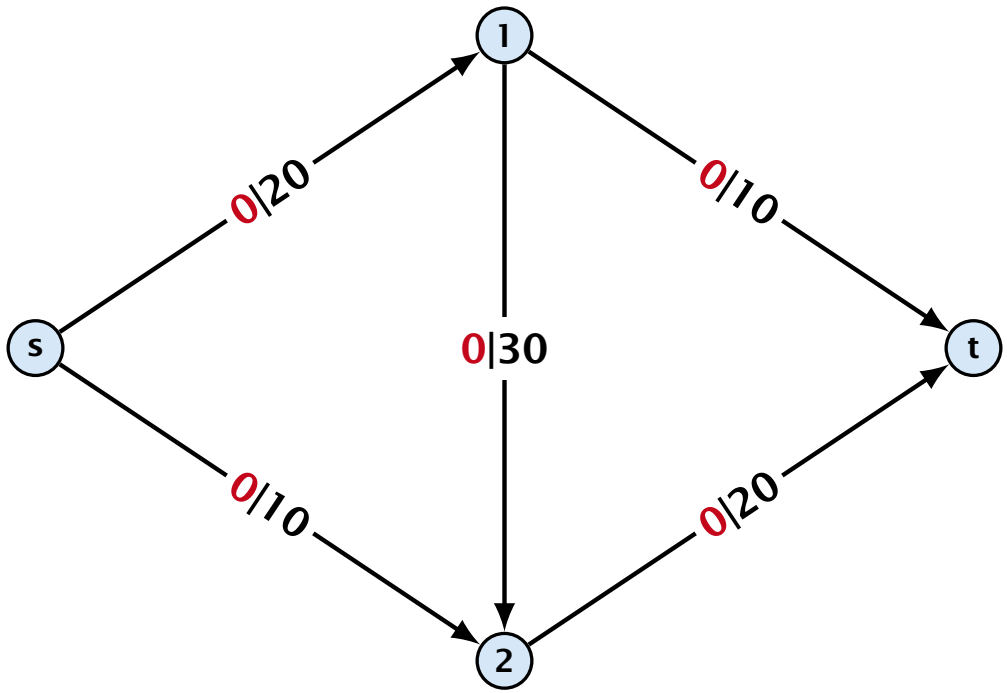
&\leq \sum_{e \in \mathrm{out}(A)} f'(e) \\
&\leq \mathrm{cap}(A, V \setminus A)
\end{aligned}$$

□

12 Augmenting Path Algorithms

Greedy-algorithm:

- start with $f(e) = 0$ everywhere
- find an s-t path with $f(e) < c(e)$ on every edge
- augment flow along the path
- repeat as long as possible



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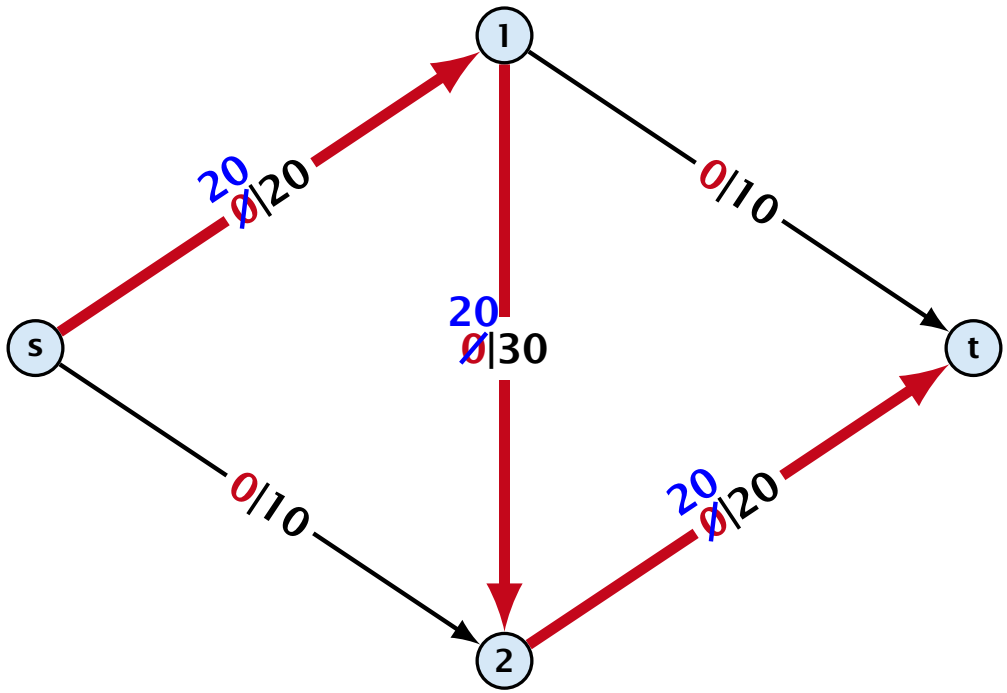

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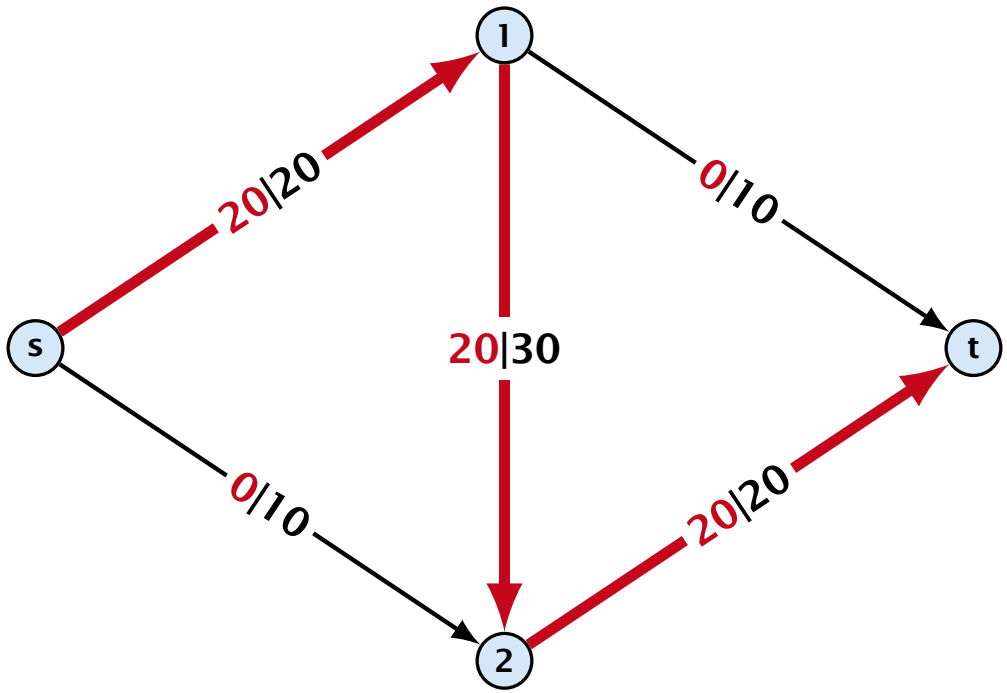

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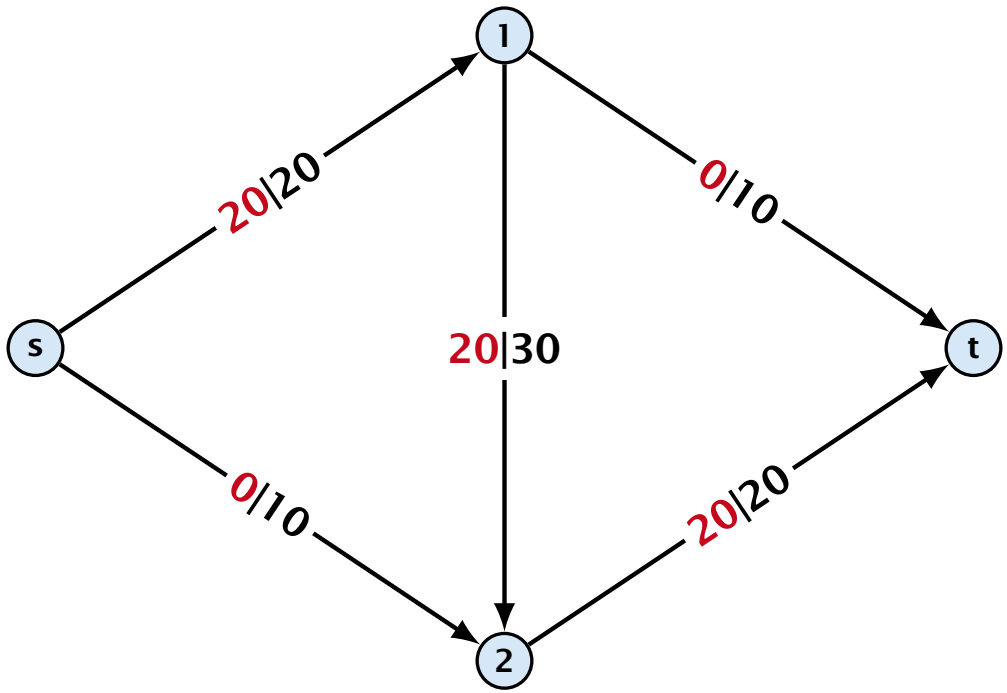

The Residual Graph

From the graph $G = (V, E, c)$ and the current flow f we construct an auxiliary graph $G_f = (V, E_f, c_f)$ (the residual graph):

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- G_f has edge e_1' with capacity $\max\{0, c(e_1) - f(e_1) + f(e_2)\}$ and e_2' with with capacity $\max\{0, c(e_2) - f(e_2) + f(e_1)\}$.

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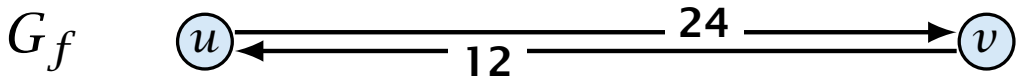

Augmenting Path Algorithm

Definition 52

An augmenting path with respect to flow f, is a path from s to t in the auxiliary graph G_f that contains only edges with non-zero capacity.

Algorithm 46 FordFulkerson$(G = (V, E, c))$

1: Initialize $f(e) \leftarrow 0$ for all edges.
2: **while** $\exists$ augmenting path p in G_f **do**
3: augment as much flow along p as possible.

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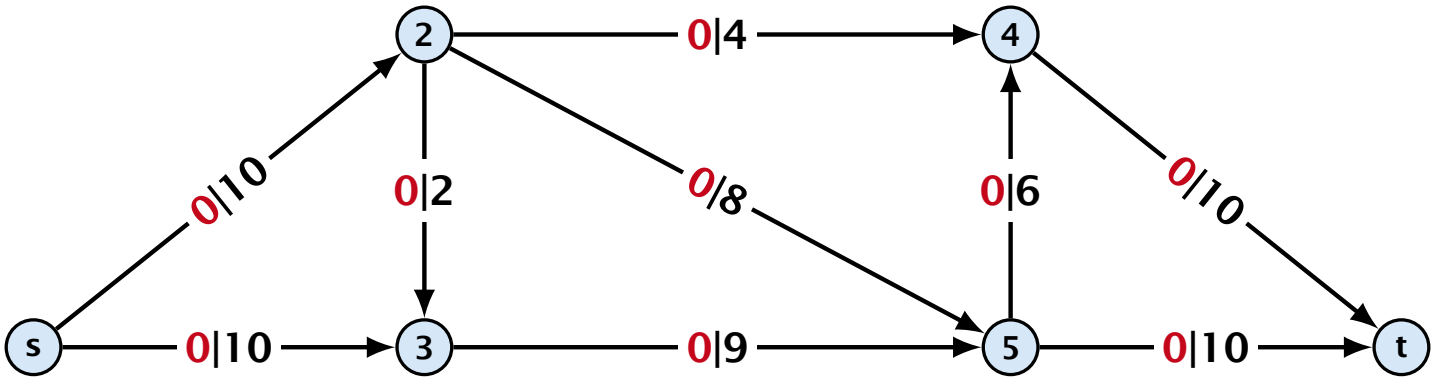
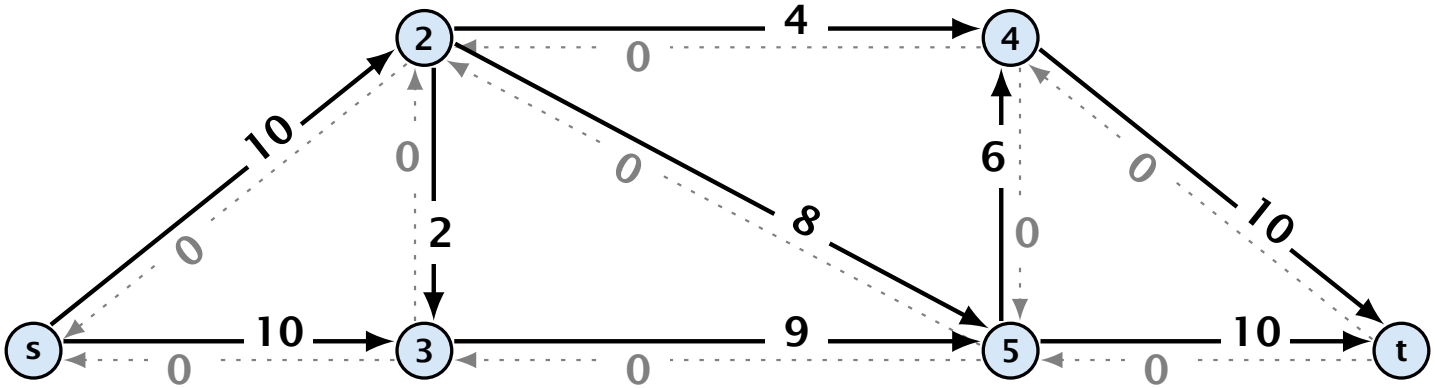
G



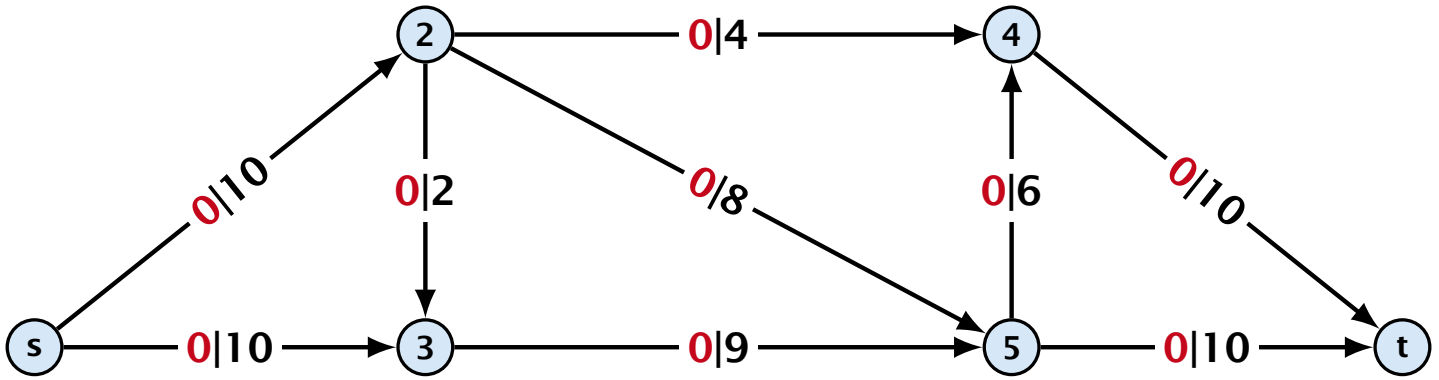
Flow value = 0

G_f



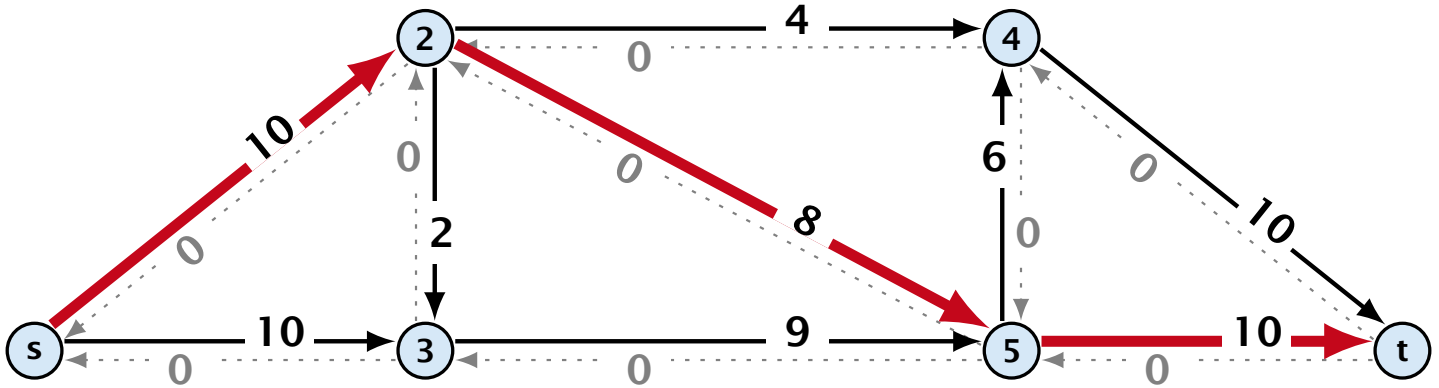
Augmenting Path Algorithm

G



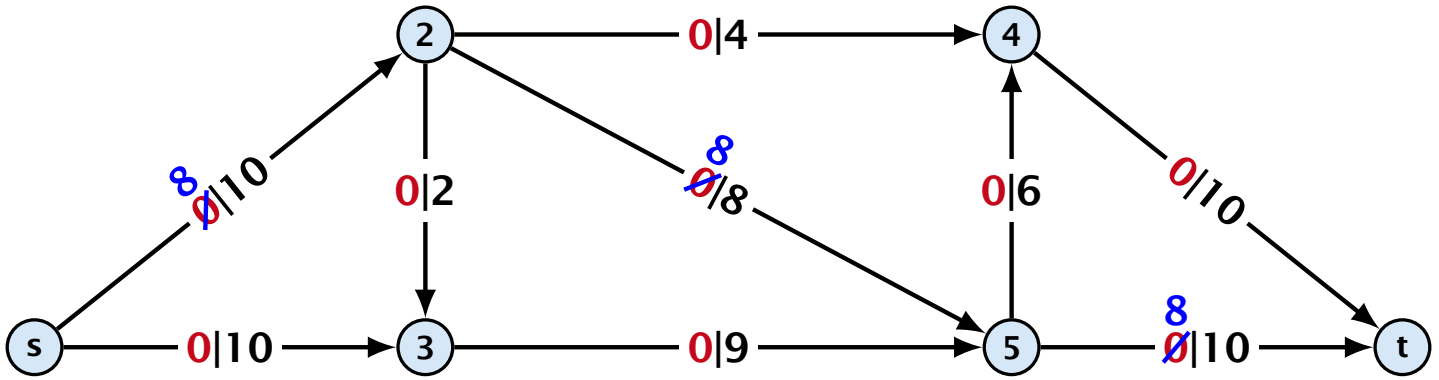
Flow value = 0

G_f



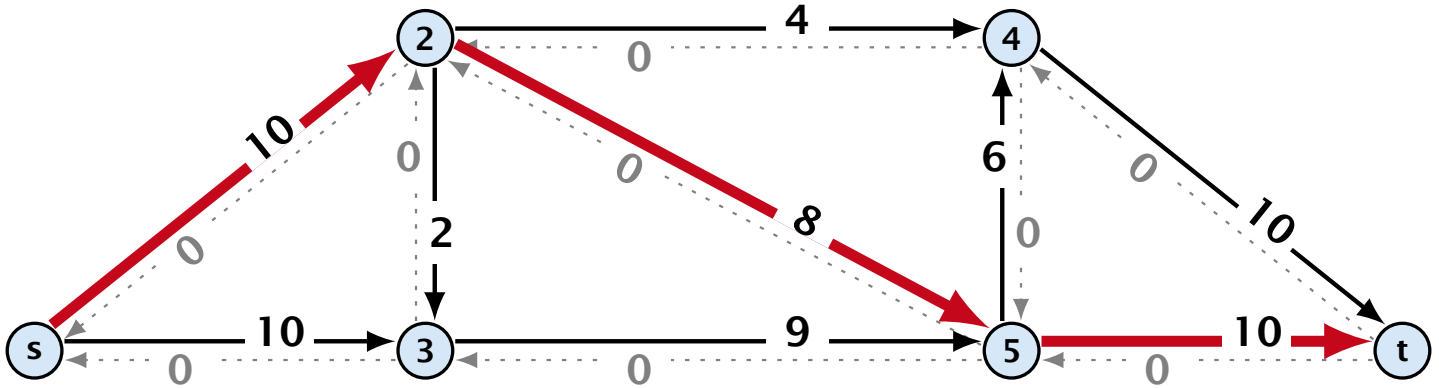
Augmenting Path Algorithm

G



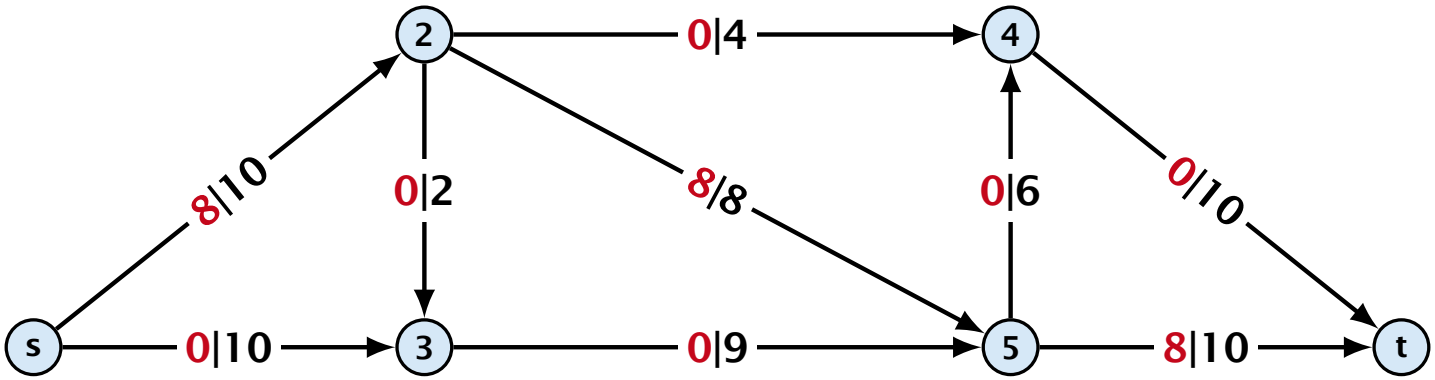
Flow value = 8

G_f



Augmenting Path Algorithm

G



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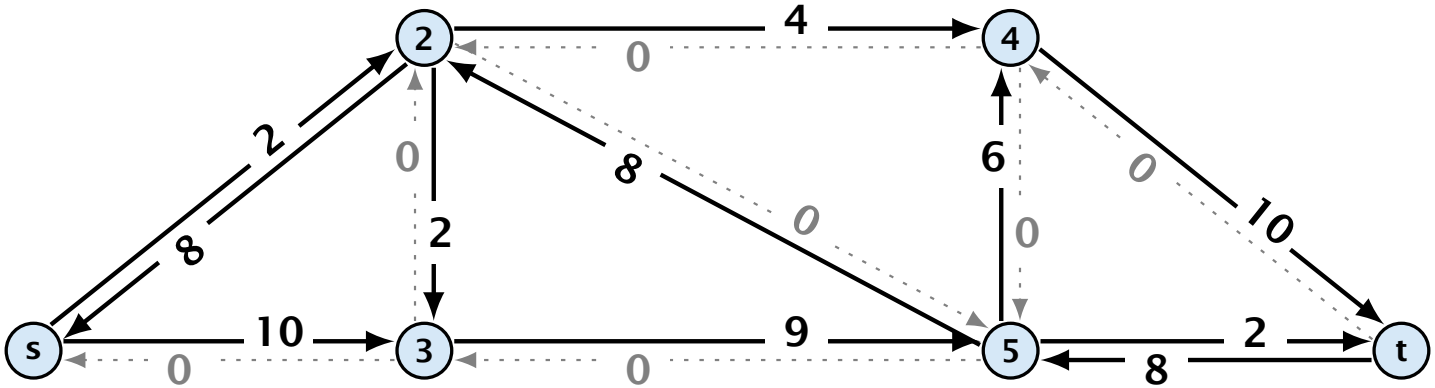
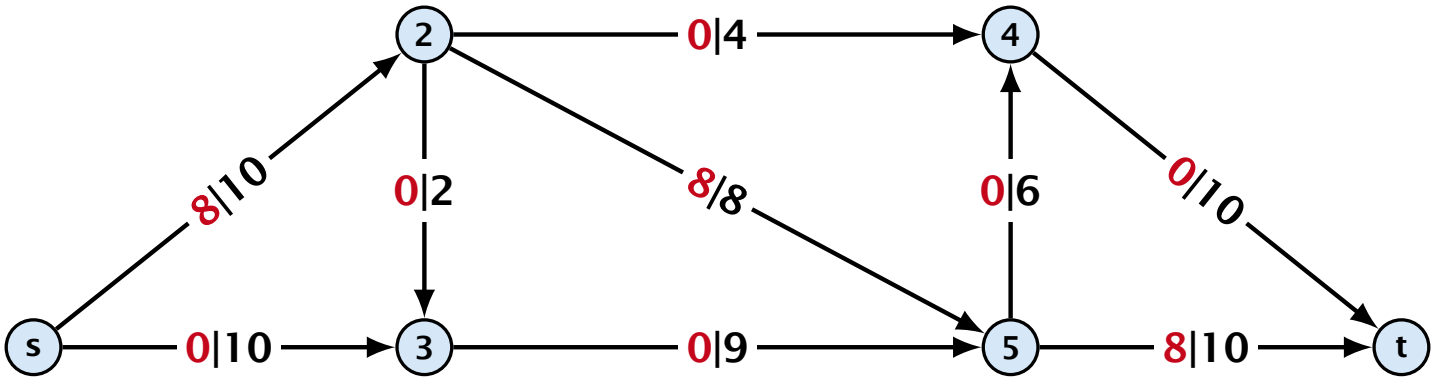


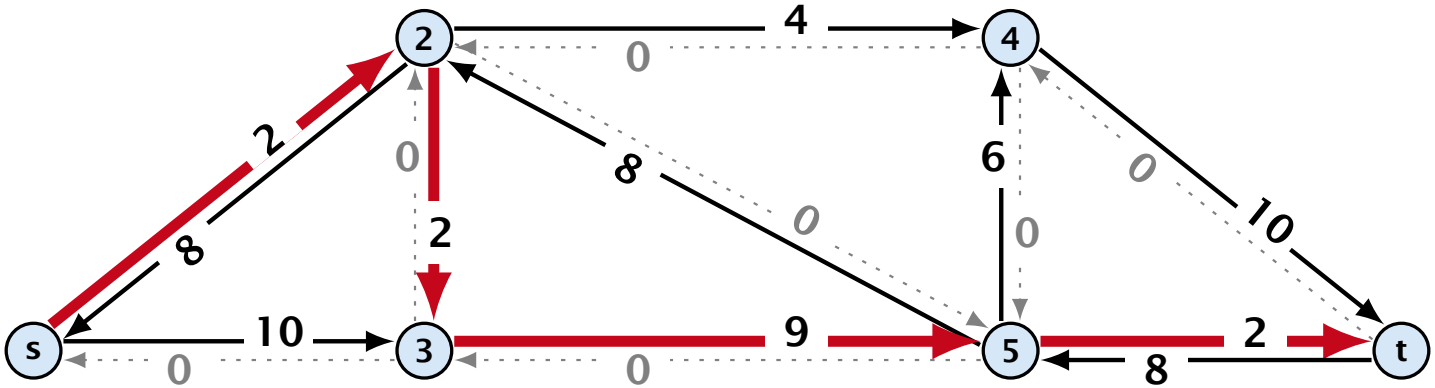
Augmenting Path Algorithm

G



Flow value = 8

G_f



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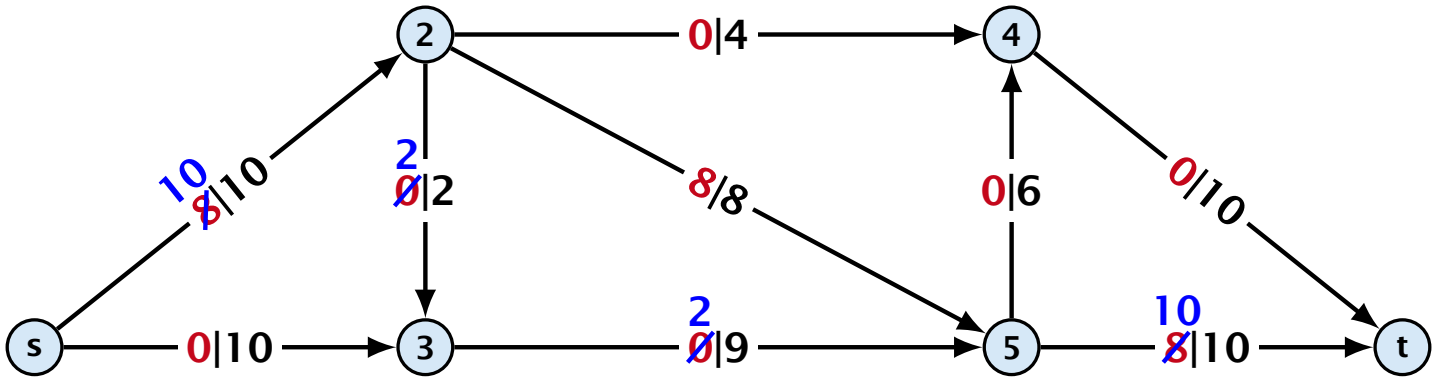
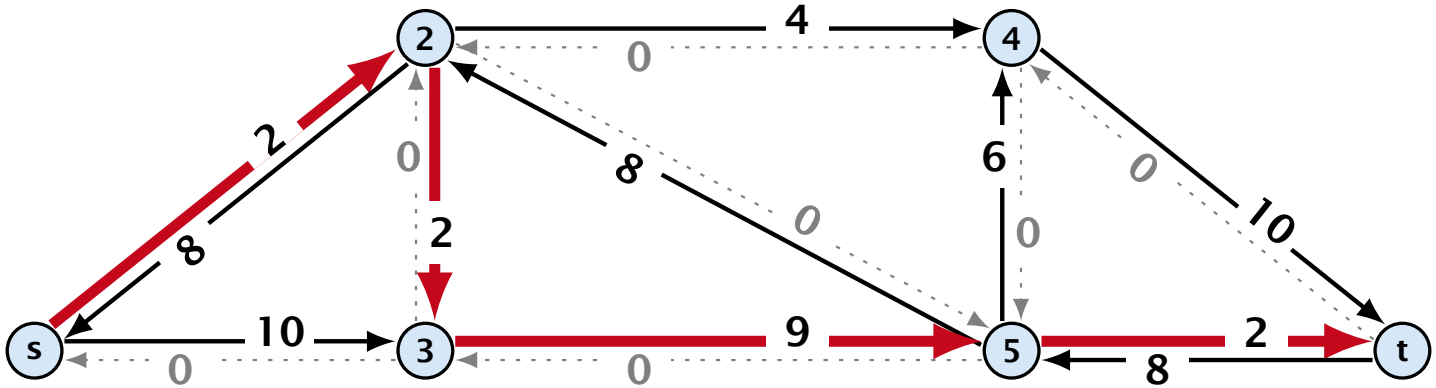


Flow value = 10

G_f



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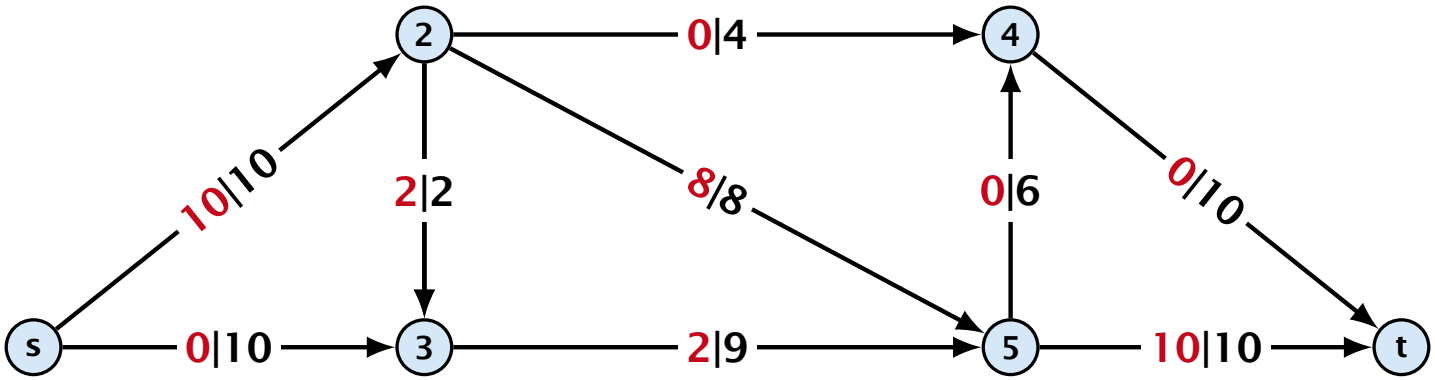
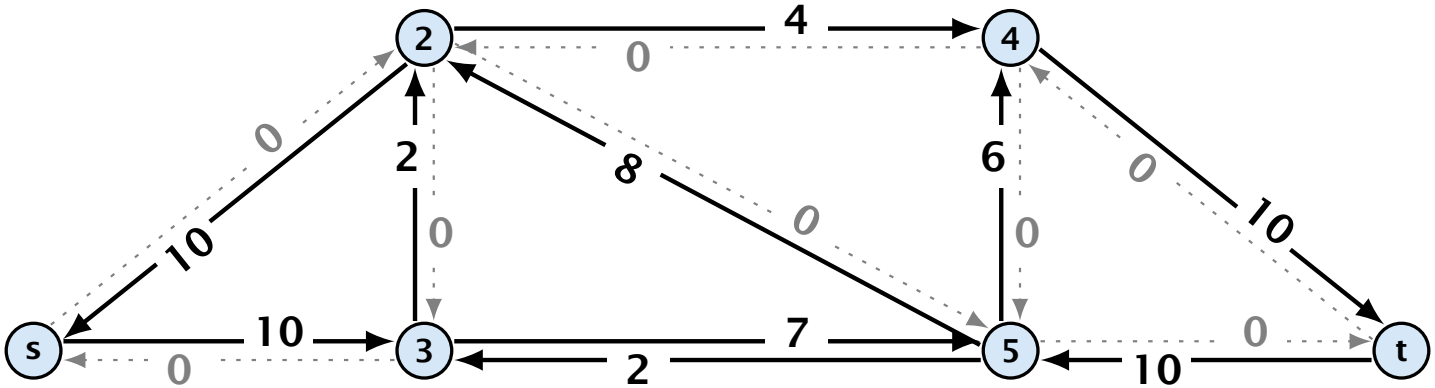


Flow value = 10

G_f



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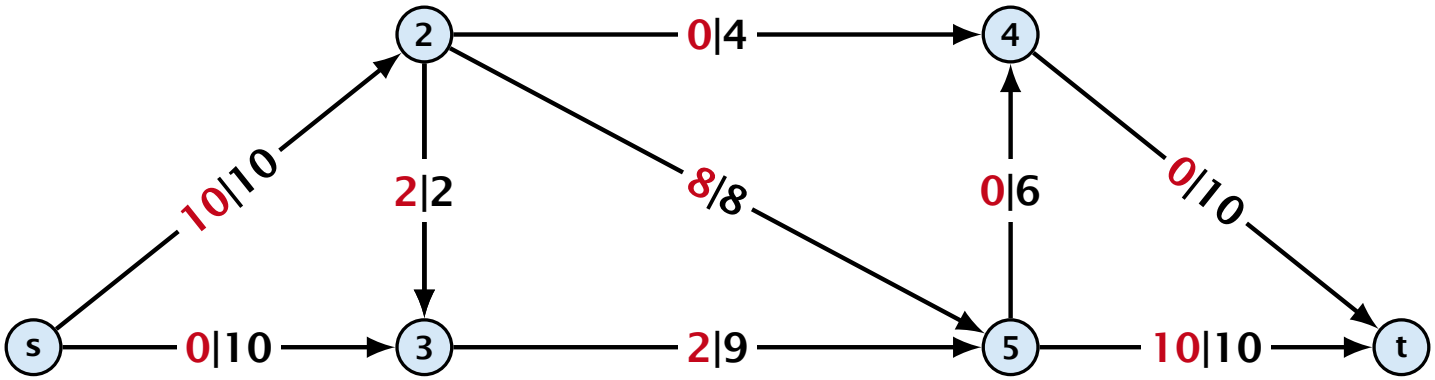

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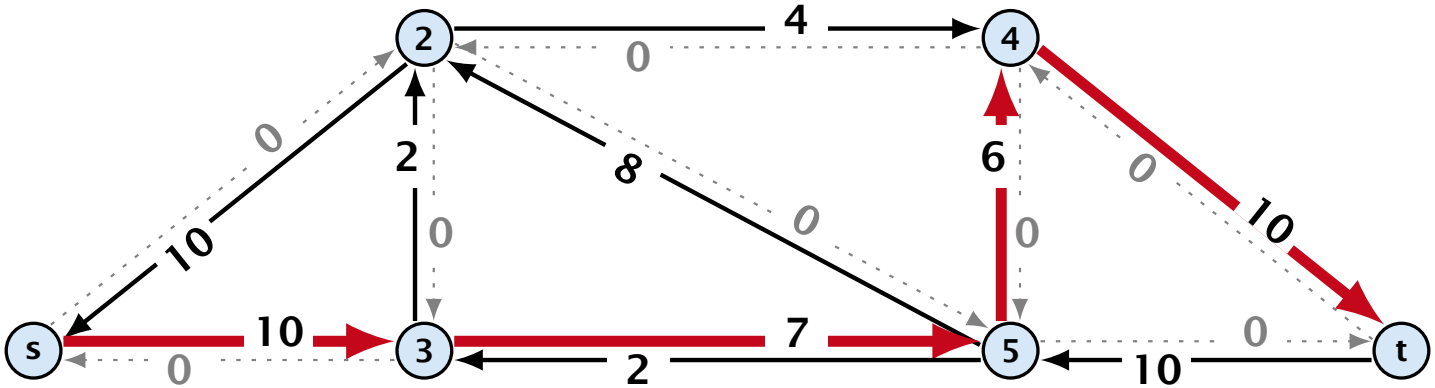

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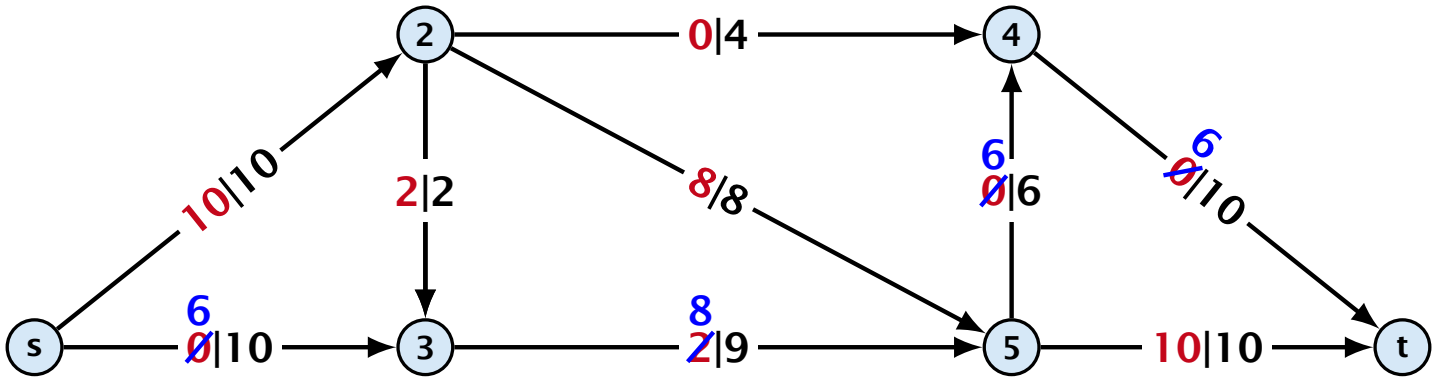
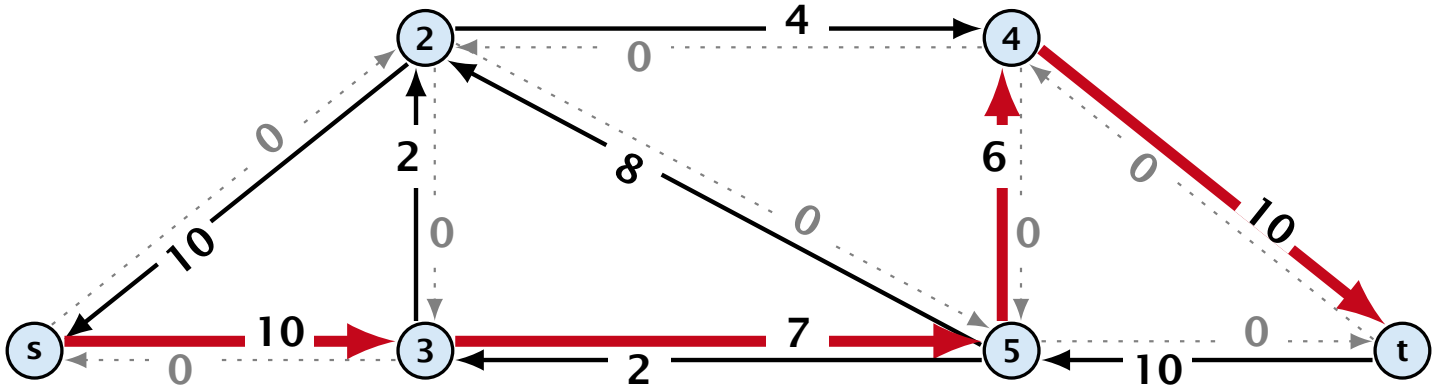


Flow value = 16

G_f



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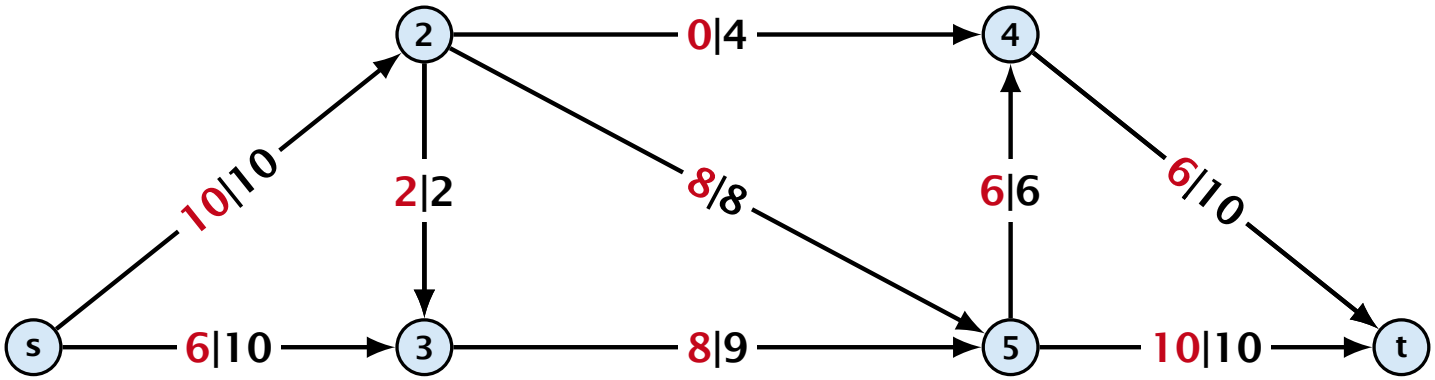
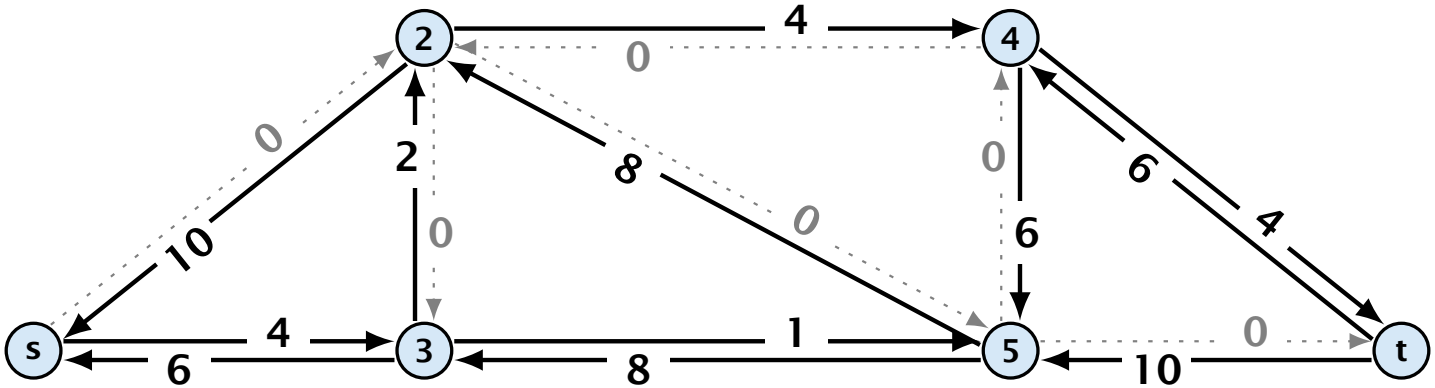


Flow value = 16

G_f



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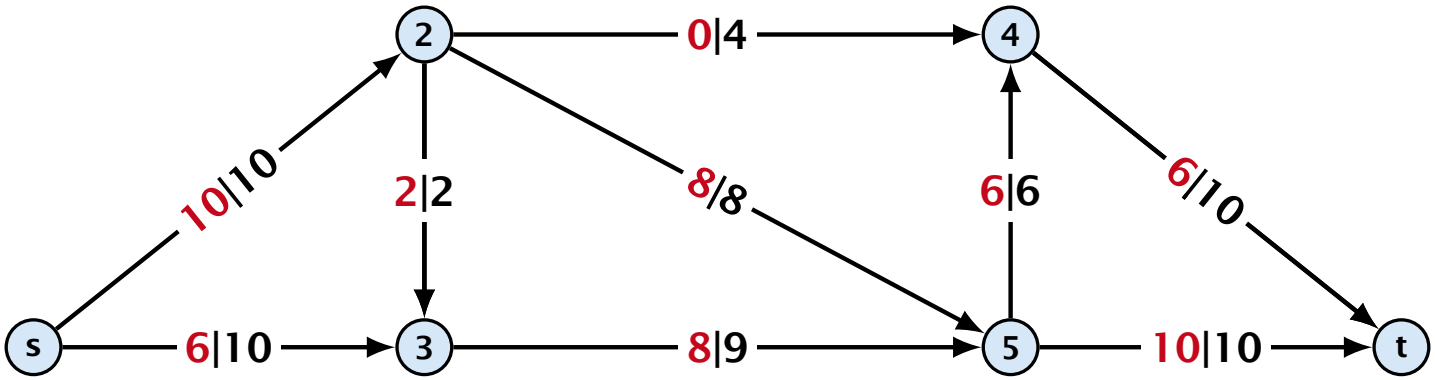

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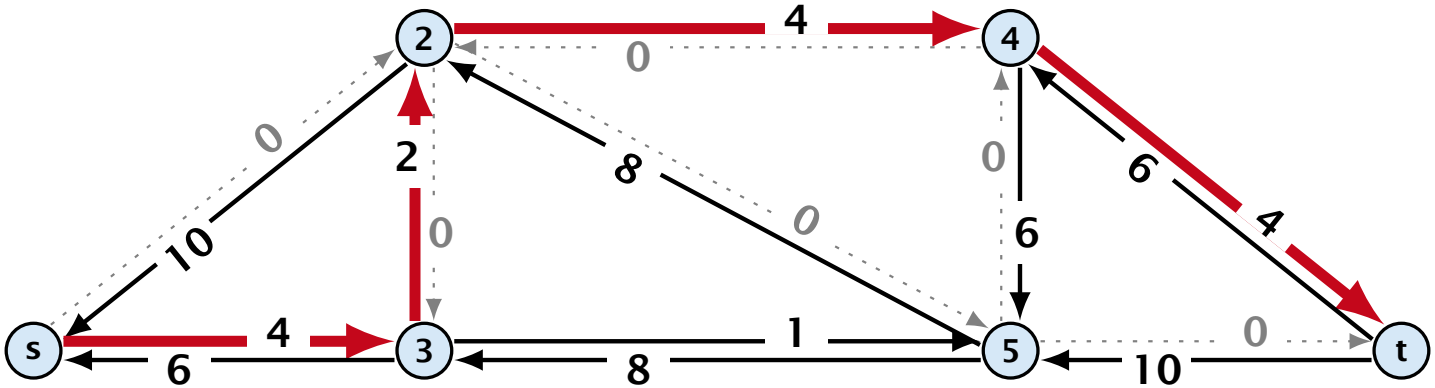

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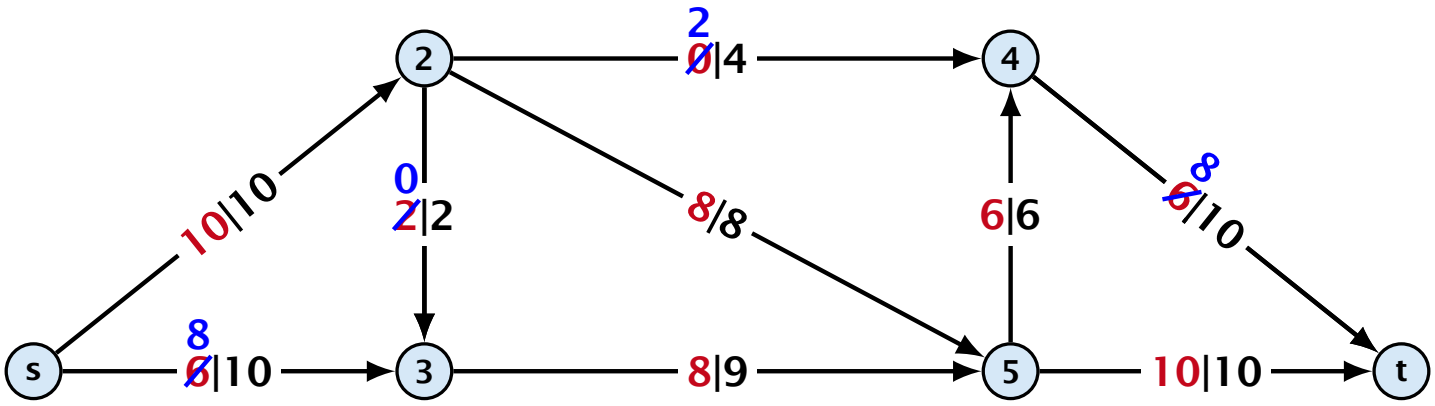
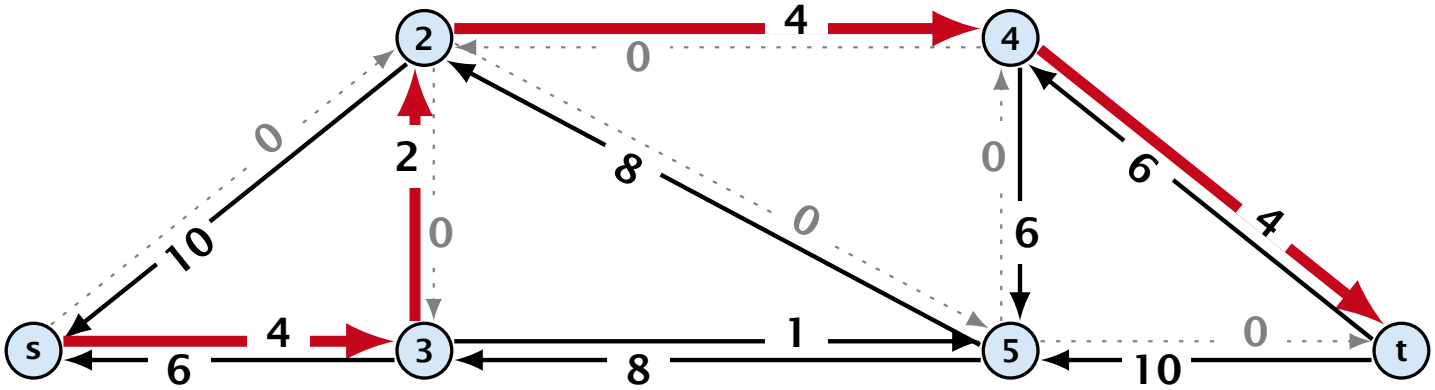


Flow value = 18

G_f



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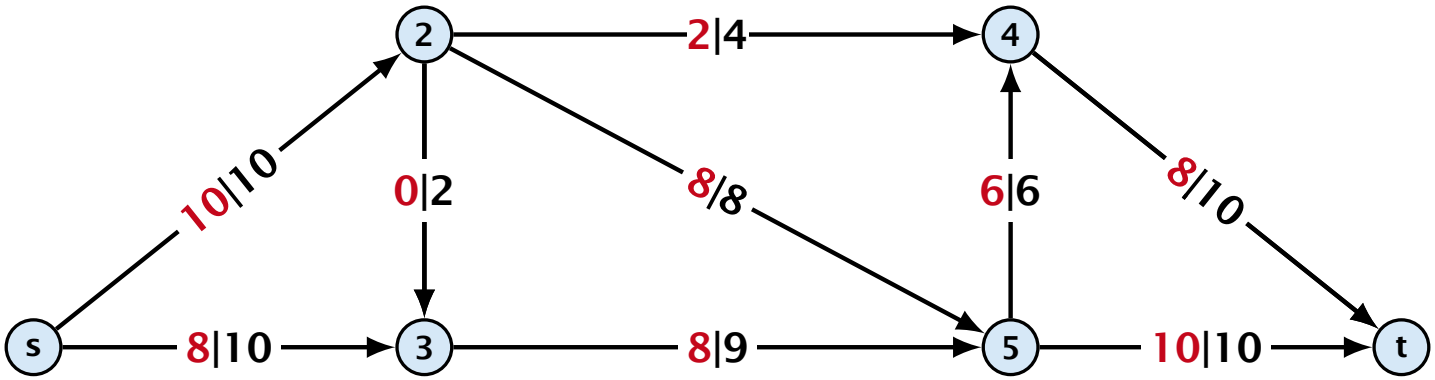
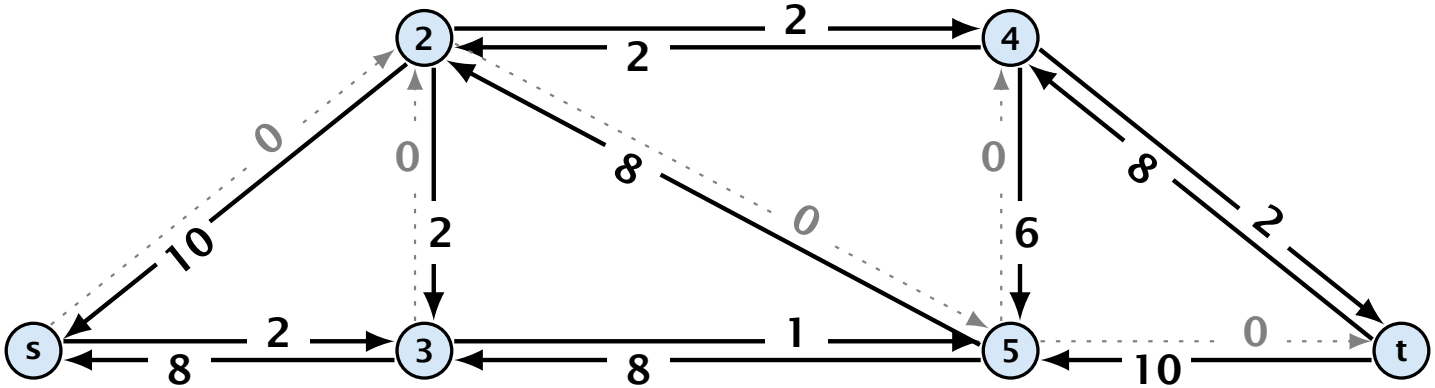


Flow value = 18

G_f



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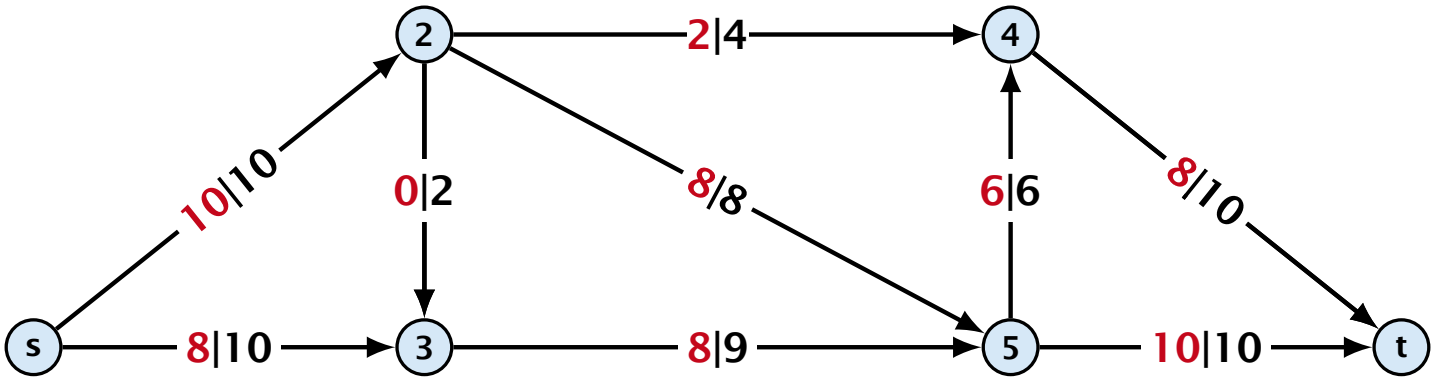

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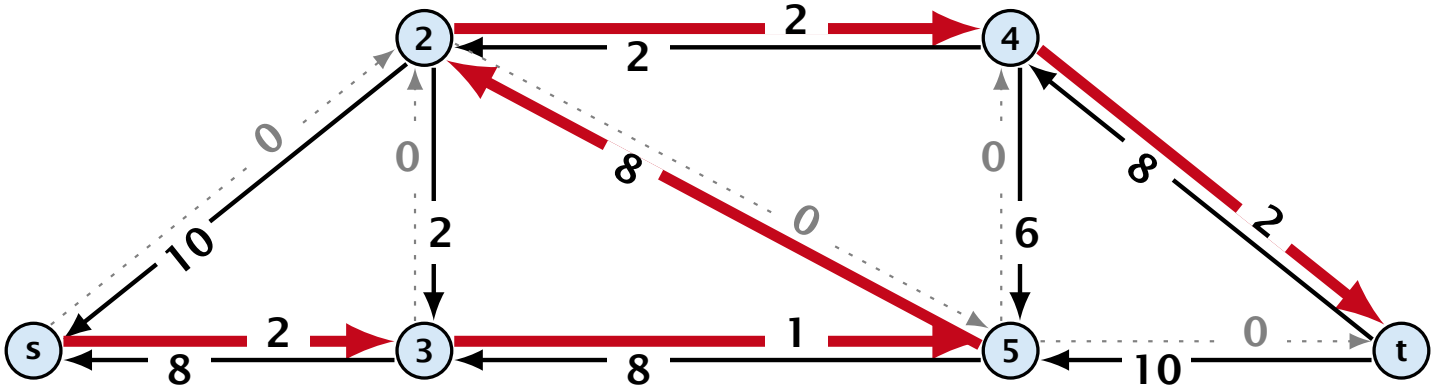

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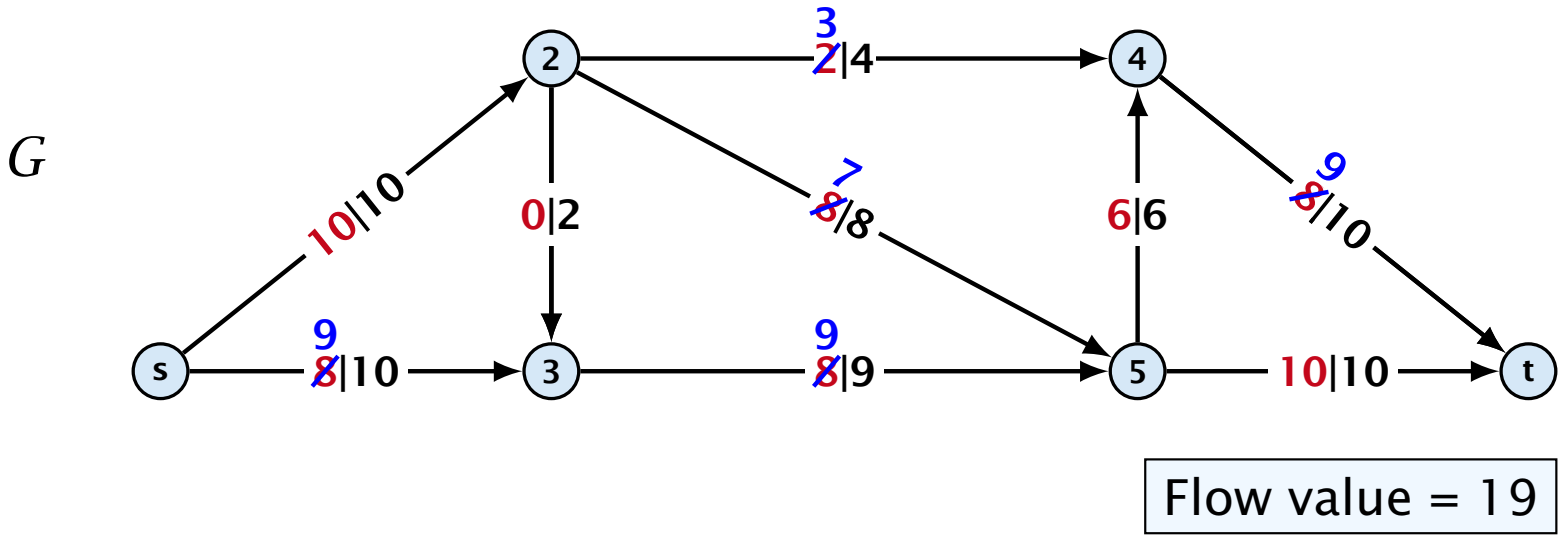

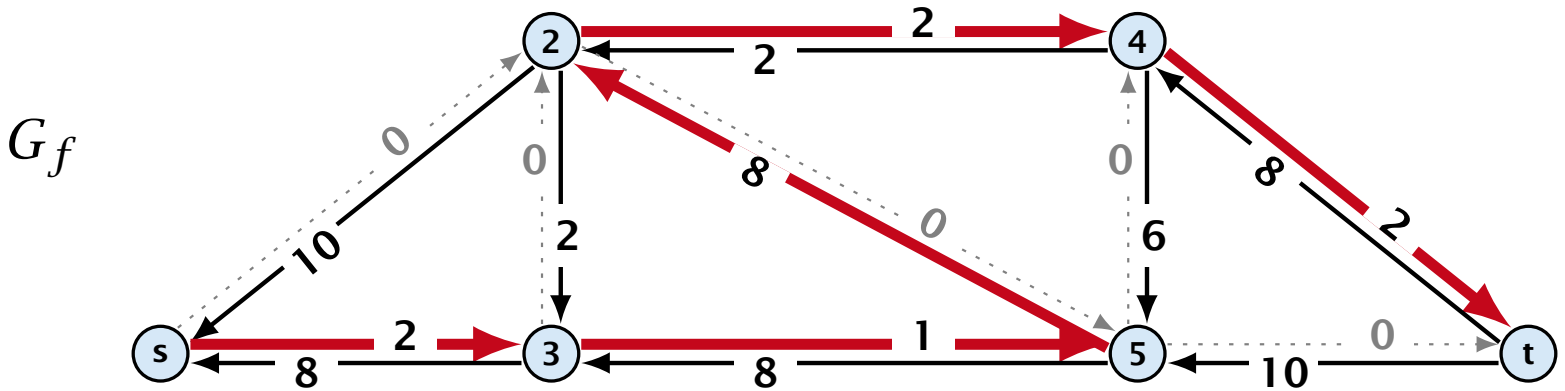

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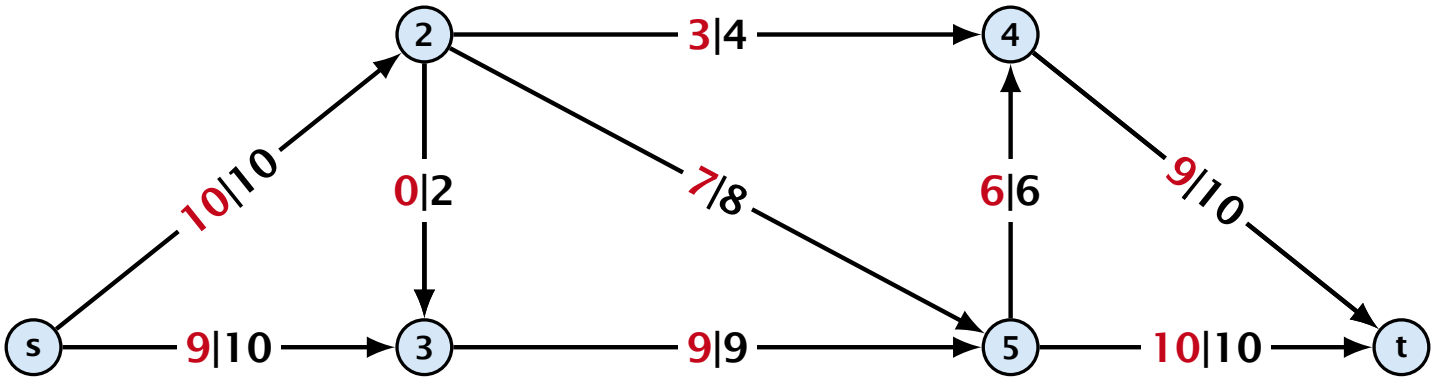
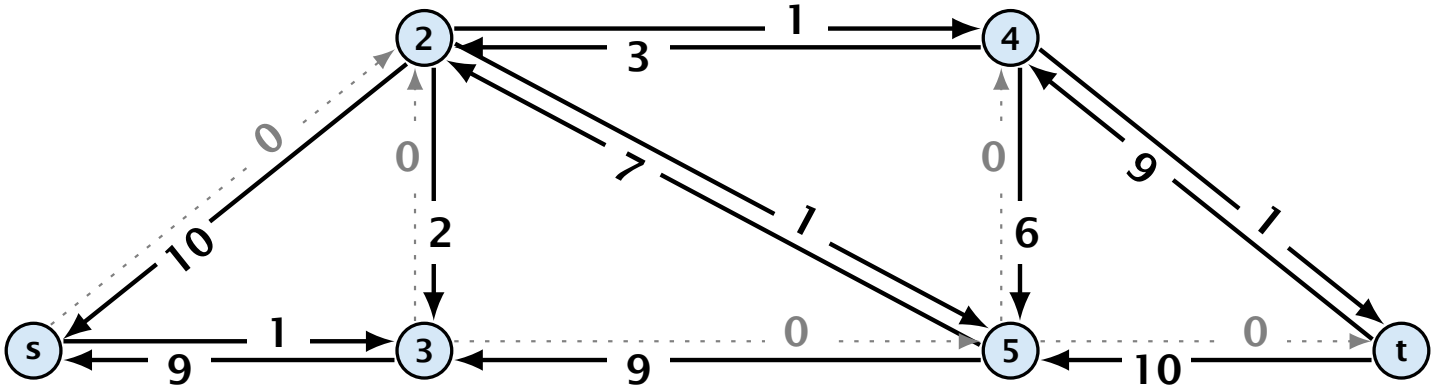


Flow value = 19

G_f



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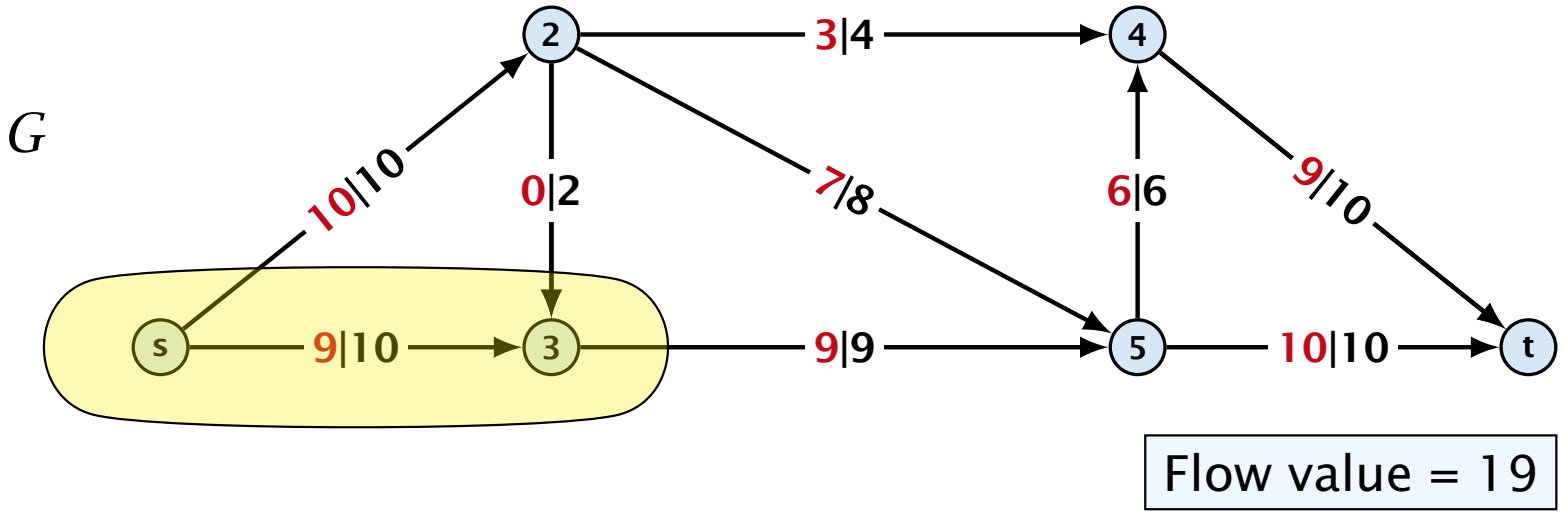

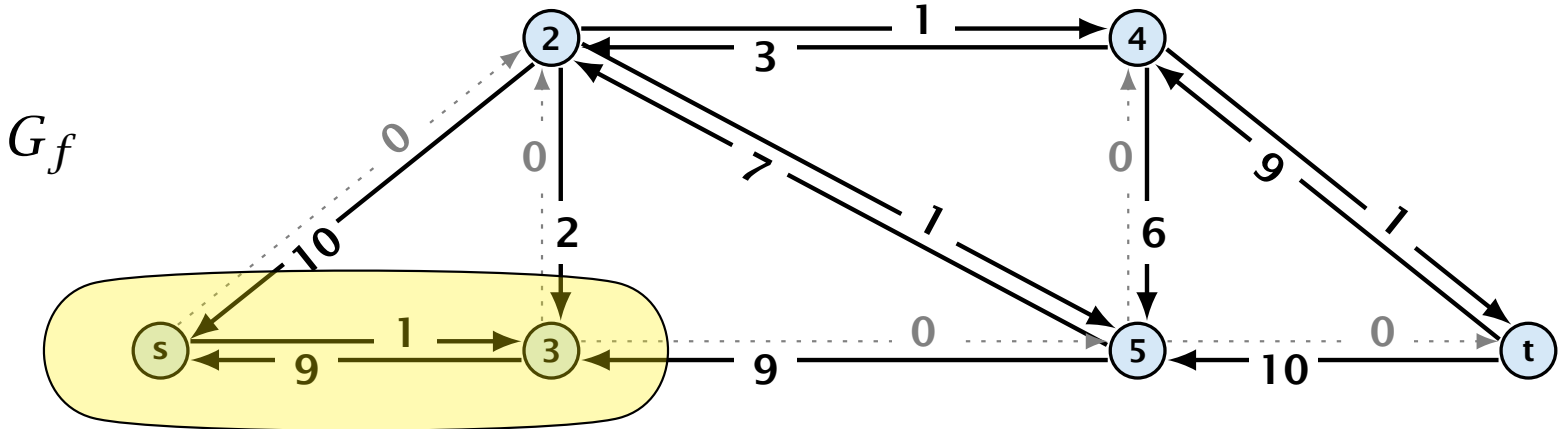

Augmenting Path Algorithm

Theorem 53
A flow f is a maximum flow iff there are no augmenting paths.

Theorem 54
The value of a maximum flow is equal to the value of a minimum cut.

Proof.
Let f be a flow. The following are equivalent:

- There exists a cut $(S, \bar{S})$ such that $|f| = c(S, \bar{S})$.
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This we already showed.

2. $\Rightarrow$ 3.
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Contradiction.

3. $\Rightarrow$ 1.

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Augmenting Path Algorithm

$\mathrm{val}(f)$

Augmenting Path Algorithm

$$\mathrm{val}(f) = \sum_{e \in \mathrm{out}(A)} f(e) - \sum_{e \in \mathrm{into}(A)} f(e)$$

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$$\begin{aligned}\mathrm{val}(f) &= \sum_{e\in \mathrm{out}(A)} f(e) - \sum_{e\in \mathrm{into}(A)} f(e) \\ &= \sum_{e\in \mathrm{out}(A)} c(e)\end{aligned}$$

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This finishes the proof.

Here the first equality uses the flow value lemma, and the second exploits the fact that the flow along incoming edges must be 0 as the residual graph does not have edges leaving A.

Analysis

Assumption:
All capacities are integers between 1 and C.

Invariant:
Every flow value $f(e)$ and every residual capacity $c_f(e)$ remains integral troughout the algorithm.

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Lemma 55

The algorithm terminates in at most $\mathrm{val}(f^*) \leq nC$ *iterations, where* f^* *denotes the maximum flow. Each iteration can be implemented in time* $\mathcal{O}(m)$*. This gives a total running time of* $\mathcal{O}(nmC)$*.*

Theorem 56

If all capacities are integers, then there exists a maximum flow for which every flow value $f(e)$ *is integral.*

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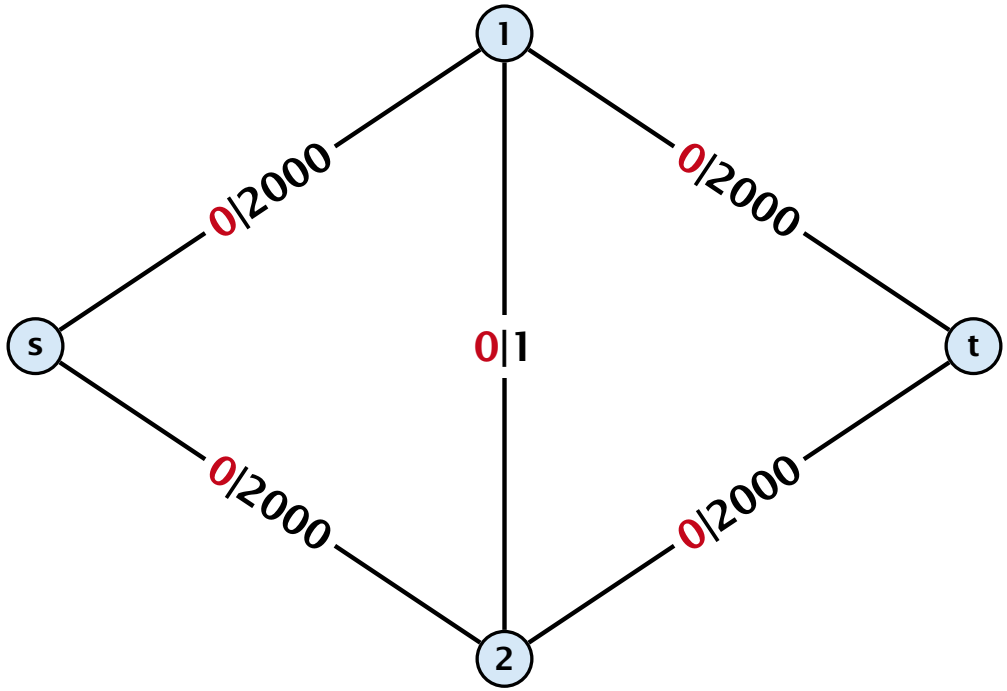
A Bad Input

Problem: The running time may not be polynomial.



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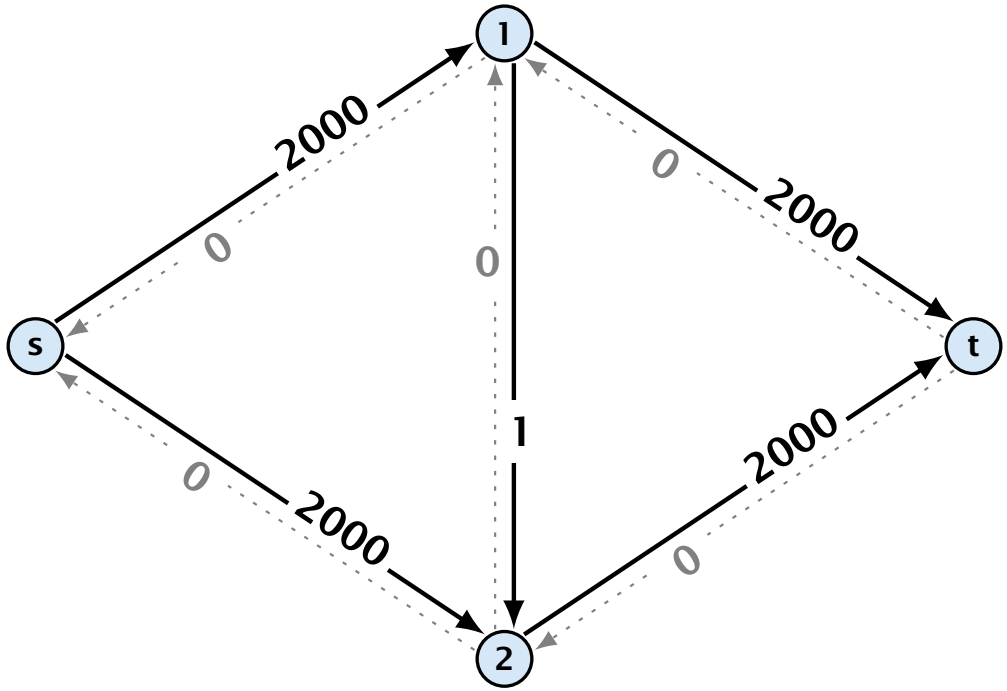

Question:
Can we tweak the algorithm so that the running time is polynomial in the input length?

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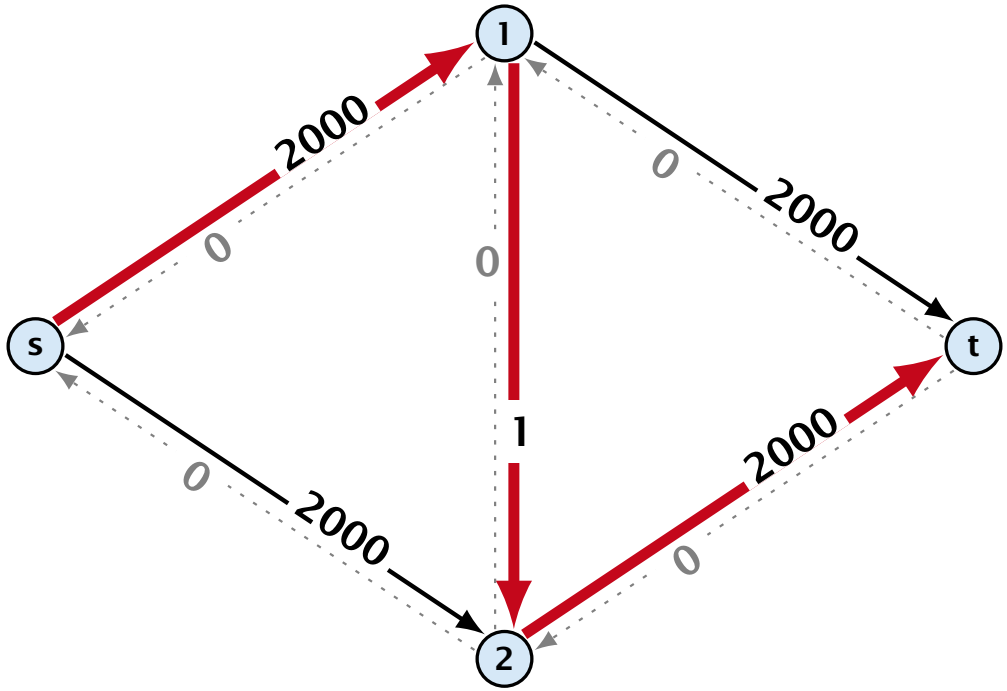

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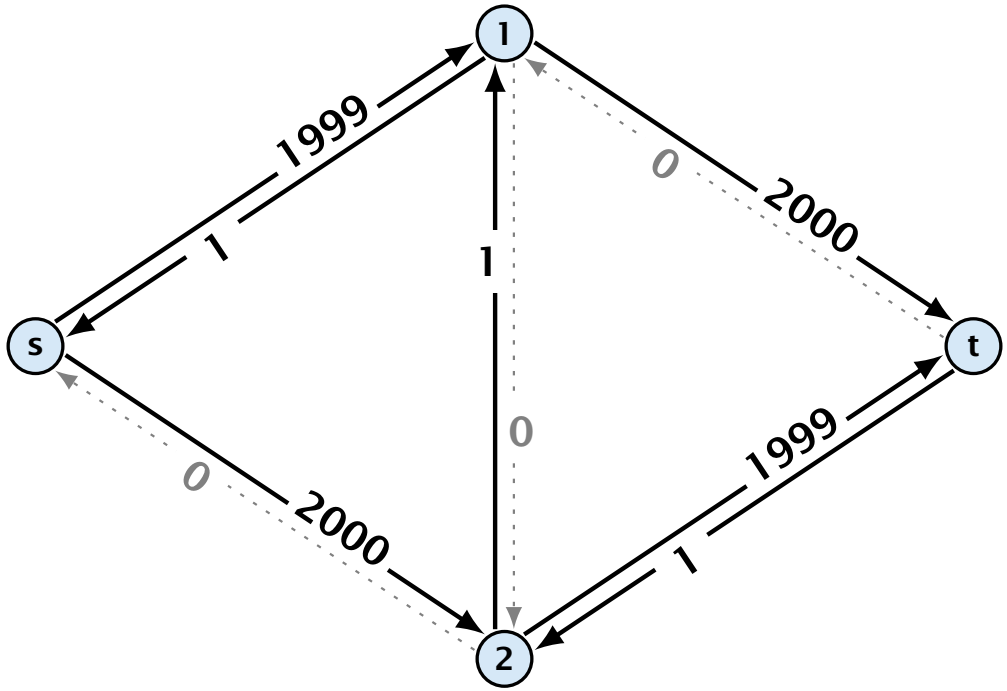

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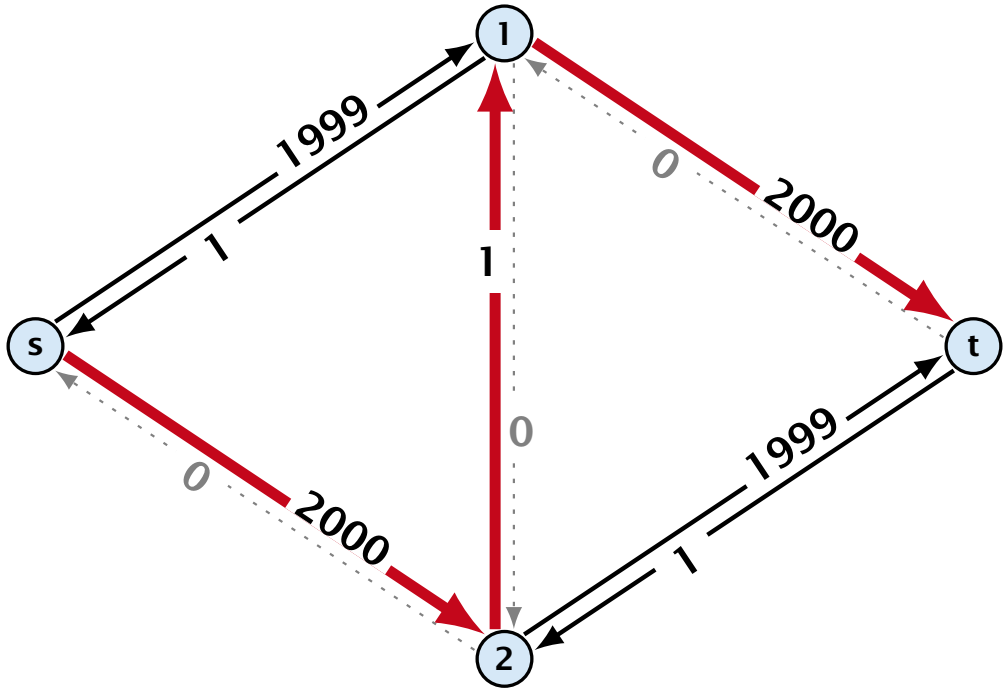

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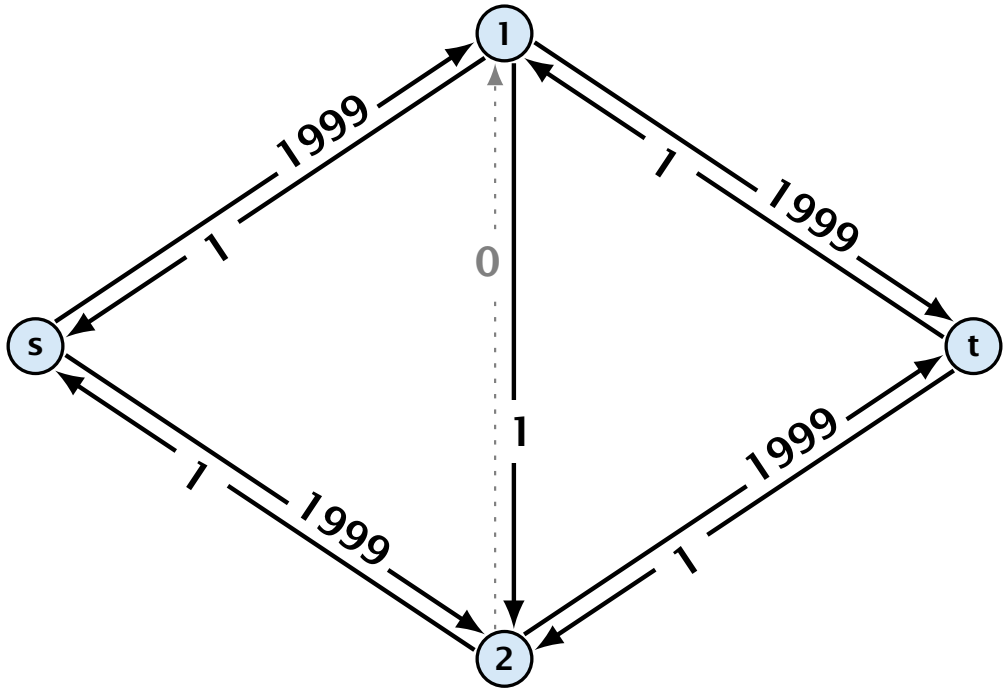

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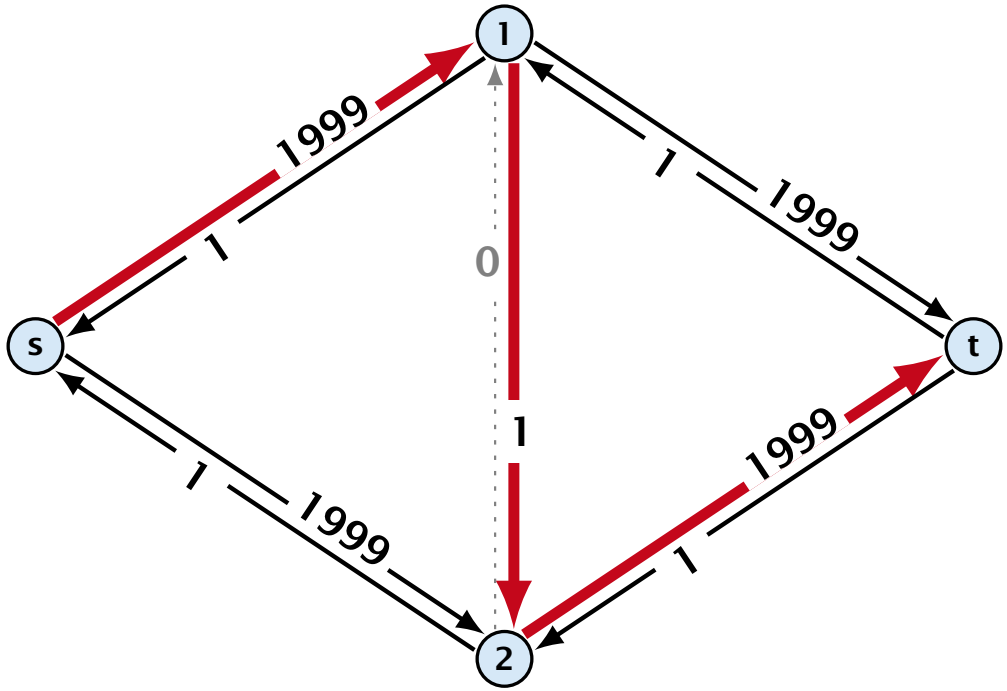

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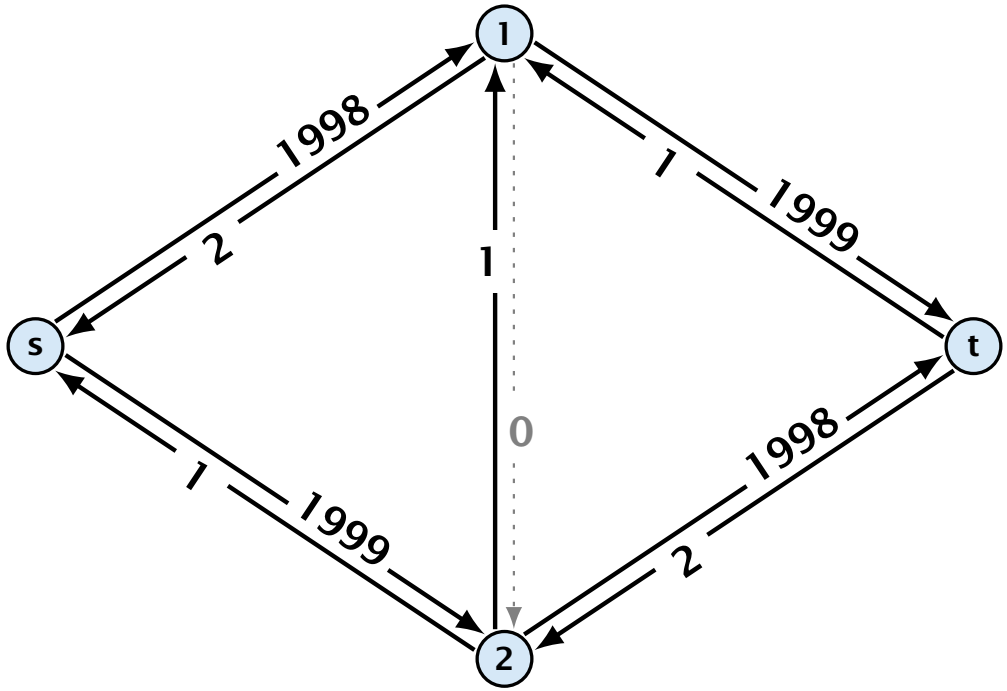

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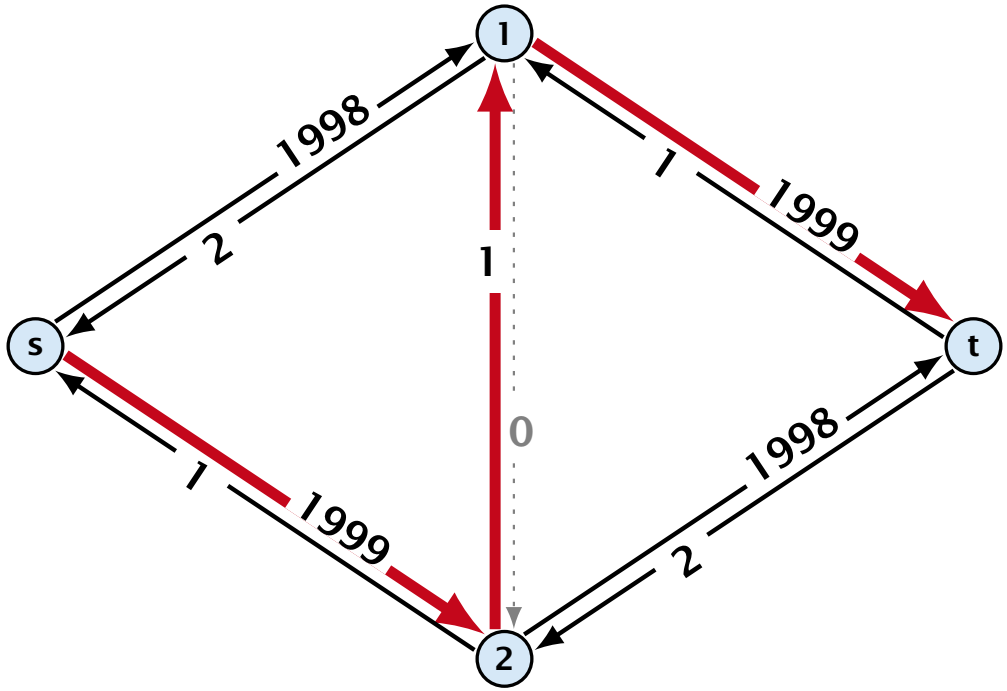

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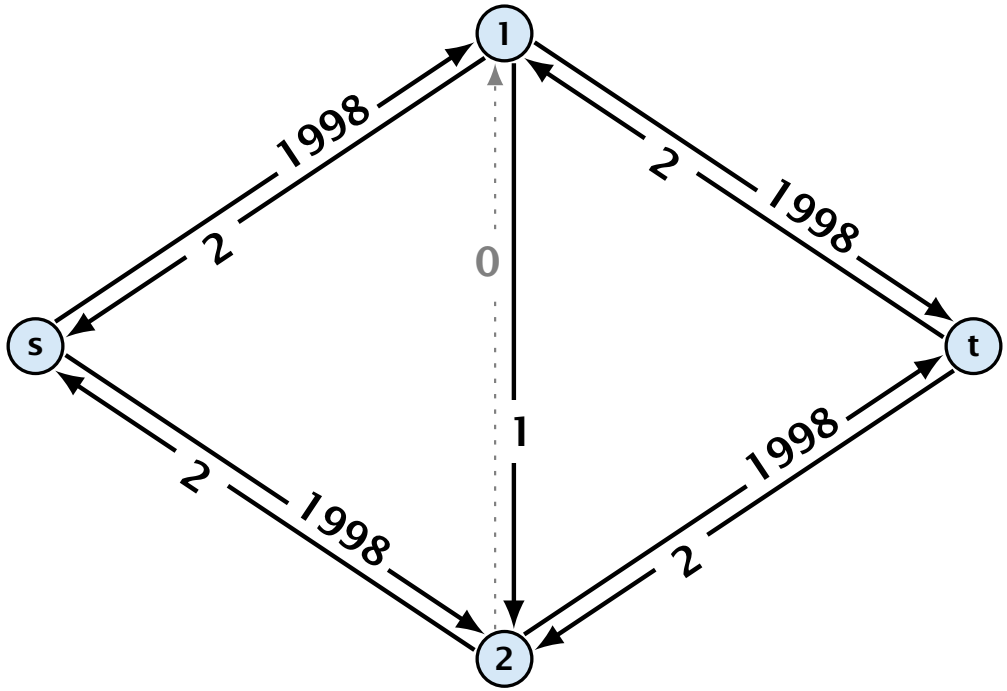

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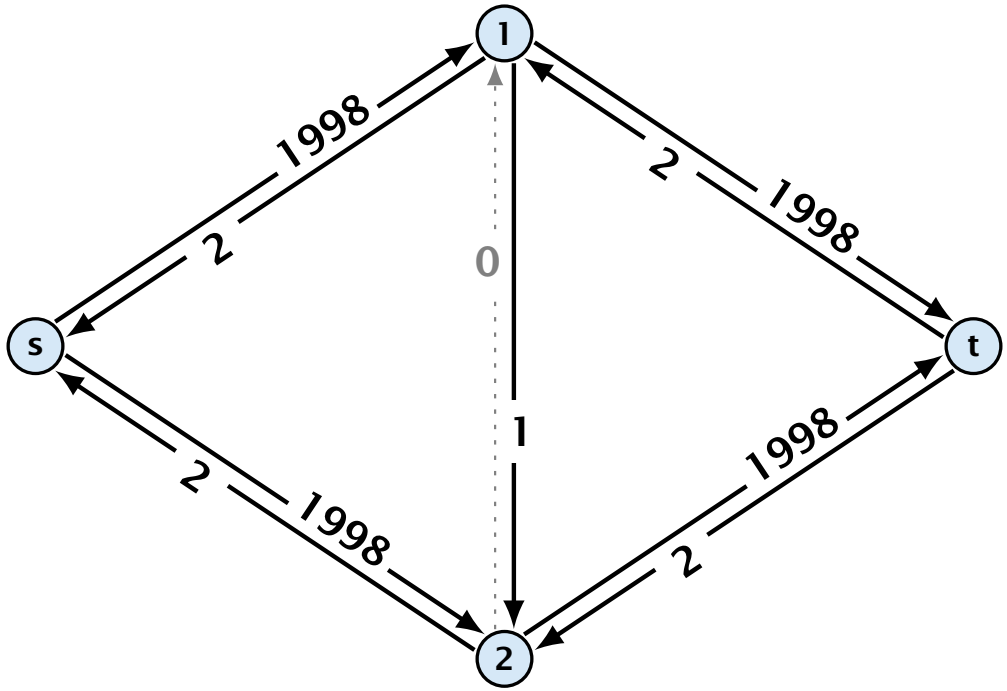

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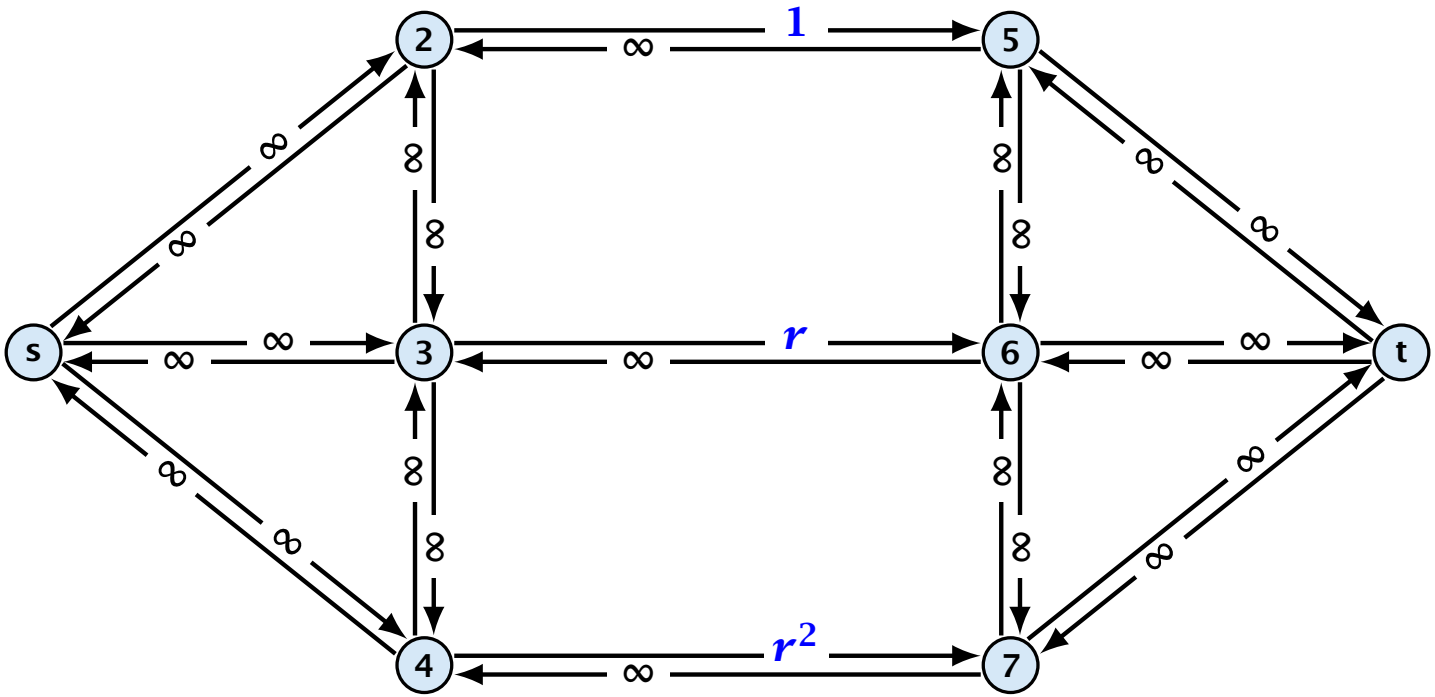

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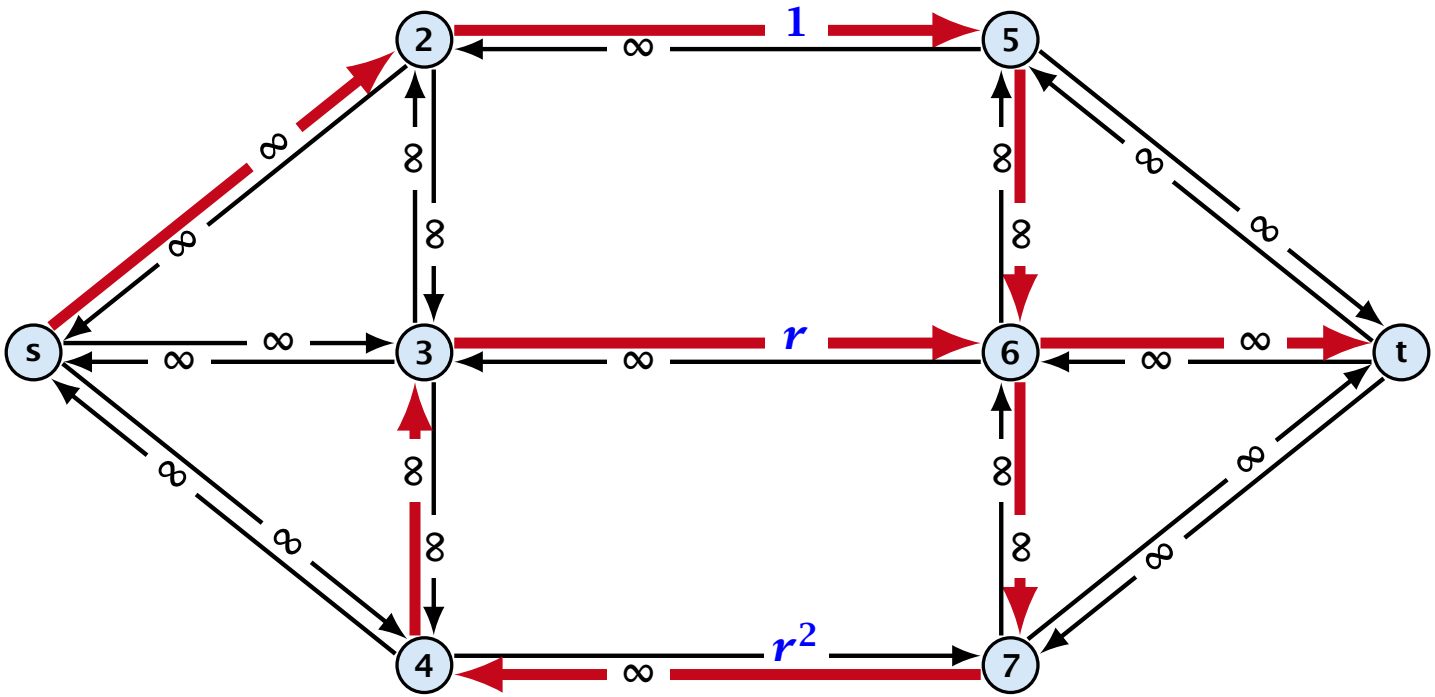

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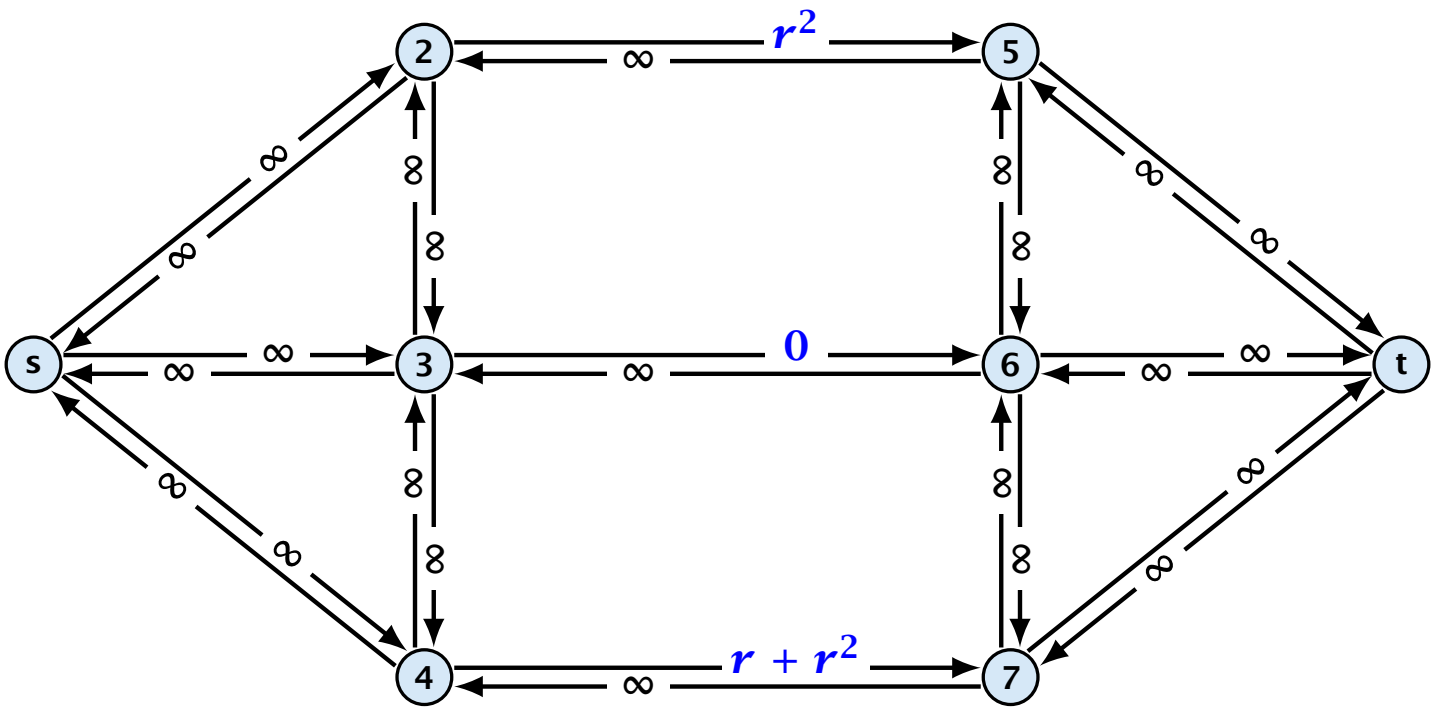

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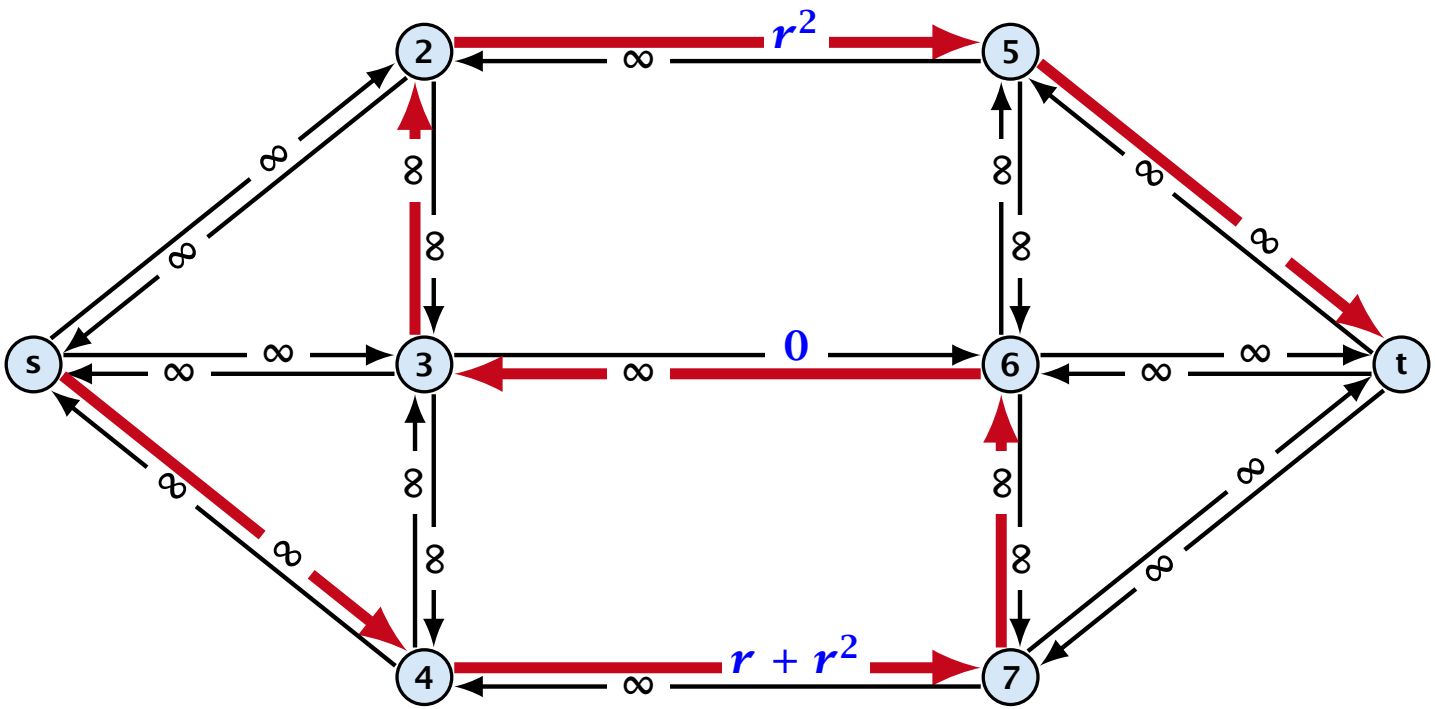

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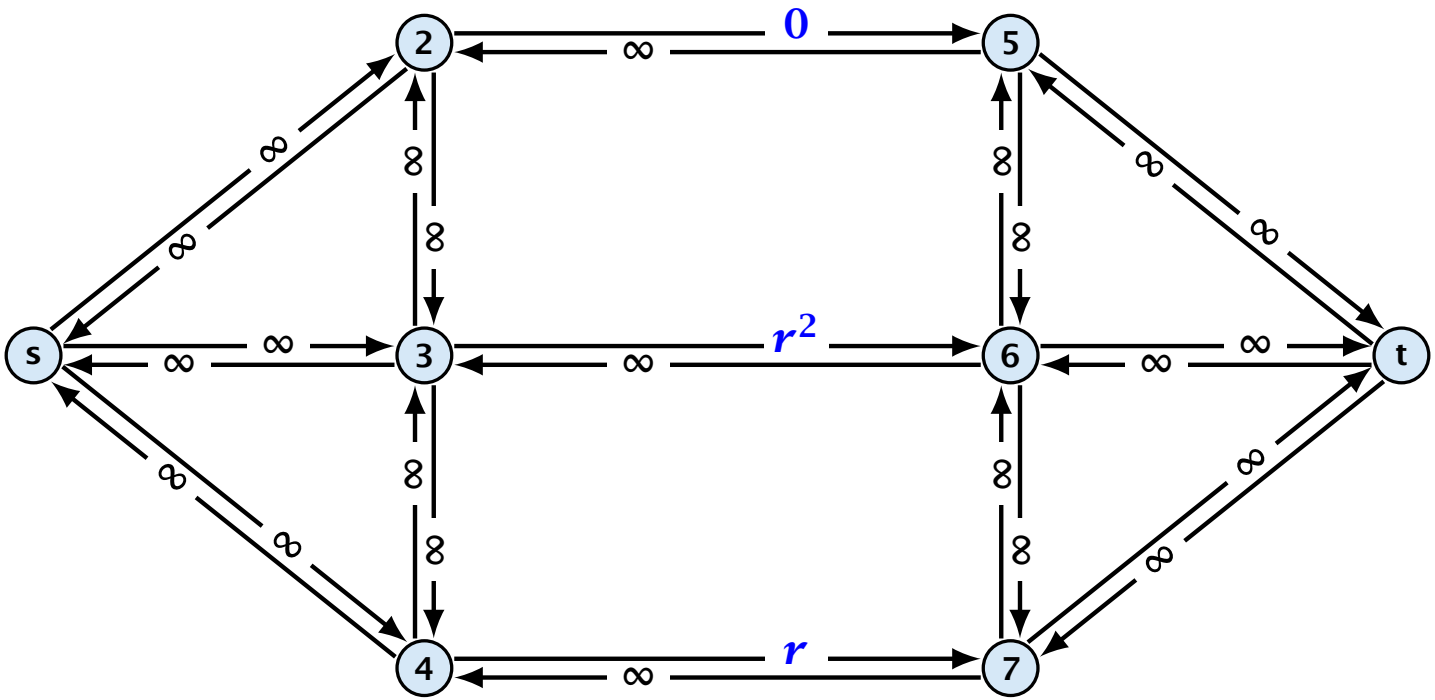

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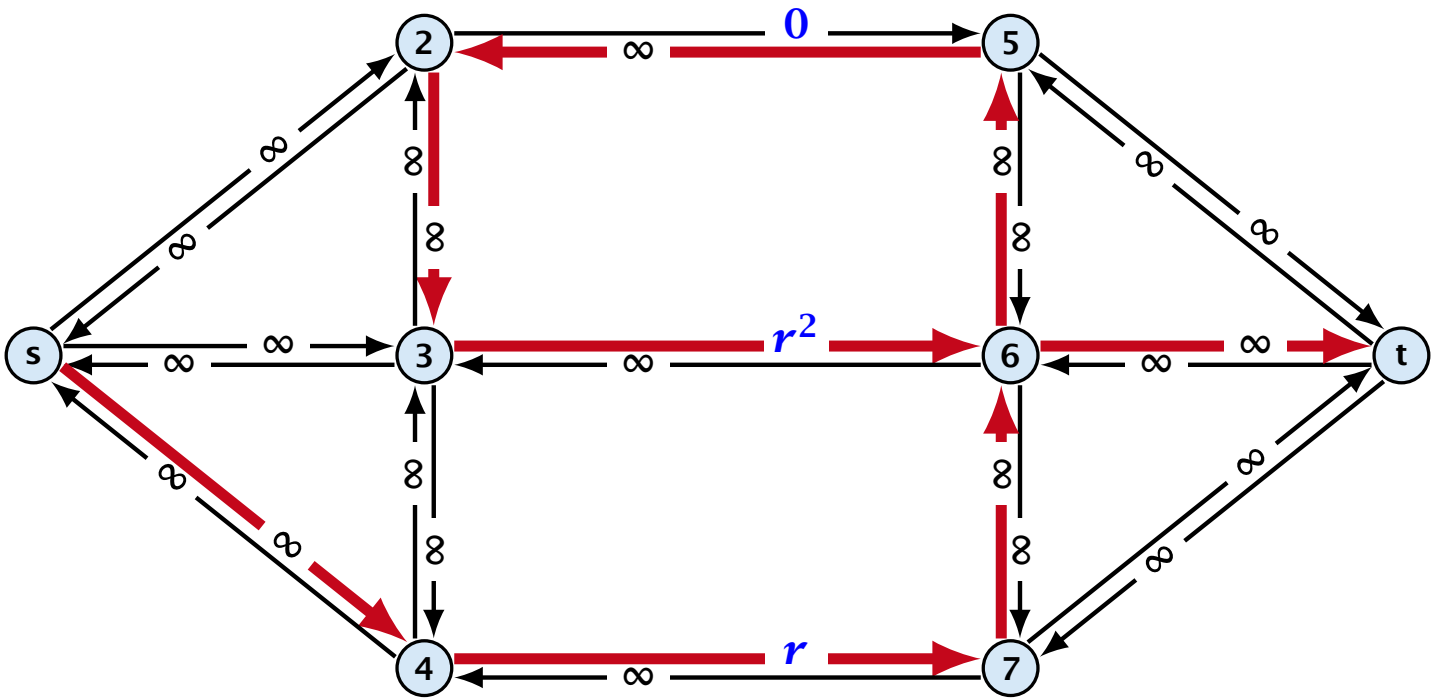

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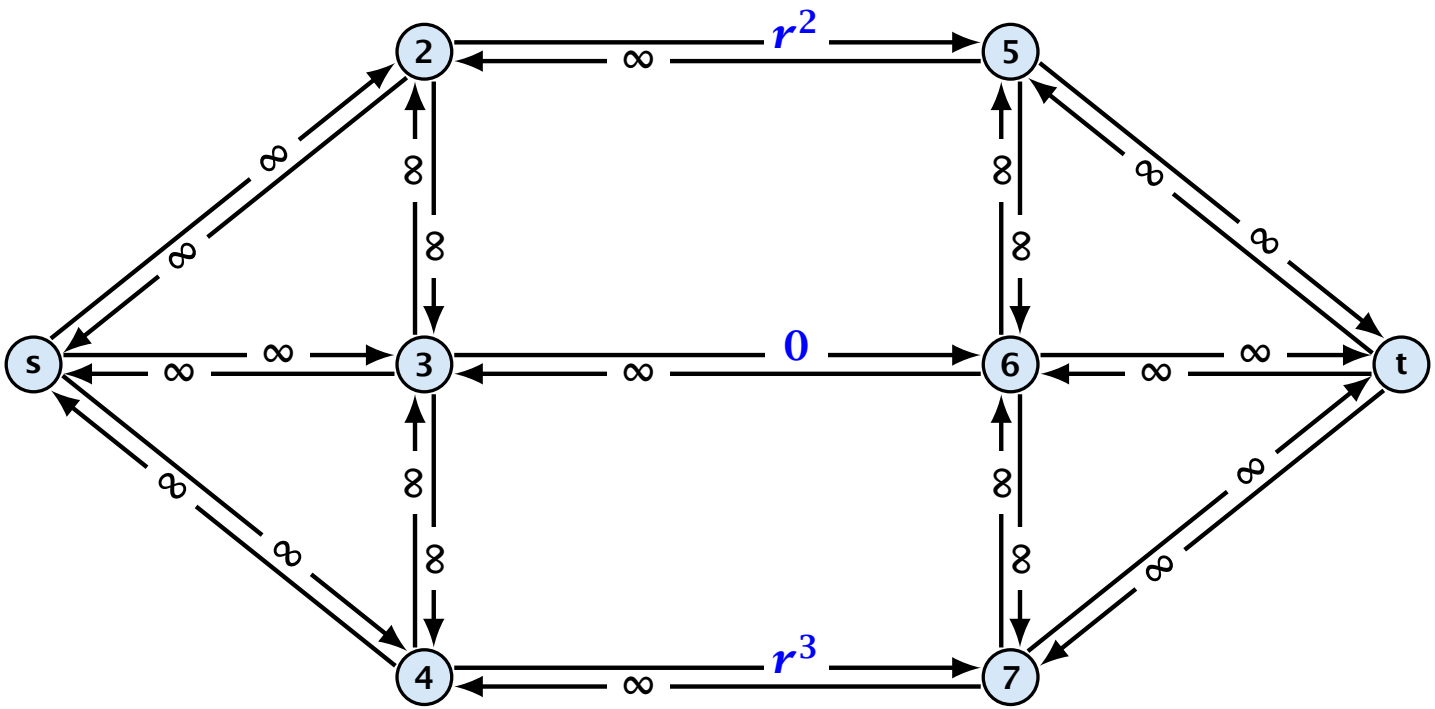

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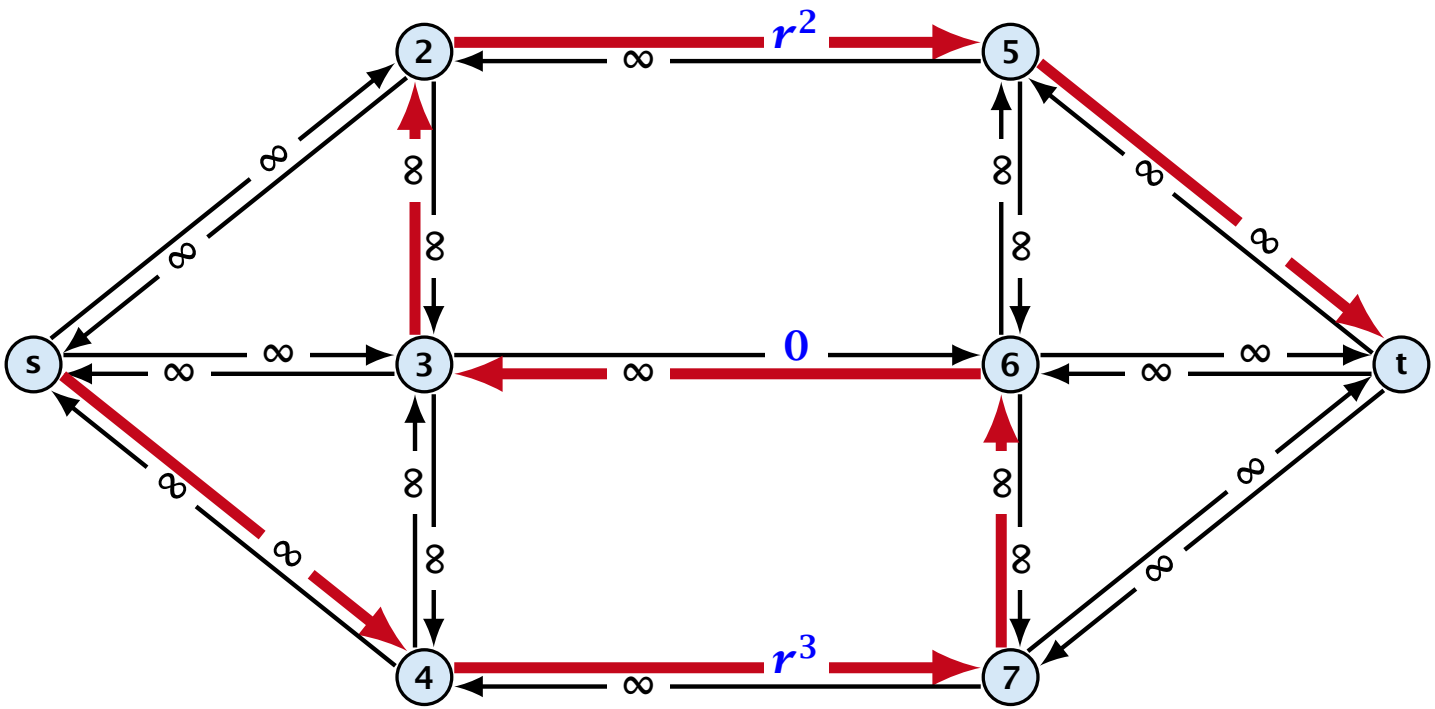

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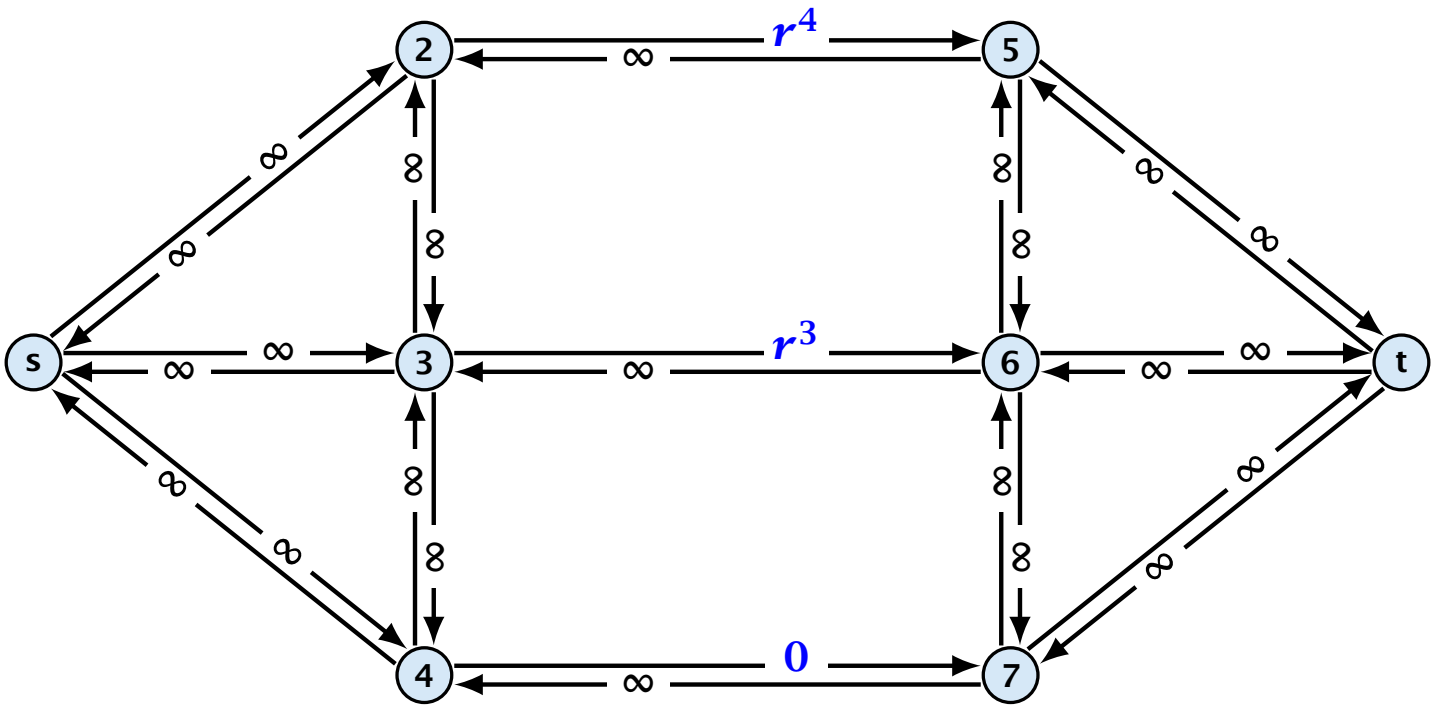

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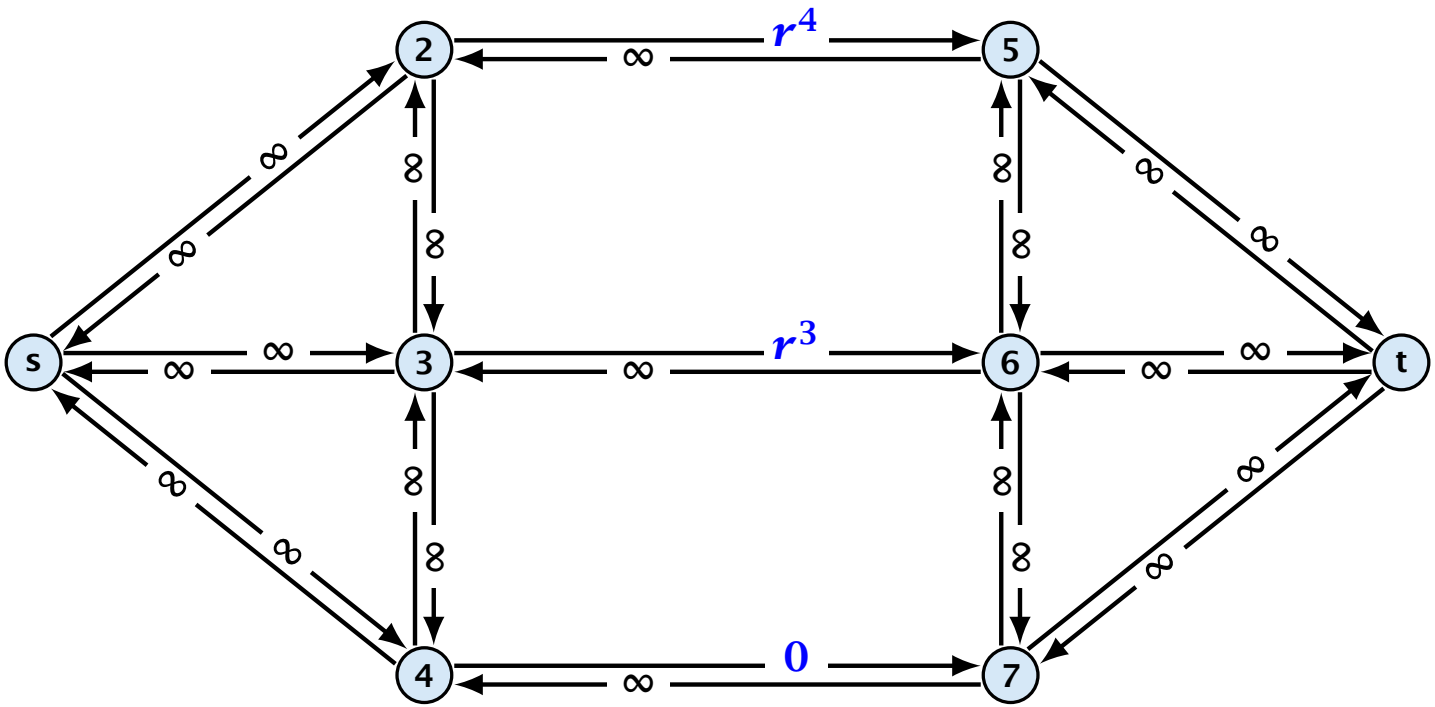

Running time may be infinite!!!

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Overview: Shortest Augmenting Paths

Lemma 57

The length of the shortest augmenting path never decreases.

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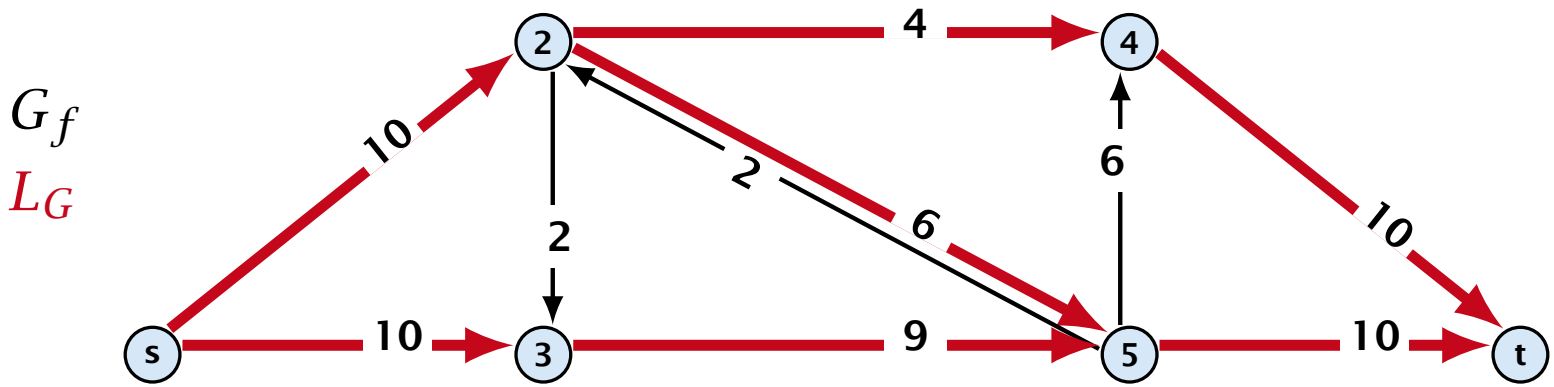

In the following we assume that the residual graph G_f does not contain zero capacity edges.

This means, we construct it in the usual sense and then delete edges of zero capacity.

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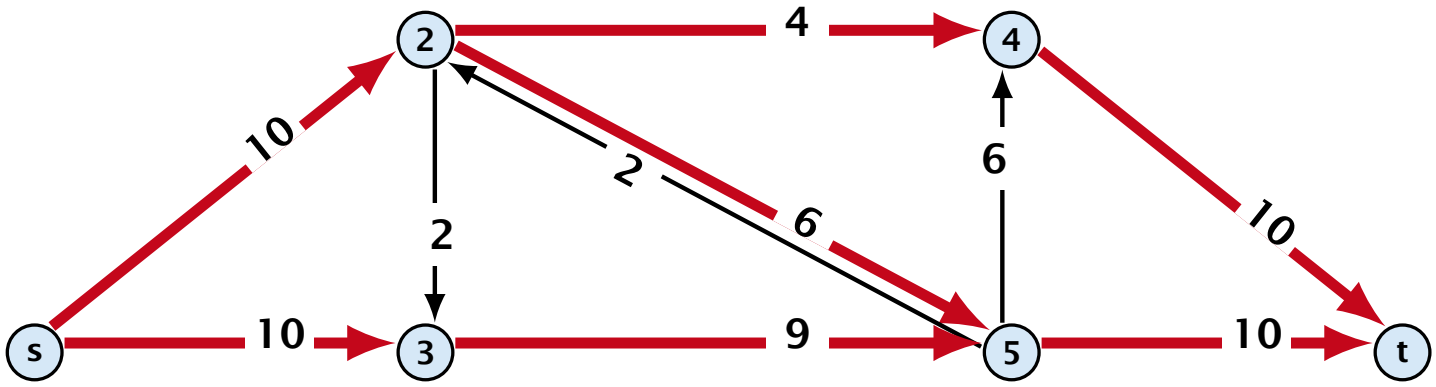

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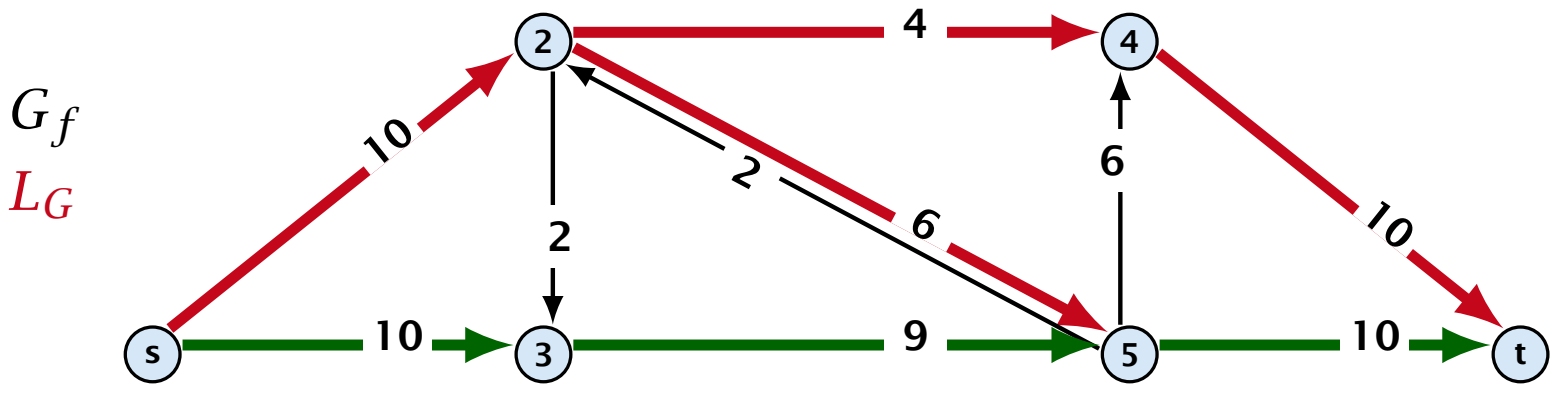

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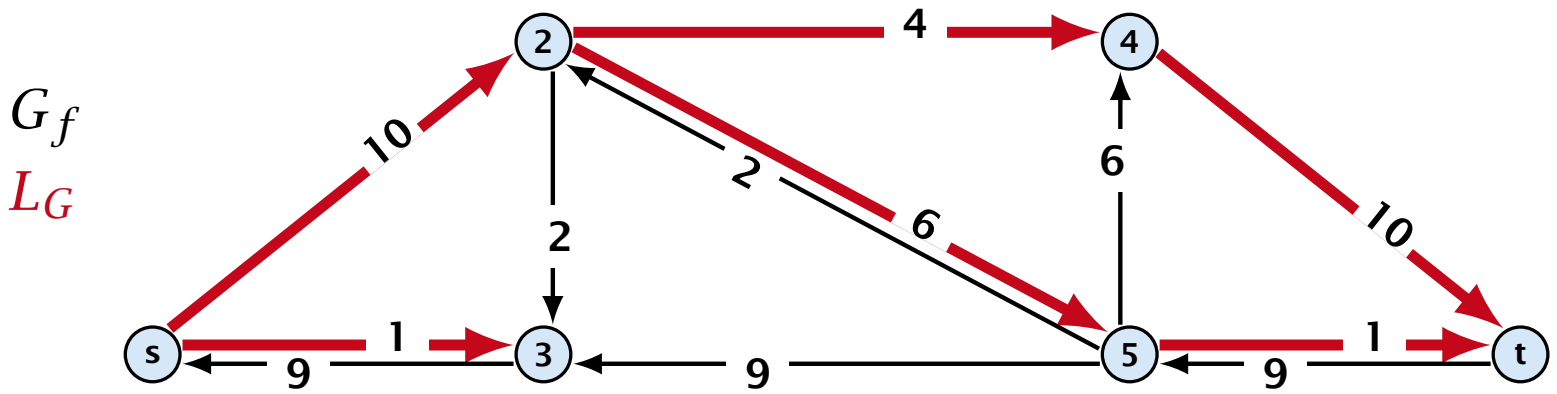

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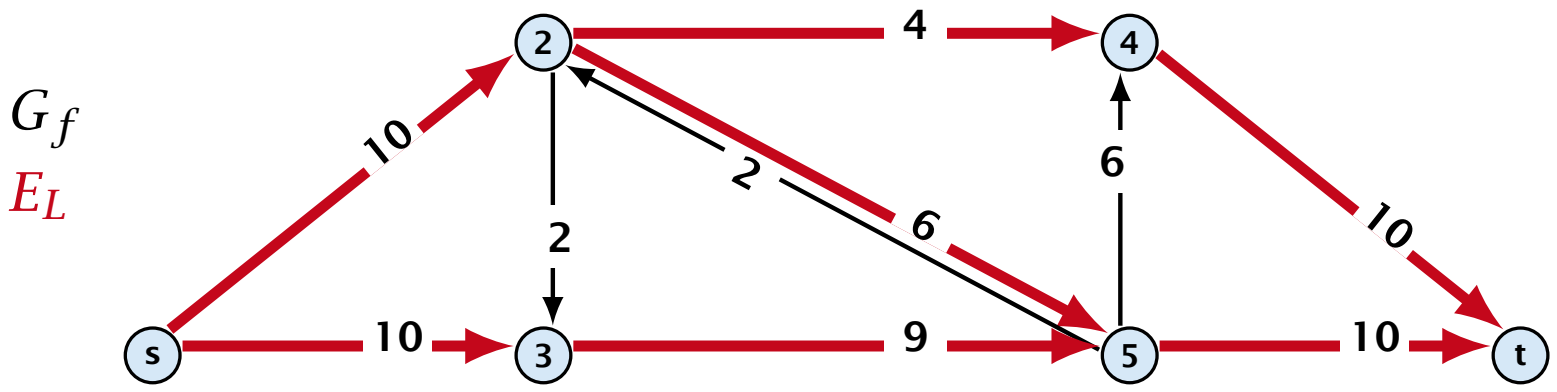

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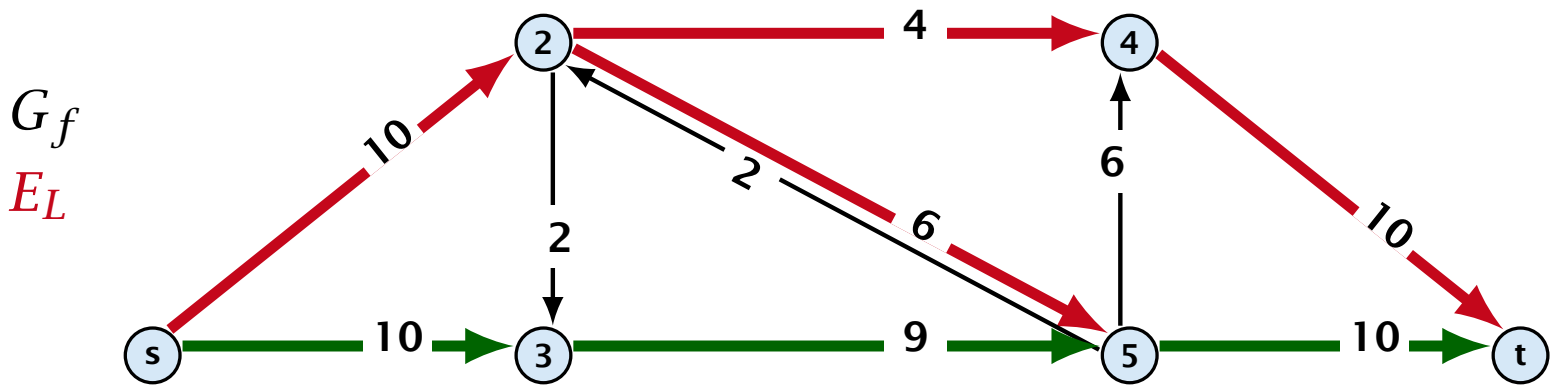

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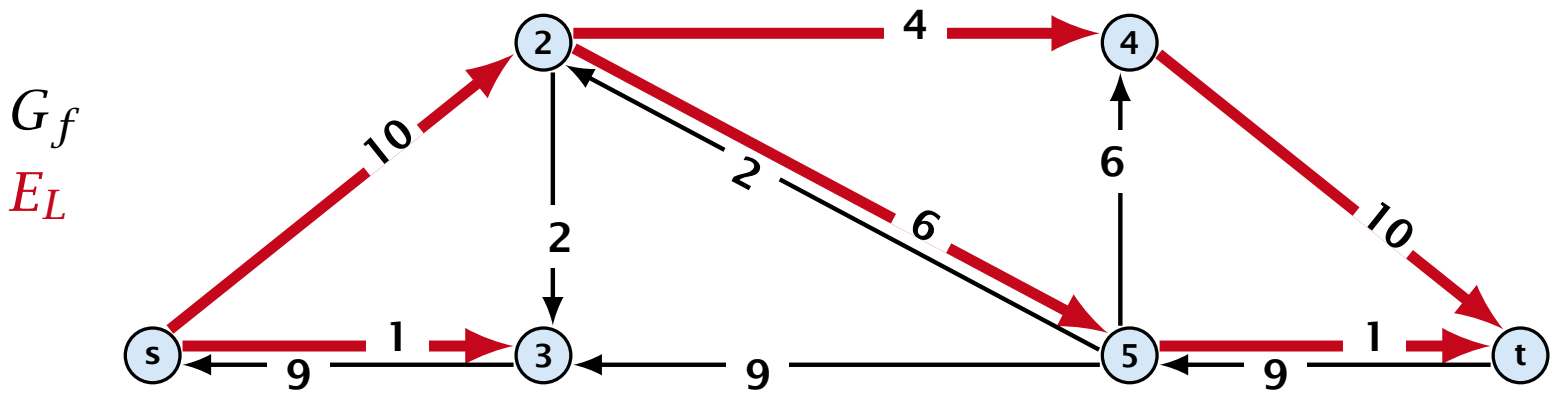

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Theorem 60

The shortest augmenting path algorithm performs at most $\mathcal{O}(mn)$ augmentations. Each augmentation can be performed in time $\mathcal{O}(m)$.

Theorem 61 (without proof)

There exist networks with $m = \Theta(n^2)$ that require $\mathcal{O}(mn)$ augmentations, when we restrict ourselves to only augment along shortest augmenting paths.

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We maintain a subset E_L of the edges of G_f with the guarantee that a shortest s-t path using only edges from E_L is a shortest augmenting path.

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Let a phase of the algorithm be defined by the time between two augmentations during which the distance between s and t strictly increases.

Initializing E_L for the phase takes time $\mathcal{O}(m)$.

The total cost for searching for augmenting paths during a phase is at most $\mathcal{O}(mn)$, since every search (successful (i.e., reaching t) or unsuccessful) decreases the number of edges in E_L and takes time $\mathcal{O}(n)$.

The total cost for performing an augmentation during a phase is only $\mathcal{O}(n)$. For every edge in the augmenting path one has to update the residual graph G_f and has to check whether the edge is still in E_L for the next search.

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The total cost for searching for augmenting paths during a phase is at most $\mathcal{O}(mn)$, since every search (successful (i.e., reaching t) or unsuccessful) decreases the number of edges in E_L and takes time $\mathcal{O}(n)$.

The total cost for performing an augmentation during a phase is only $\mathcal{O}(n)$. For every edge in the augmenting path one has to update the residual graph G_f and has to check whether the edge is still in E_L for the next search.

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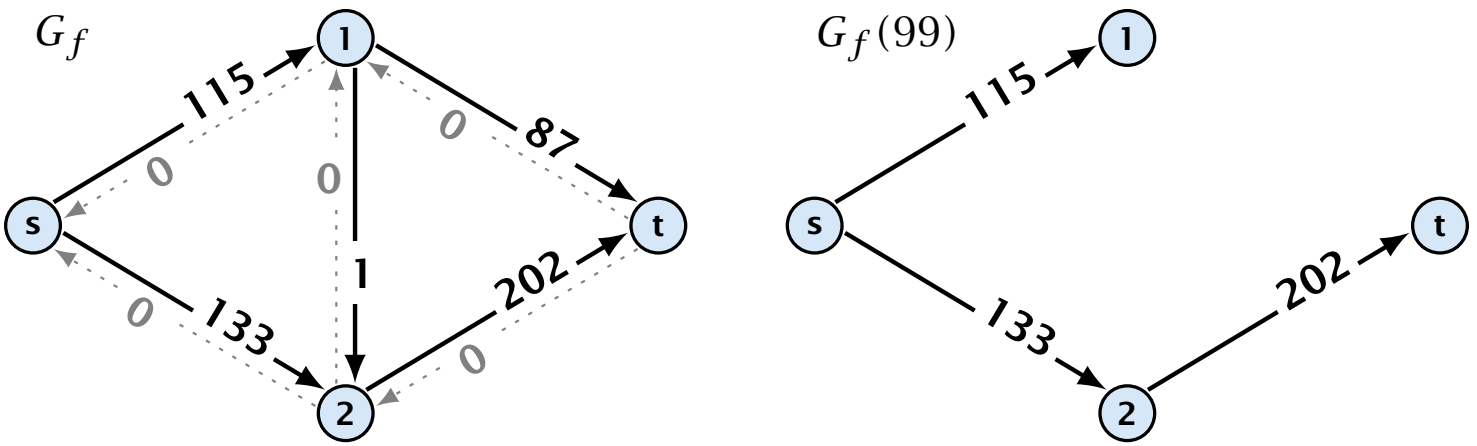

Capacity Scaling

Algorithm 2 maxflow(G, s, t, c)

```
1: foreach e ∈ E do f_e ← 0;
2: Δ ← 2^⌈log_2 C⌉
3: while Δ ≥ 1 do
4:         G_f(Δ) ← Δ-residual graph
5:         while there is augmenting path P in G_f(Δ) do
6:                 f ← augment(f, c, P)
7:                 update(G_f(Δ))
8:         Δ ← Δ/2
9: return f
```

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- this means we have a maximum flow.

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- There must exist an s-t cut in $G_f(\Delta)$ of zero capacity.
- In G_f this cut can have capacity at most $m\Delta$.
- This gives me an upper bound on the flow that I can still add.

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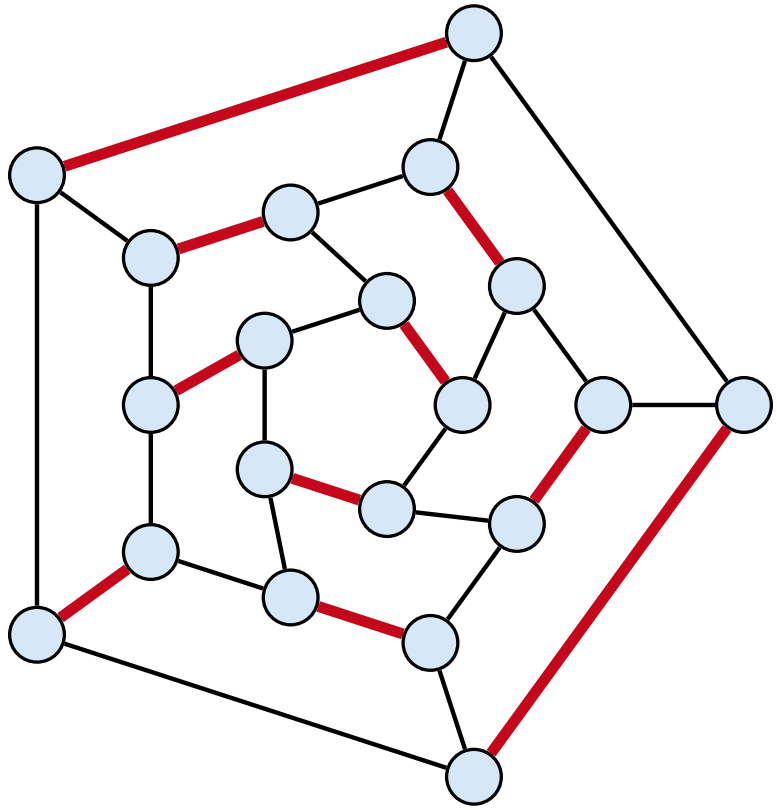
Theorem 65

We need $\mathcal{O}(m \log C)$ augmentations. The algorithm can be implemented in time $\mathcal{O}(m^2 \log C)$.

Matching

- Input: undirected graph $G = (V, E)$.
- $M \subseteq E$ is a matching if each node appears in at most one edge in M.
- Maximum Matching: find a matching of maximum cardinality



Bipartite Matching

- Input: undirected, bipartite graph $G = (L \uplus R, E)$.
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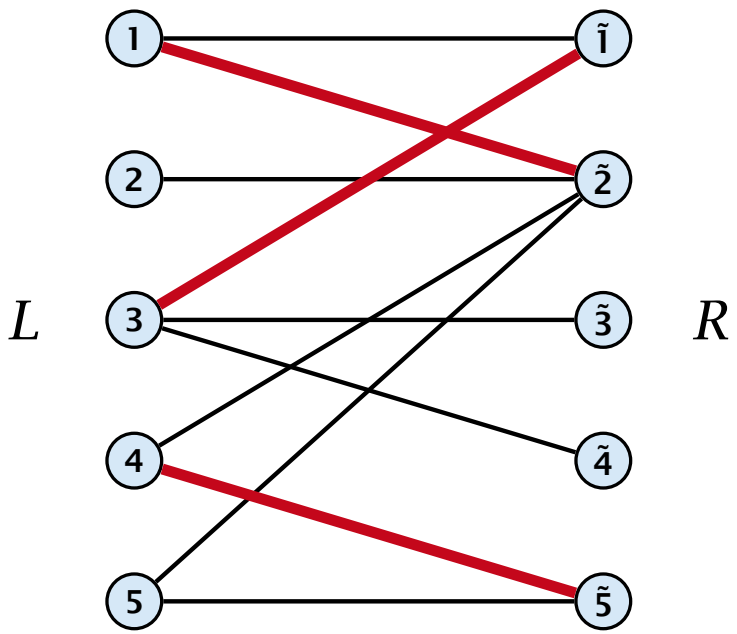

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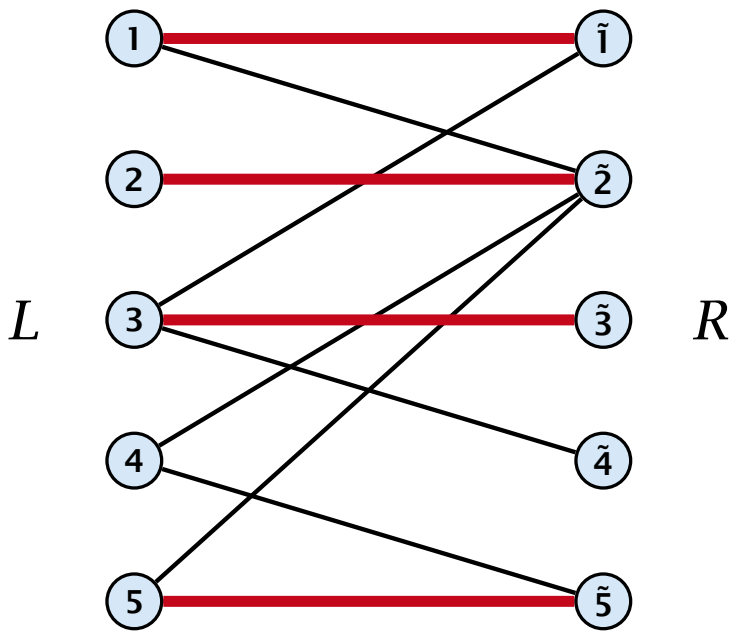
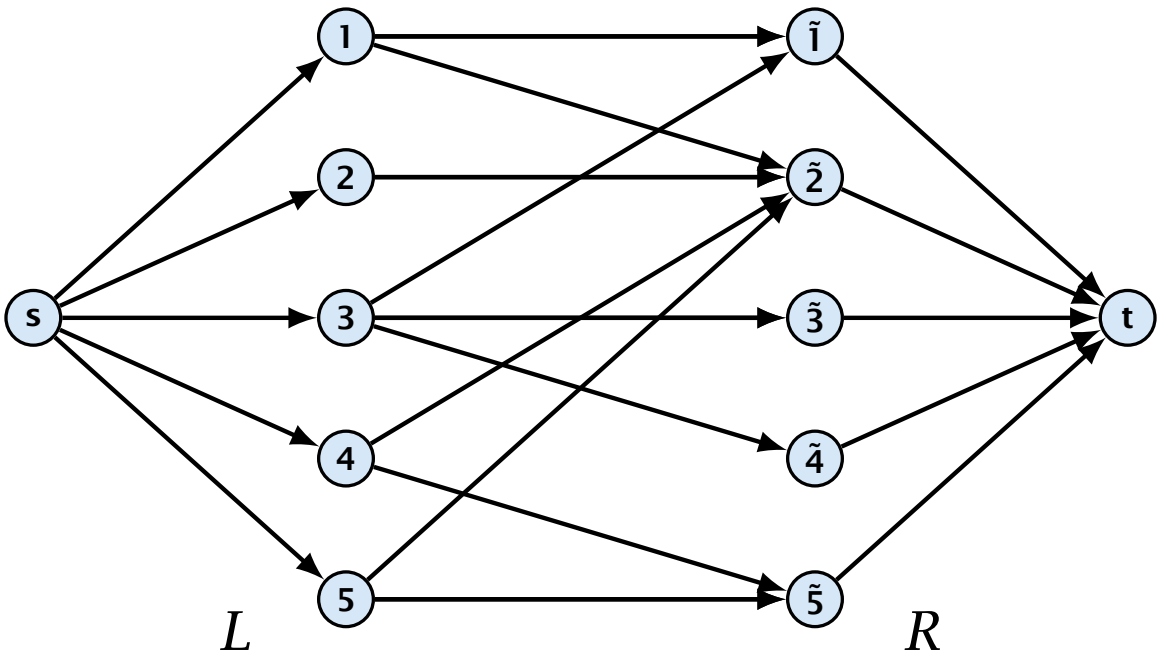


Maxflow Formulation

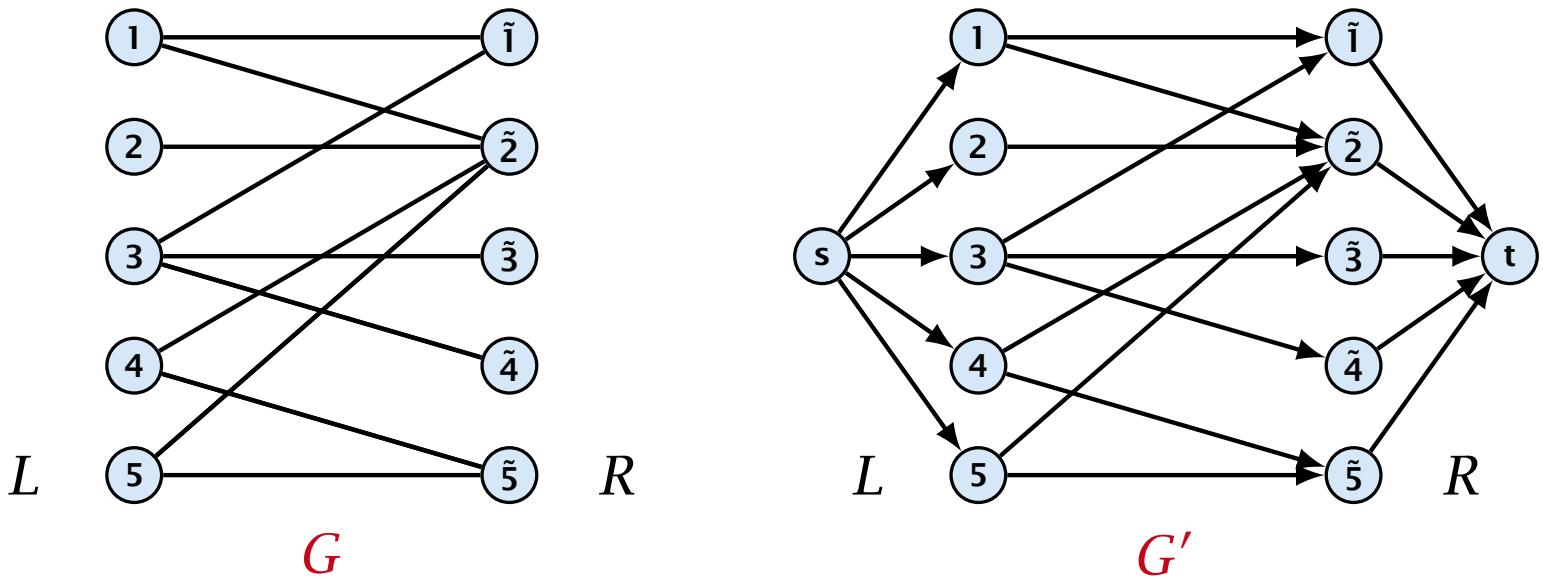
- Input: undirected, bipartite graph $G = (L \uplus R \uplus \{s,t\}, E')$.
- Direct all edges from L to R.
- Add source s and connect it to all nodes on the left.
- Add t and connect all nodes on the right to t.
- All edges have unit capacity.



Proof

Max cardinality matching in $G \leq$ value of maxflow in G'

- Given a maximum matching M of cardinality k.
- Consider flow f that sends one unit along each of k paths.
- f is a flow and has cardinality k.



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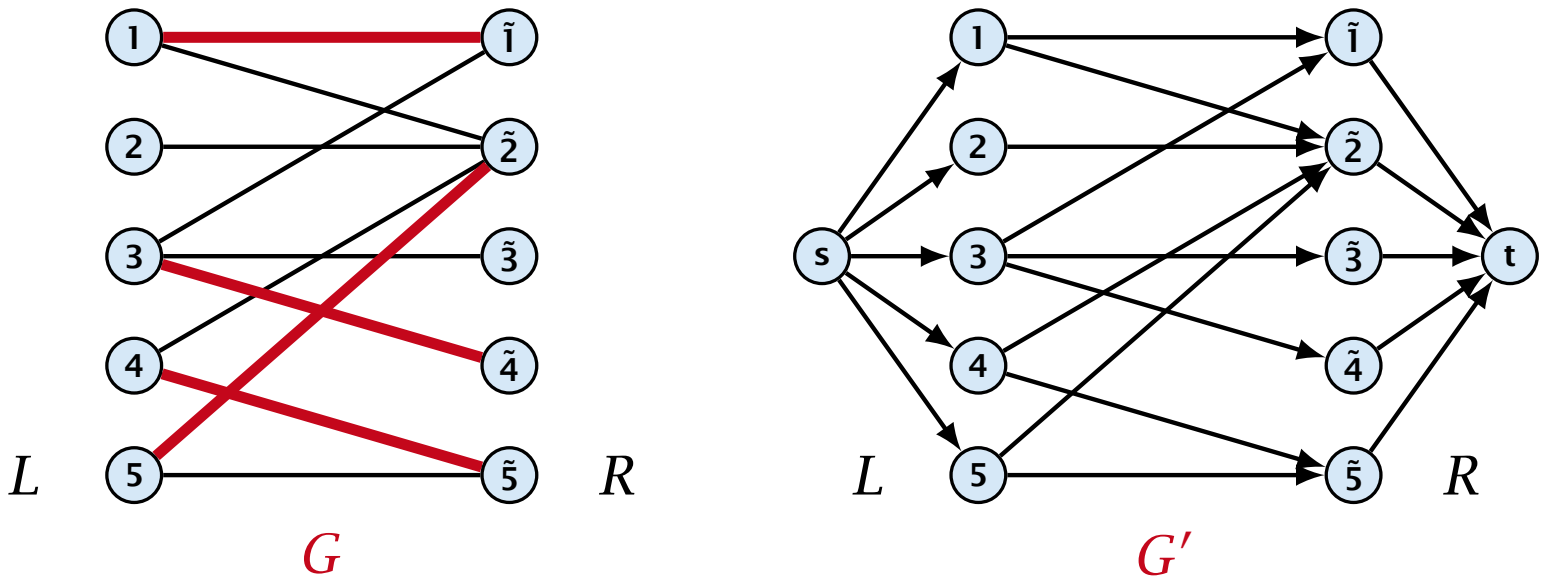

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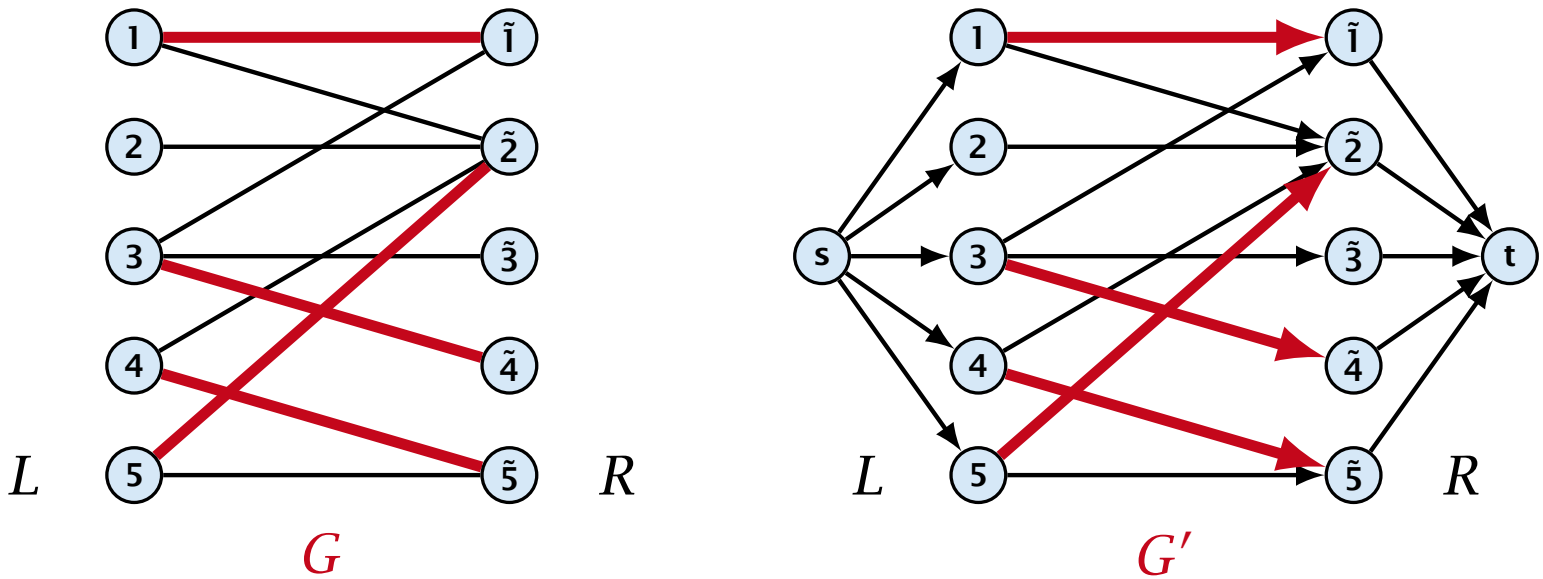

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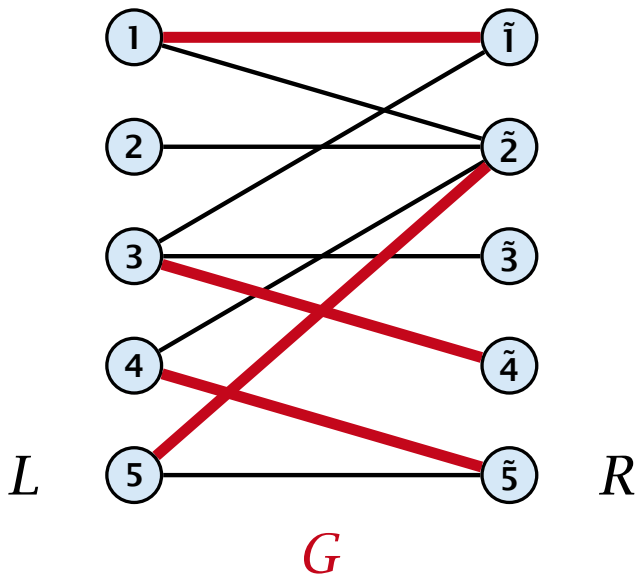

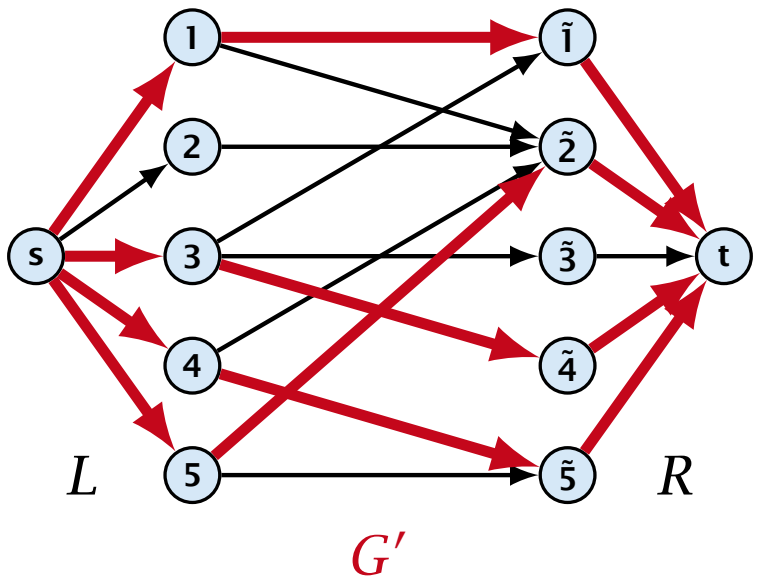

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- Let f be a maxflow in G' of value k
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- Consider M= set of edges from L to R with $f(e) = 1$.
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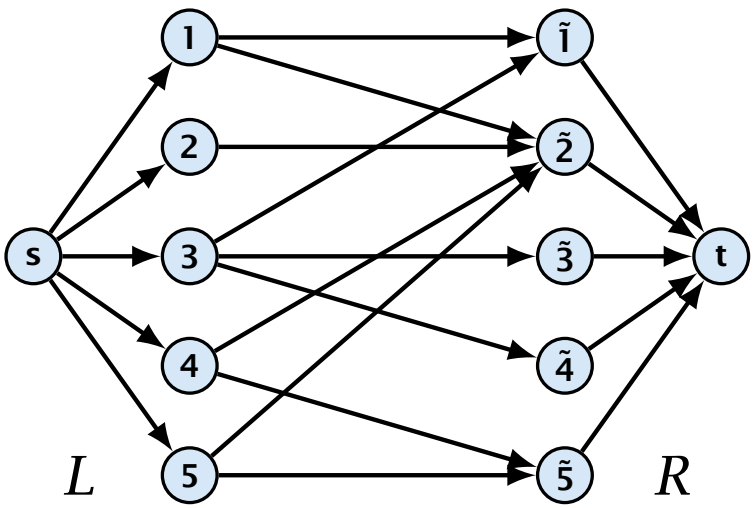

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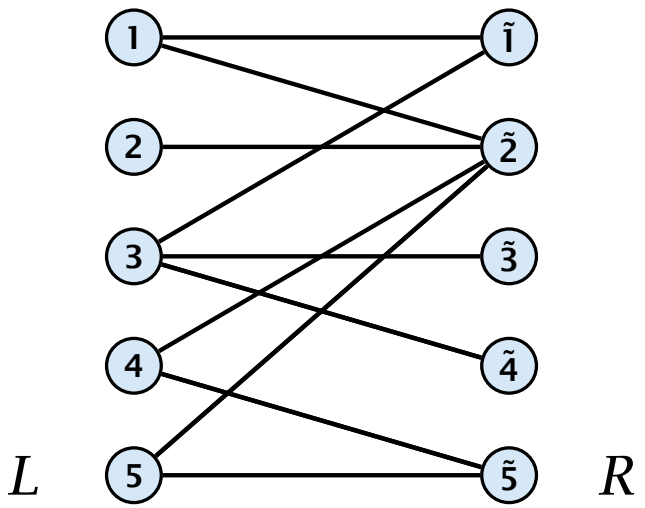

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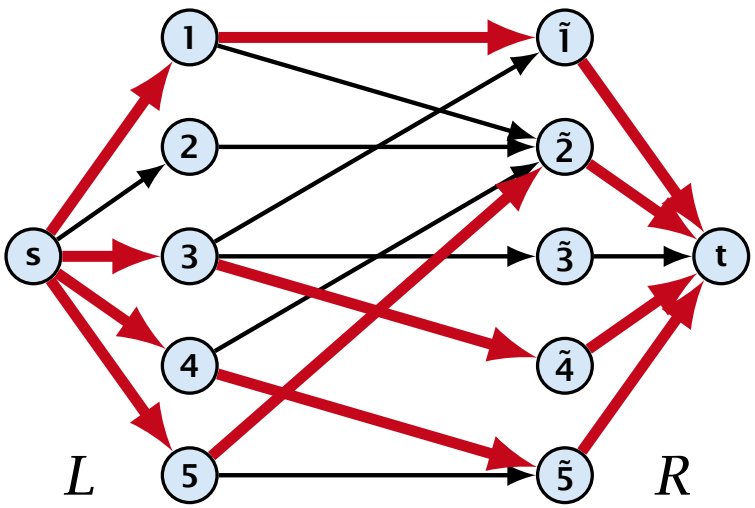

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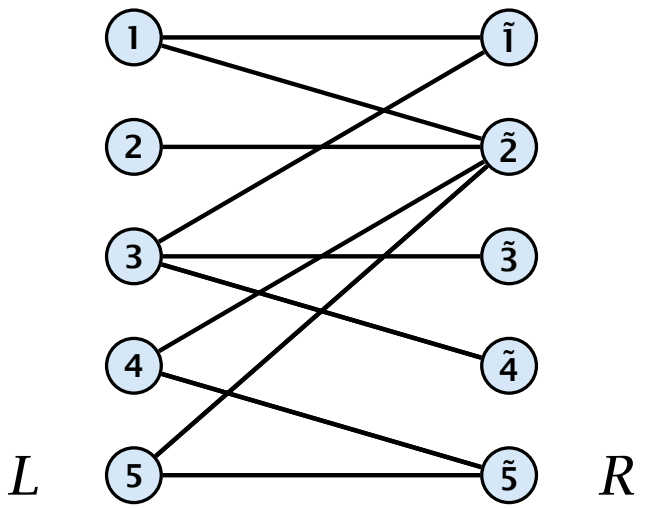

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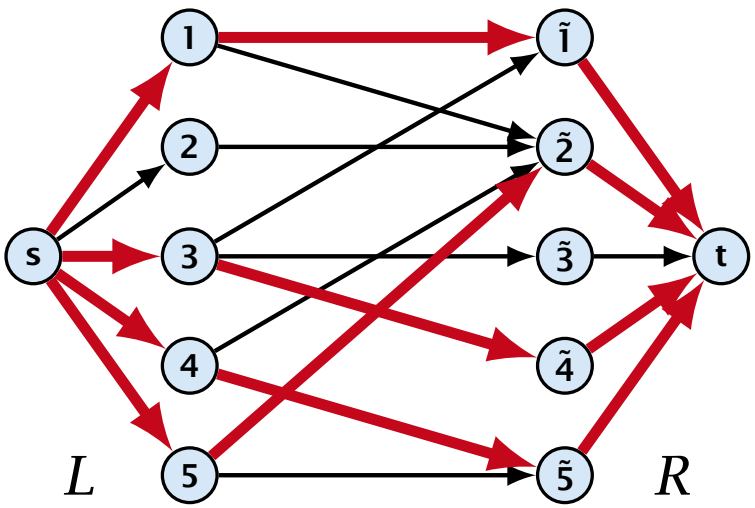

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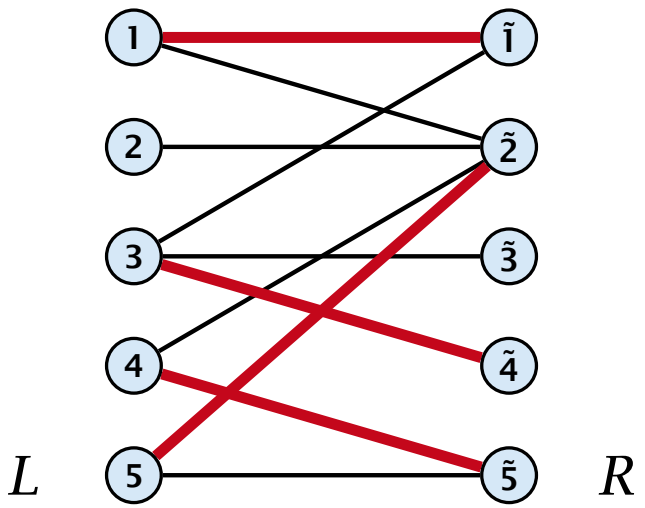


G

13.1 Matching

Which flow algorithm to use?

- Generic augmenting path: $\mathcal{O}(m \operatorname{val}(f^*)) = \mathcal{O}(mn)$.
- Capacity scaling: $\mathcal{O}(m^2 \log C) = \mathcal{O}(m^2)$.

Baseball Elimination

team i	*wins* w_i	*losses* ℓ_i	*remaining games*			
			Atl	*Phi*	*NY*	*Mon*
Atlanta	83	71	–	1	6	1
Philadelphia	80	79	1	–	0	2
New York	78	78	6	0	–	0
Montreal	77	82	1	2	0	–

Which team can end the season with most wins?

- Montreal is eliminated, since even after winning all remaining games there are only 80 wins.
- But also Philadelphia is eliminated. Why?

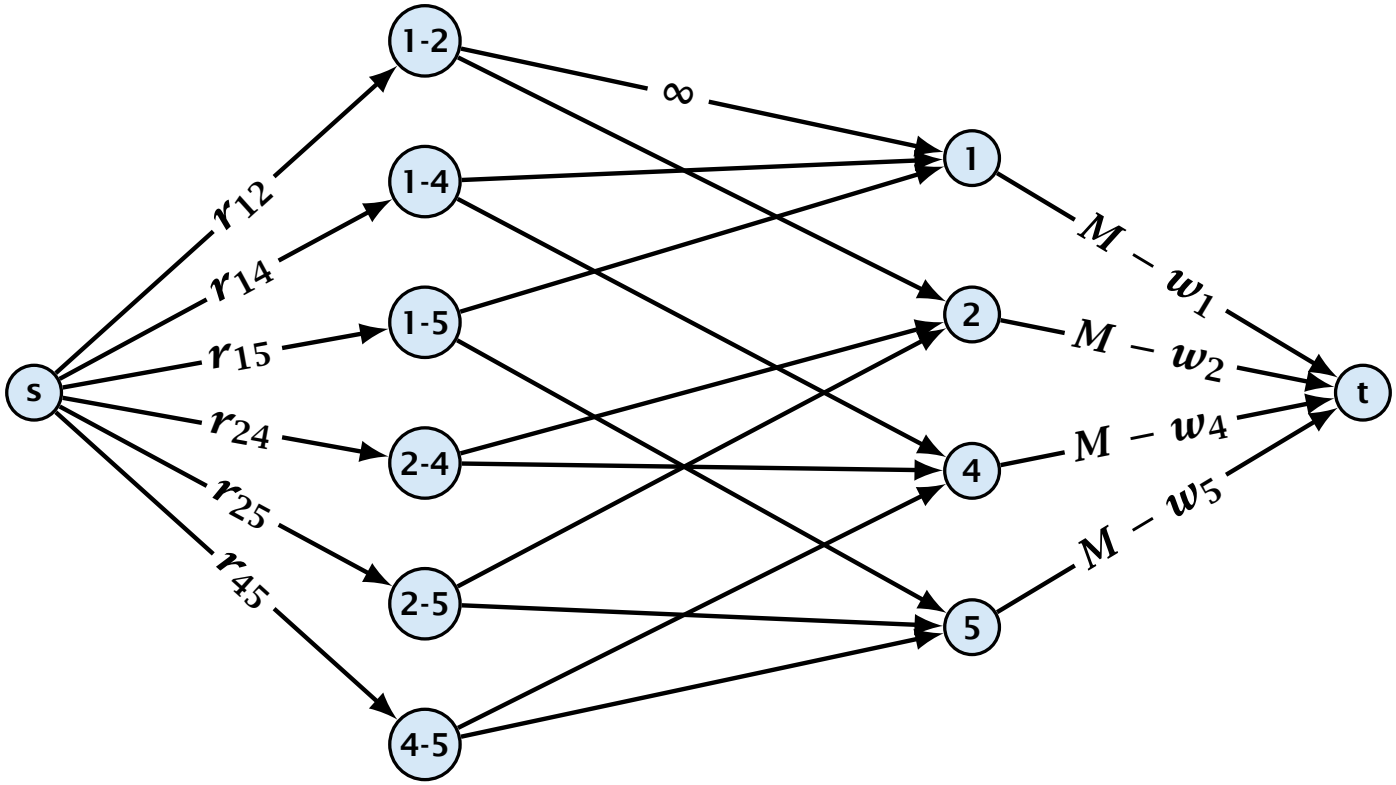
Baseball Elimination

Formal definition of the problem:

- Given a set S of teams, and one specific team $z \in S$.
- Team x has already won w_x games.
- Team x still has to play team y, r_{xy} times.
- Does team z still have a chance to finish with the most number of wins.

Baseball Elimination

Flow network for $z = 3$. M is number of wins Team 3 can still obtain.



Idea. Distribute the results of remaining games in such a way that no team gets too many wins.

Certificate of Elimination

Let $T \subseteq S$ be a subset of teams. Define

$$w(T) := \sum_{i \in T} w_i, \qquad r(T) := \sum_{i,j \in T, i<j} r_{ij}$$

$w(T)$: wins of teams in T

$r(T)$: remaining games among teams in T

If $\frac{w(T)+r(T)}{|T|} > M$ then one of the teams in T will have more than M wins in the end. A team that can win at most M games is therefore eliminated.

Theorem 66

A team z is eliminated if and only if the flow network for z does not allow a flow of value $\sum_{ij \in S \setminus \{z\}, i<j} r_{ij}$.

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$$r(S \setminus \{z\})$$

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- This gives $M < (w(T) + r(T))/|T|$, i.e., z is eliminated.

Baseball Elimination

Proof (⇒)

- Suppose we have a flow that saturates all source edges.
- We can assume that this flow is integral.
- For every pairing x-y it defines how many games team x and team y should win.
- The flow leaving the team-node x can be interpreted as the additional number of wins that team x will obtain.
- This is less than $M - w_x$ because of capacity constraints.
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- We can assume that this flow is integral.
- For every pairing x-y it defines how many games team x and team y should win.
- The flow leaving the team-node x can be interpreted as the additional number of wins that team x will obtain.
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Project Selection

Project selection problem:

- Set P of possible projects. Project v has an associated profit p_v (can be positive or negative).
- Some projects have requirements (taking course EA2 requires course EA1).
- Dependencies are modelled in a graph. Edge (u, v) means "can't do project u without also doing project v."
- A subset A of projects is feasible if the prerequisites of every project in A also belong to A.

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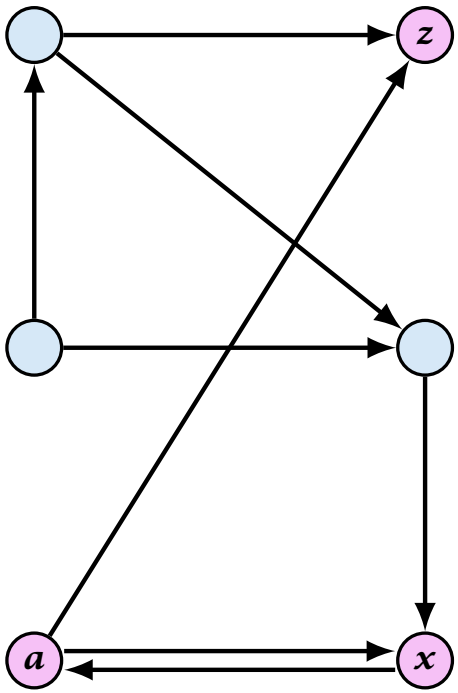
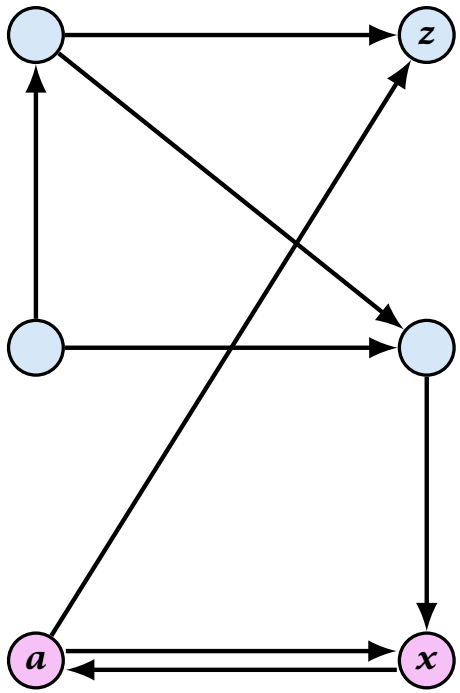
Project Selection

The prerequisite graph:

- $\{x, a, z\}$ is a feasible subset.
- $\{x, a\}$ is infeasible.

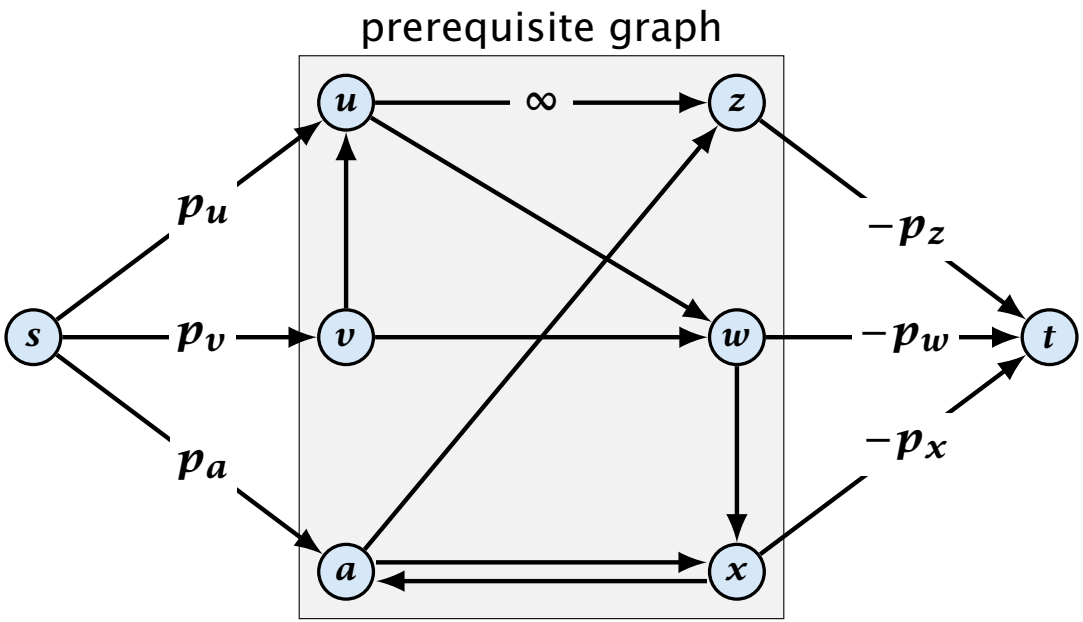




Project Selection

Mincut formulation:

- Edges in the prerequisite graph get infinite capacity.
- Add edge (s, v) with capacity p_v for nodes v with positive profit.
- Create edge (v, t) with capacity $-p_v$ for nodes v with negative profit.



Theorem 67

A is a mincut if $A \setminus \{s\}$ is the optimal set of projects.

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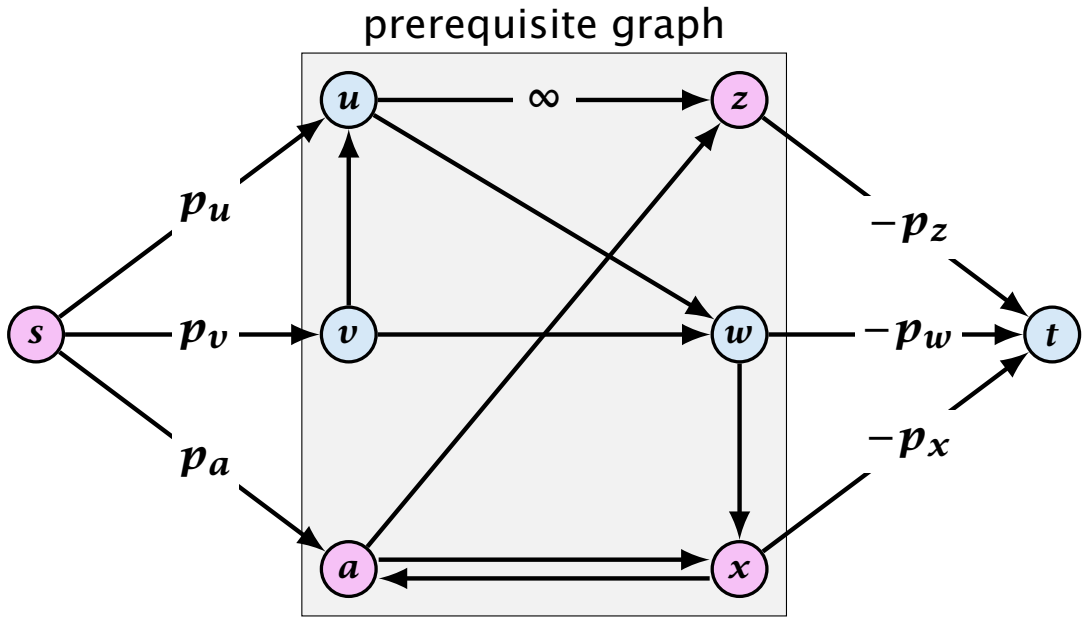
- A is feasible because of capacity infinity edges.



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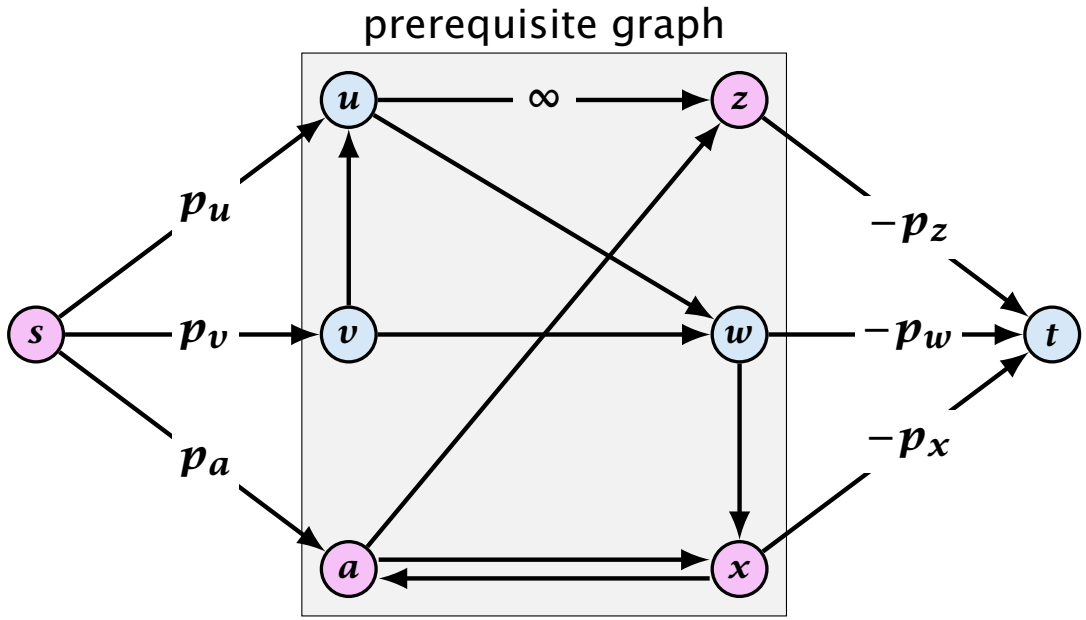

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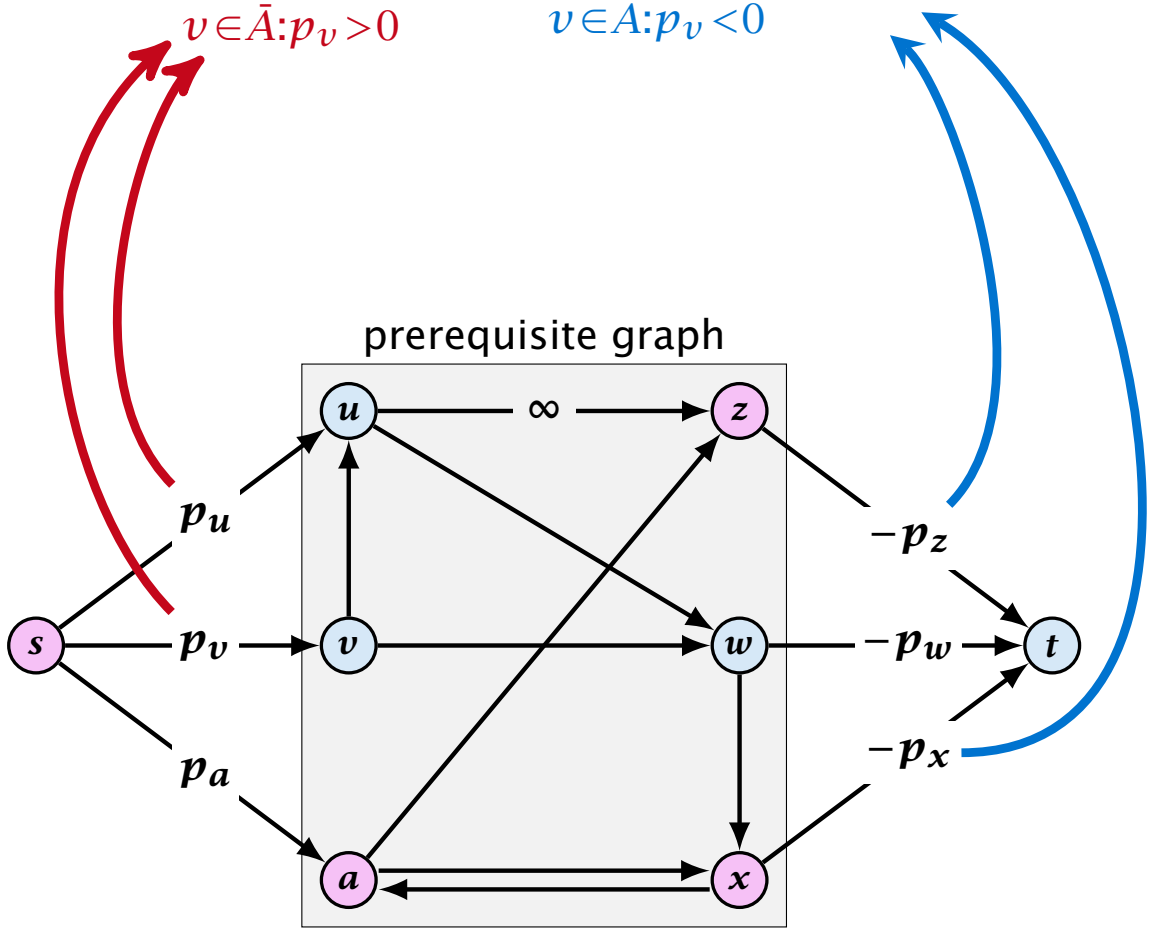

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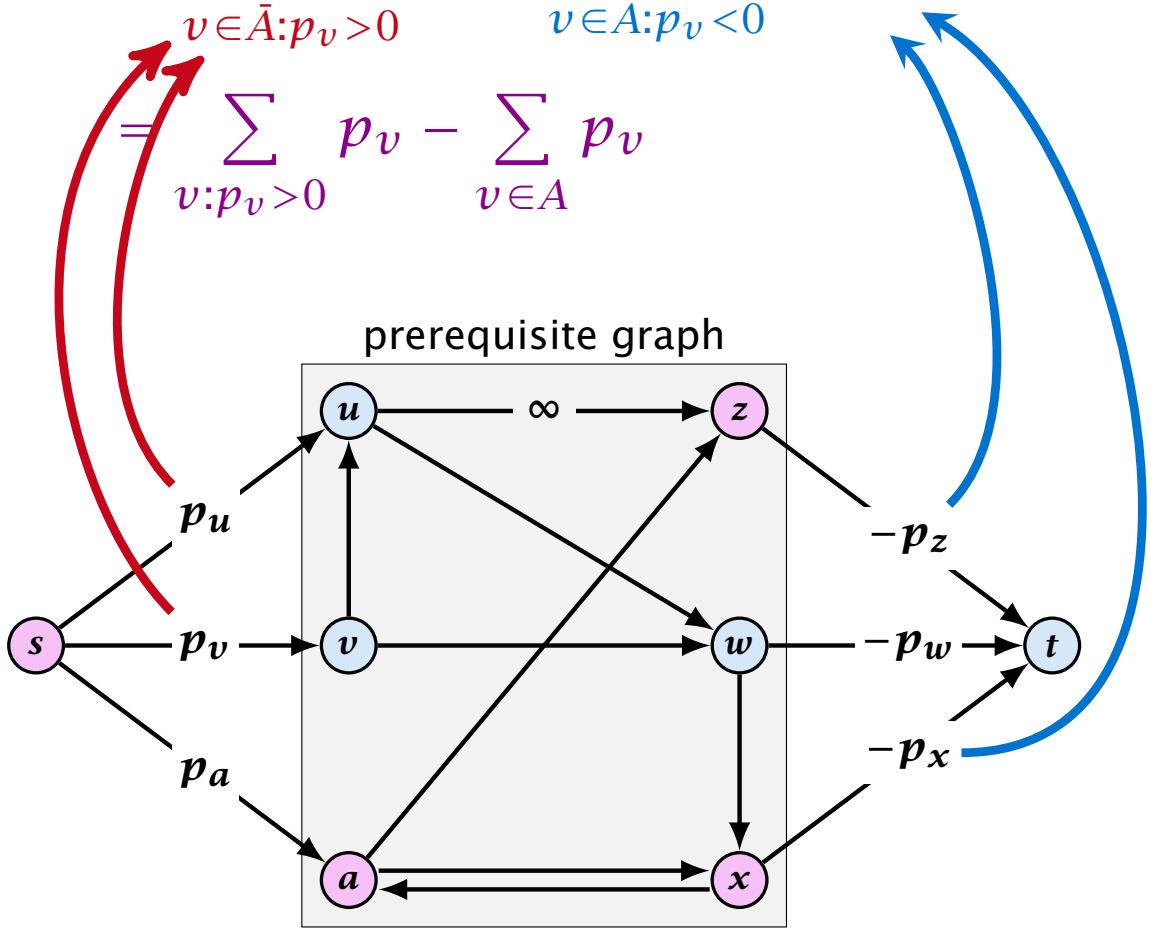
$$= \sum_{v: p_v > 0} p_v - \sum_{v \in A} p_v$$



Preflows

Definition 68

An (s,t)-preflow is a function $f : E \mapsto \mathbb{R}^+$ that satisfies

- For each edge
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1. For each edge e

$$0 \leq f(e) \leq c(e) \ .$$

(capacity constraints)

2. For each $v \in V \setminus \{s,t\}$

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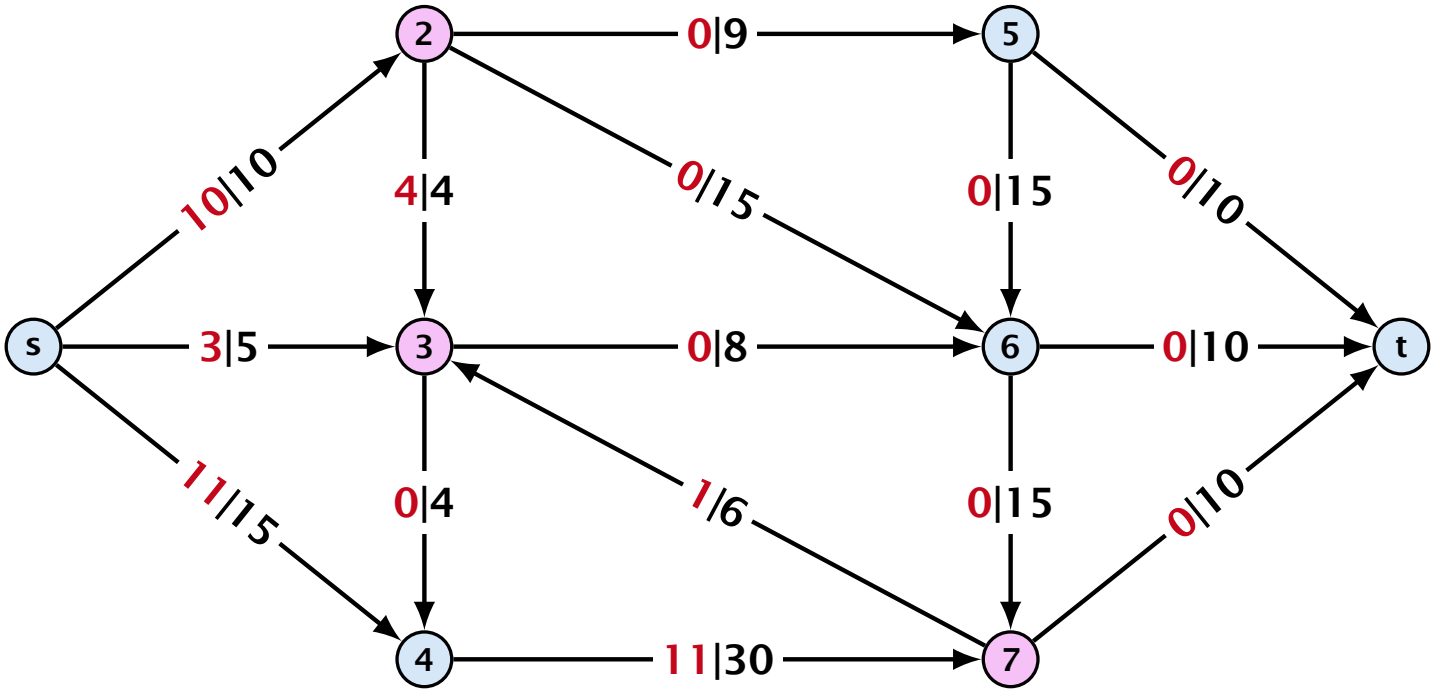
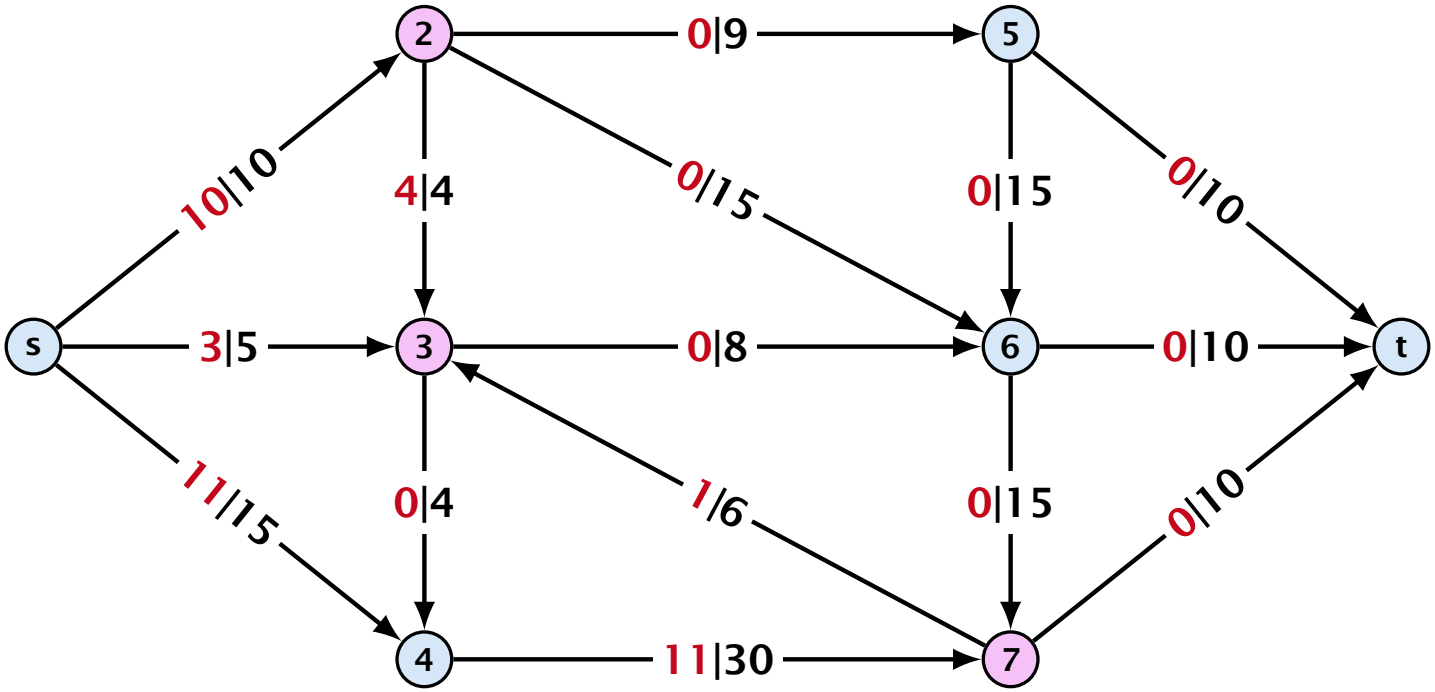


Preflows

Example 69



A node that has $\sum_{e \in \text{out}(v)} f(e) < \sum_{e \in \text{into}(v)} f(e)$ is called an active node.

Preflows

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Definition:

A labelling is a function $\ell : V \to \mathbb{N}$. It is valid for preflow f if

- $\ell(u) \leq \ell(v) + 1$ for all edges in the residual graph G_f (only non-zero capacity edges!!!)

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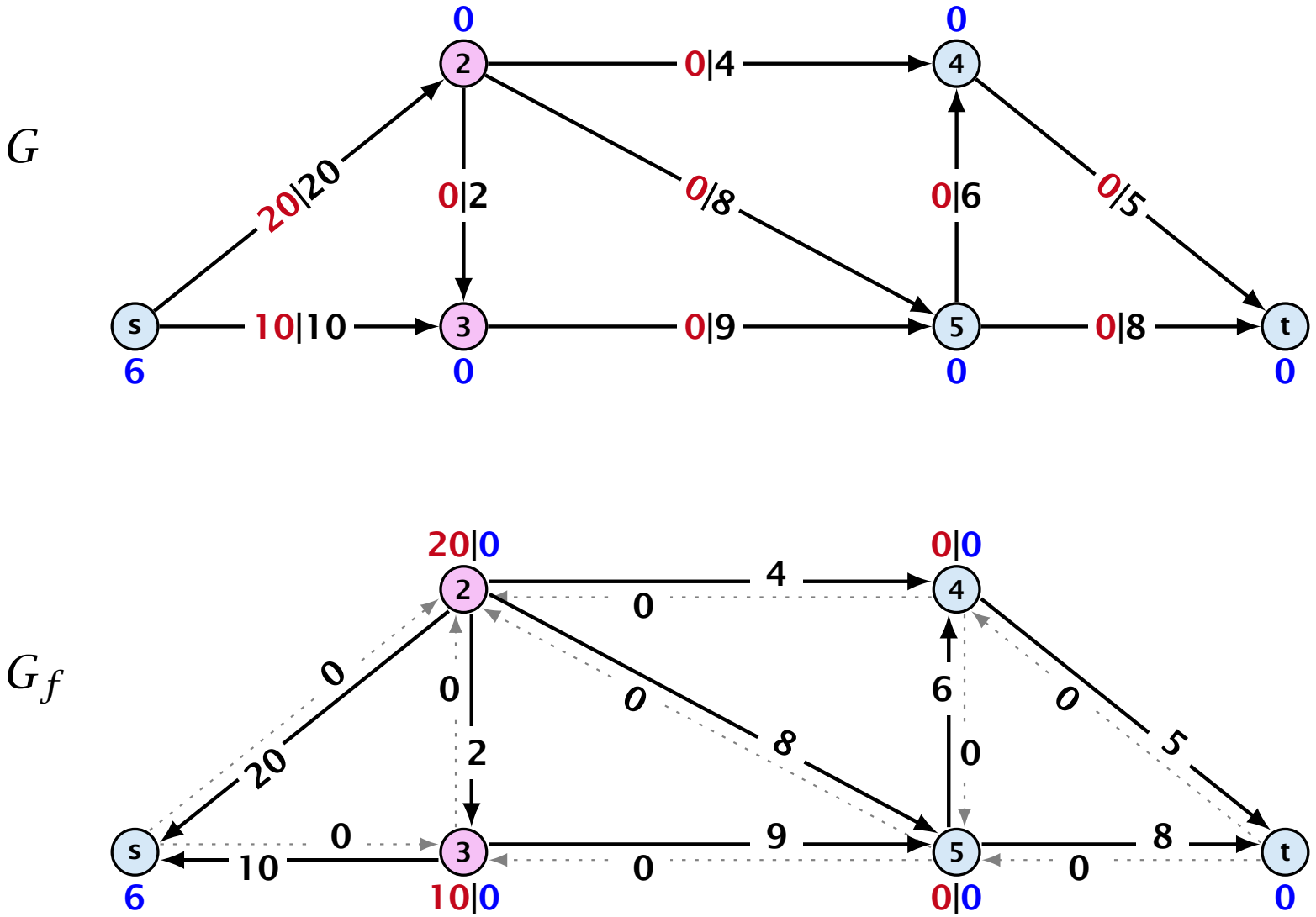
Intuition:

The labelling can be viewed as a height function. Whenever the height from node u to node v decreases by more than 1 (i.e., it goes very steep downhill from u to v), the corresponding edge must be saturated.

Preflows


G
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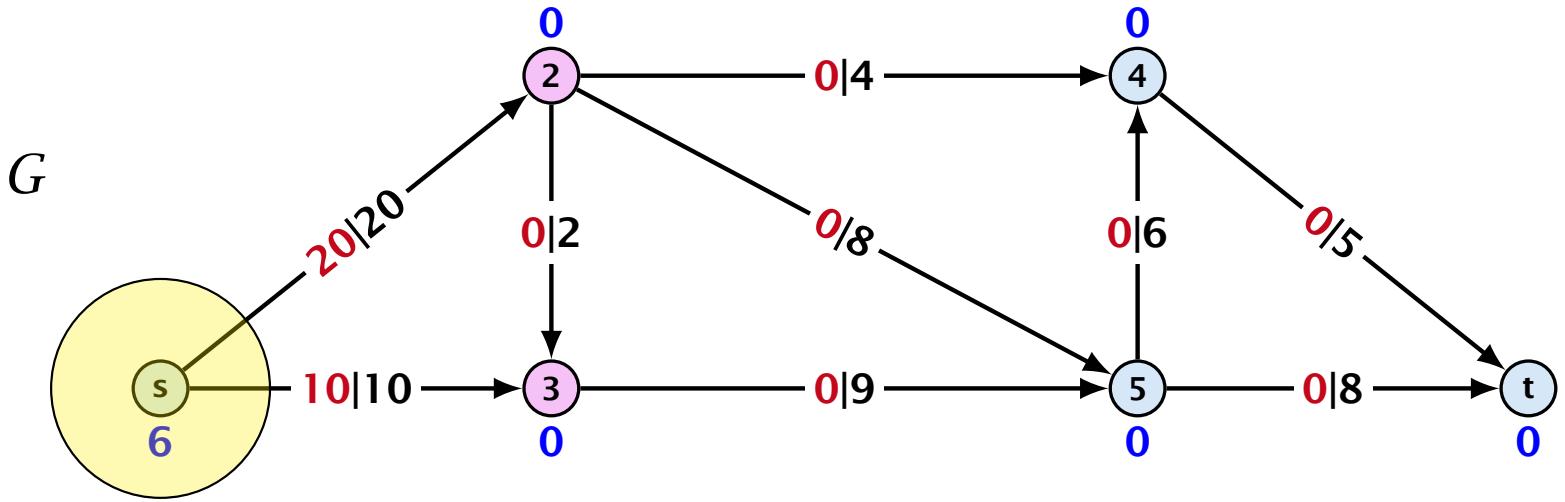
Preflows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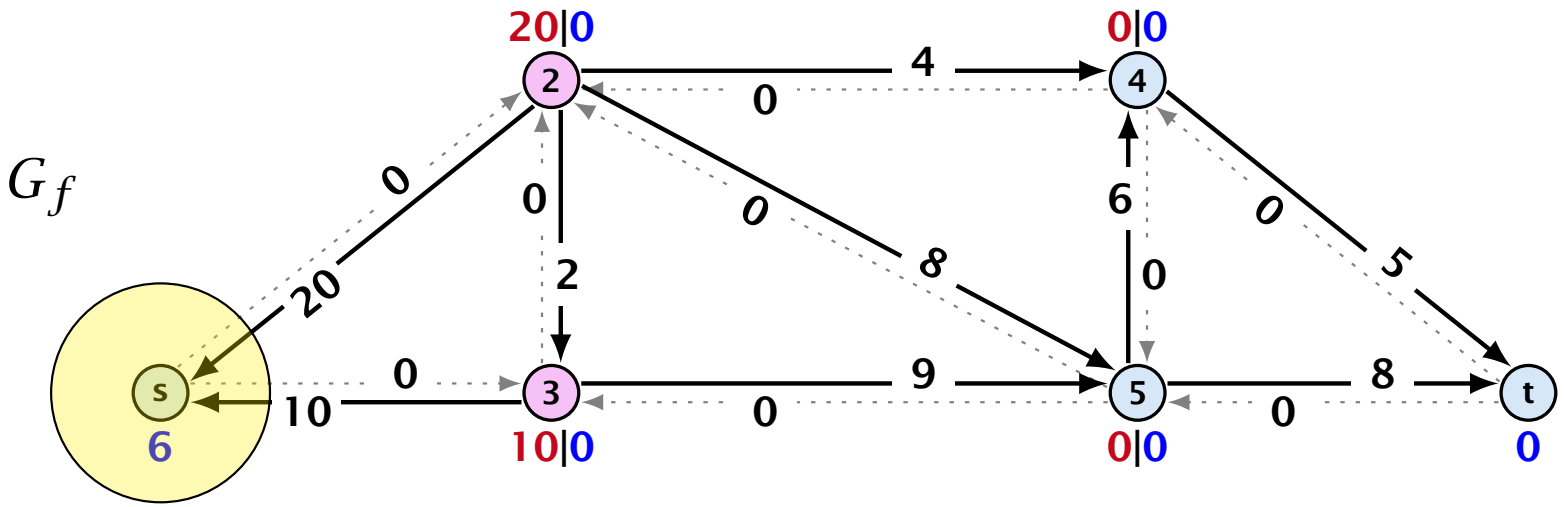
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Preflows

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Lemma 70

A preflow that has a valid labelling saturates a cut.

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Lemma 71

A flow that has a valid labelling is a maximum flow.

Push Relabel Algorithms

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Idea:

- start with some preflow and some valid labelling

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- successively change the preflow while maintaining a valid labelling

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- start with some preflow and some valid labelling
- successively change the preflow while maintaining a valid labelling
- stop when you have a flow (i.e., no more active nodes)

Changing a Preflow

An arc (u, v) with $c_f(u, v) > 0$ in the residual graph is admissable if $\ell(u) = \ell(v) + 1$ (i.e., it goes downwards w.r.t. labelling ℓ).

The push operation

Consider an active node u with excess flow $f(u) = \sum_{e \in \text{into}(u)} f(e) - \sum_{e \in \text{out}(u)} f(e)$ and suppose $e = (u, v)$ is an admissable arc with residual capacity $c_f(e)$.

We can send flow $\min\{c_f(e), f(u)\}$ along e and obtain a new preflow. The old labelling is still valid (!!!).

- the arc e is deleted from the residual graph
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- saturating push: $\min\{f(u), c_f(e)\} = c_f(e)$
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- non-saturating push: $\min\{f(u), c_f(e)\} = f(u)$
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Push Relabel Algorithms

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Push Relabel Algorithms

The relabel operation

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Increasing the label of u by 1 results in a valid labelling.

- Edges (w,u) incoming to u still fulfill their constraint $\ell(w) \leq \ell(u)+1$.
- An outgoing edge (u,w) had $\ell(u) < \ell(w)+1$ before since it was not admissable. Now: $\ell(u) \leq \ell(w)+1$.

Push Relabel Algorithms

Intuition:
We want to send flow downwards, since the source has a height/label of n and the target a height/label of 0. If we see an active node u with an admissible arc we push the flow at u towards the other end-point that has a lower height/label. If we do not have an admissible arc but excess flow into u it should roughly mean that the level/height/label of u should rise. (If we consider the flow to be water than this would be natural).

Note that the above intuition is very incorrect as the labels are integral, i.e., they cannot really be seen as the height of a node.

Reminder

- In a preflow nodes may not fulfill conservation constraints; a node may have more incoming flow than outgoing flow.
- Such a node is called active.
- A labelling is valid if for every edge (u,v) in the residual graph $\ell(u) \leq \ell(v)+1$.
- An arc (u,v) in residual graph is admissable if $\ell(u) = \ell(v)+1$.
- A saturation push along e pushes an amount of $c(e)$ flow along the edge, thereby saturating the edge (and making it dissappear from the residual graph).
- A non-saturating push along $e=(u,v)$ pushes a flow of $f(u)$, where $f(u)$ is the excess flow of u. This makes u inactive.

Push Relabel Algorithms

Algorithm 3 maxflow(G, s, t, c)

```
1: find initial preflow f
2: while there is active node u do
3:       if there is admiss. arc e out of u then
4:             push(G, e, f, c)
5:       else
6:             relabel(u)
7: return f
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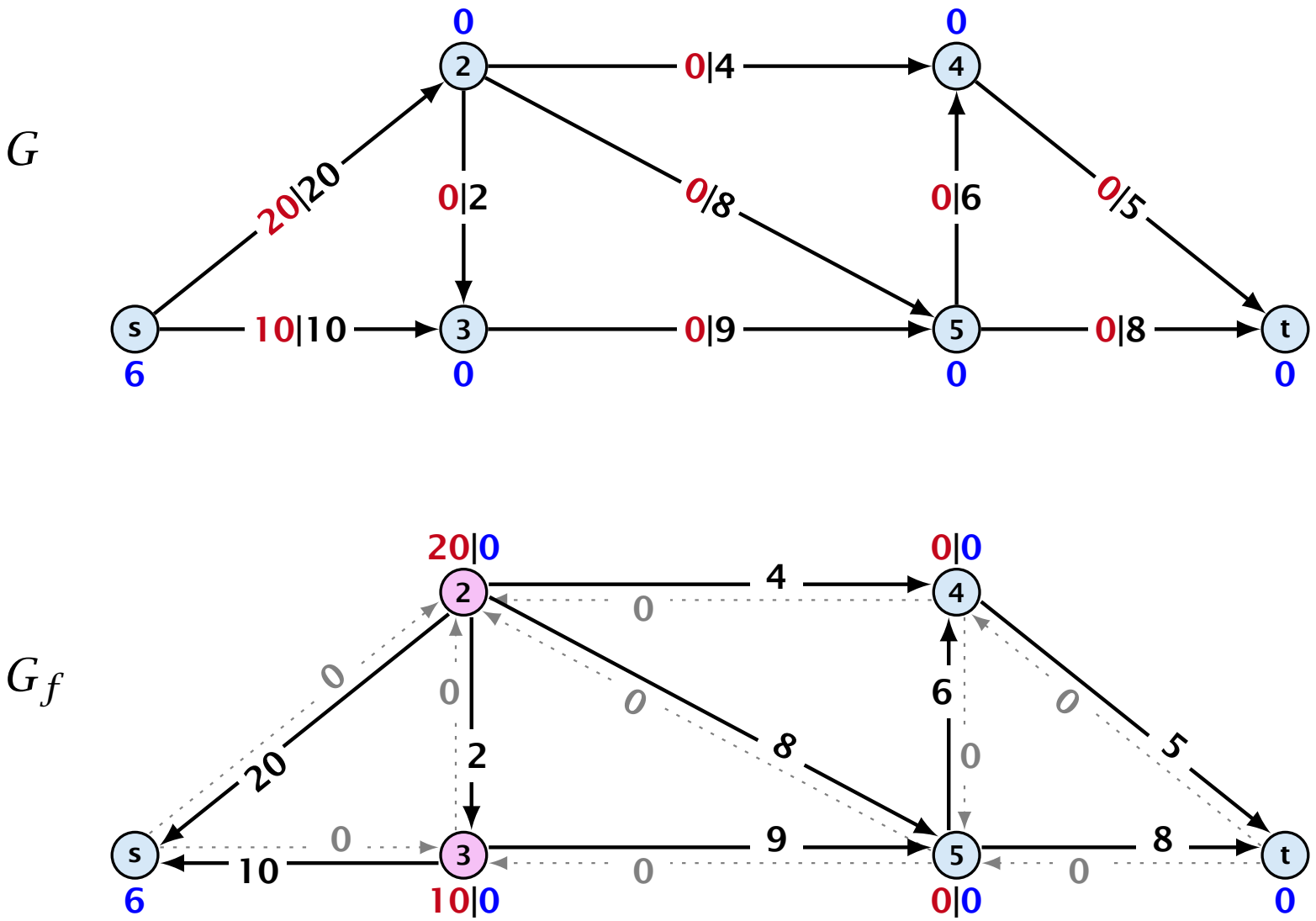
In the following example we always stick to the same active node u until it becomes inactive but this is not required.

Preflow Push Algorithm



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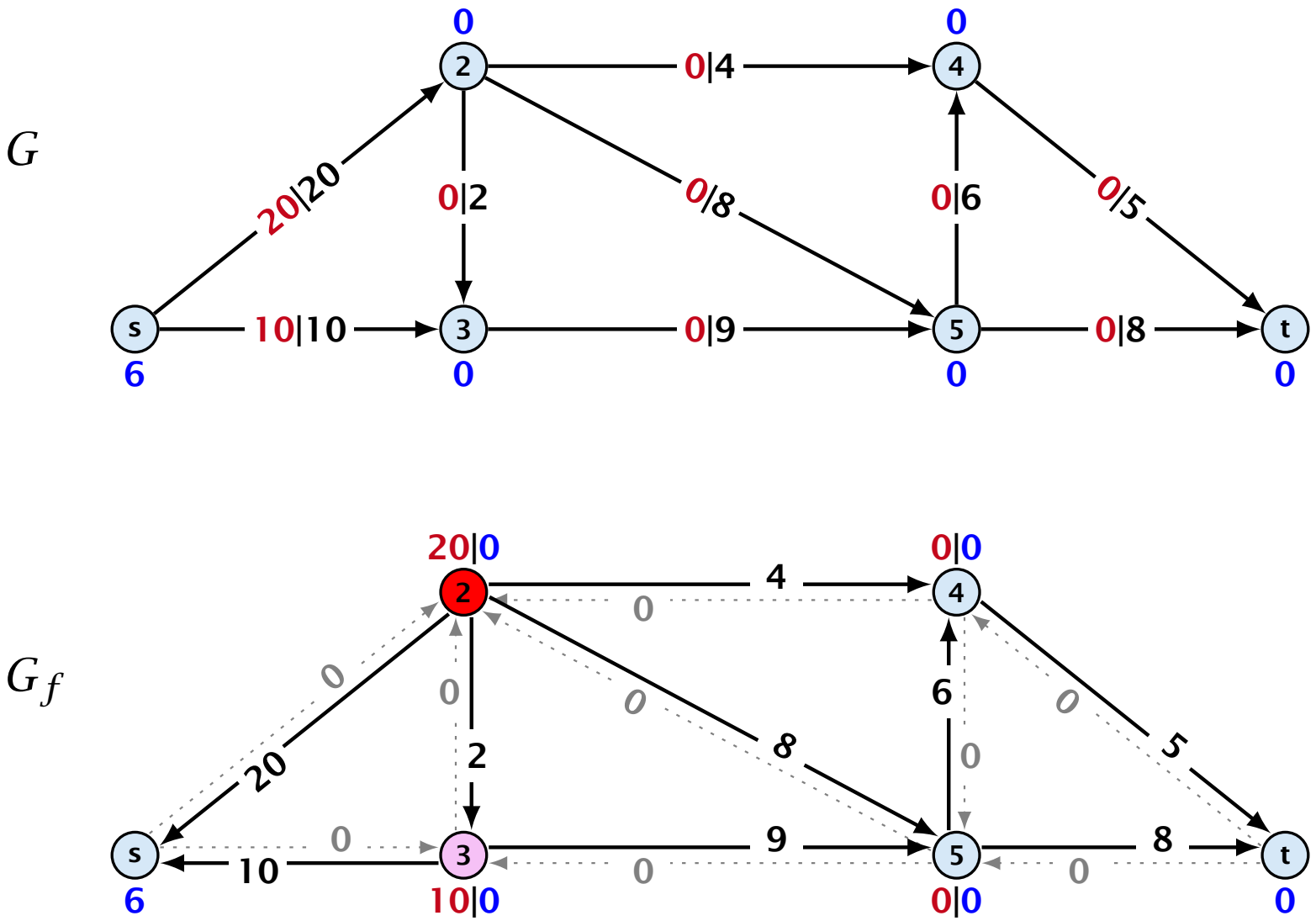
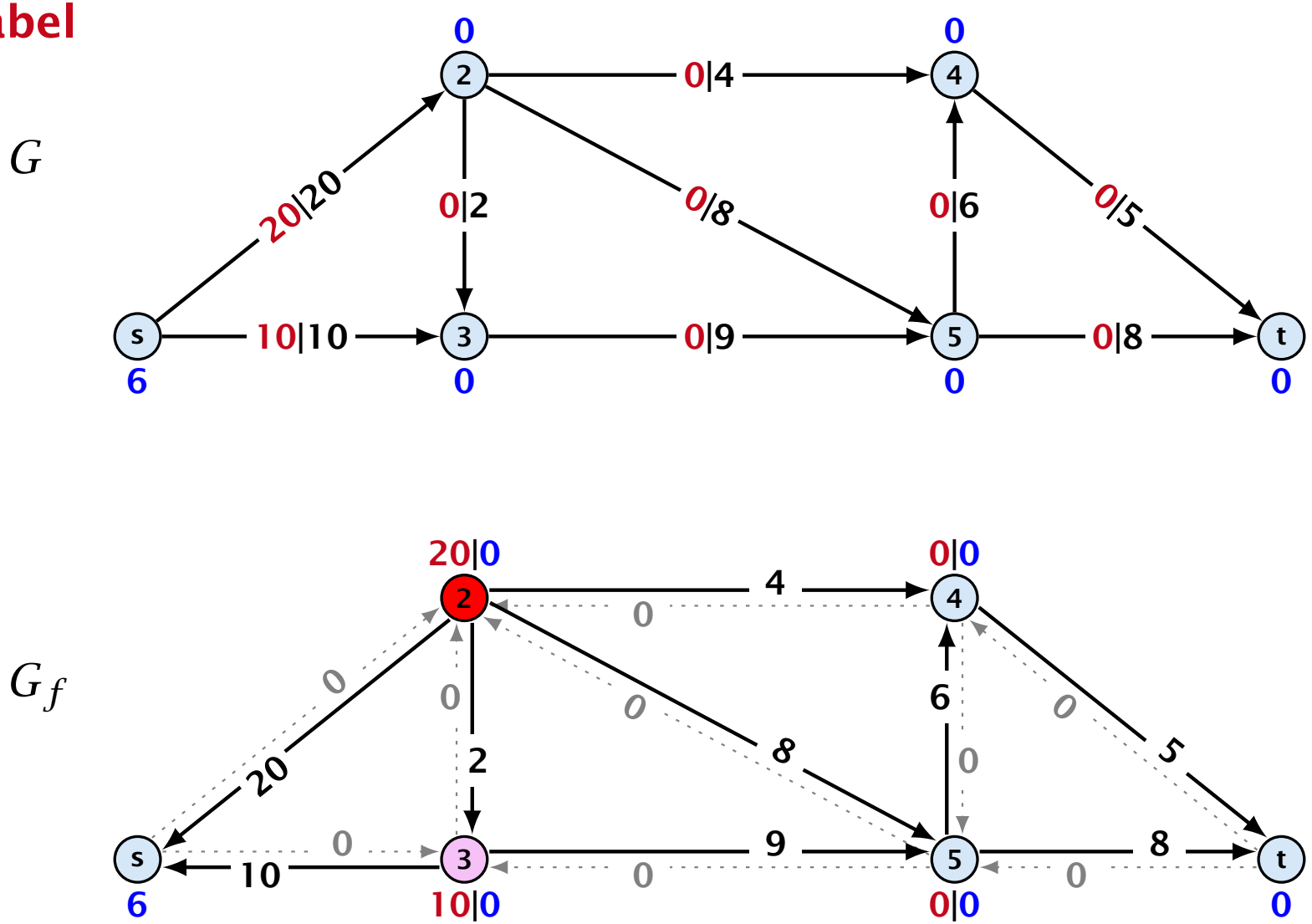
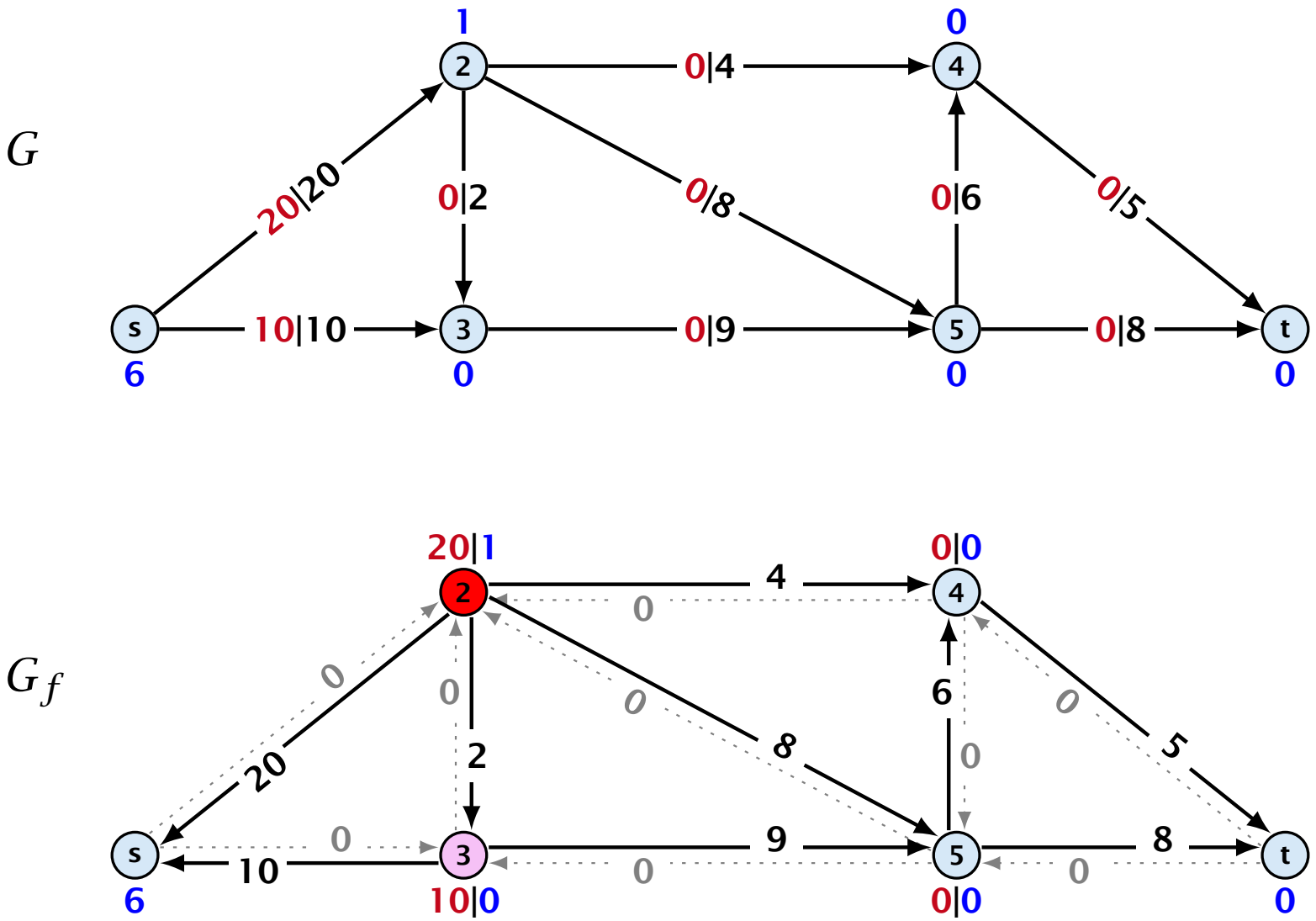


Preflow Push Algorithm

relabel

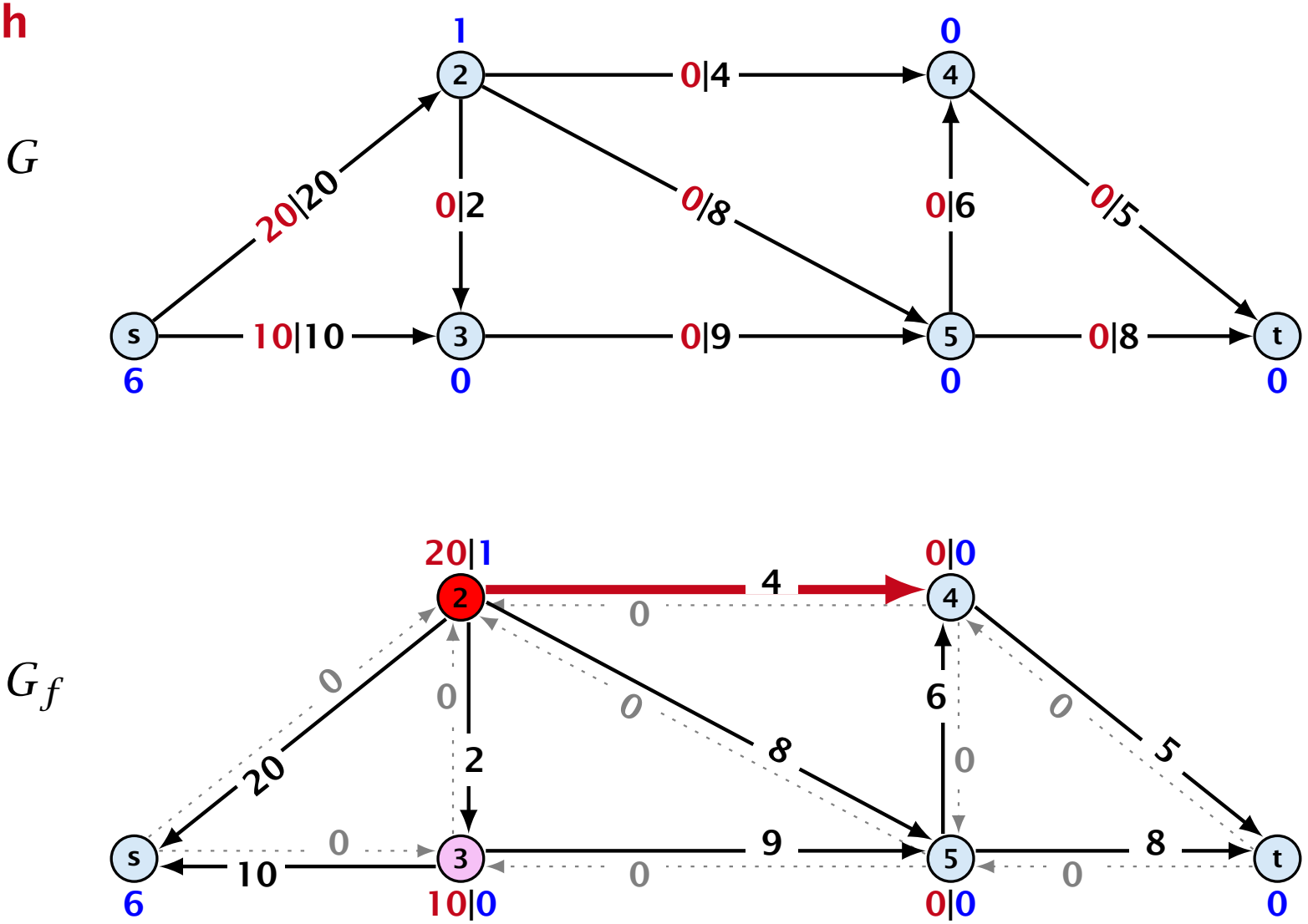


Preflow Push Algorithm

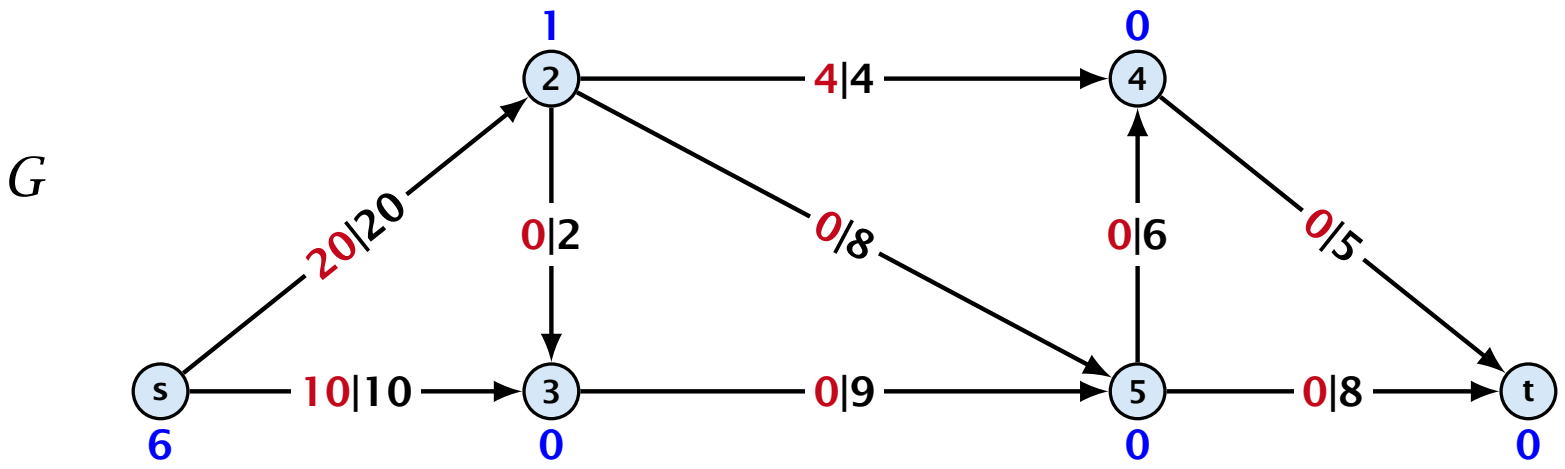


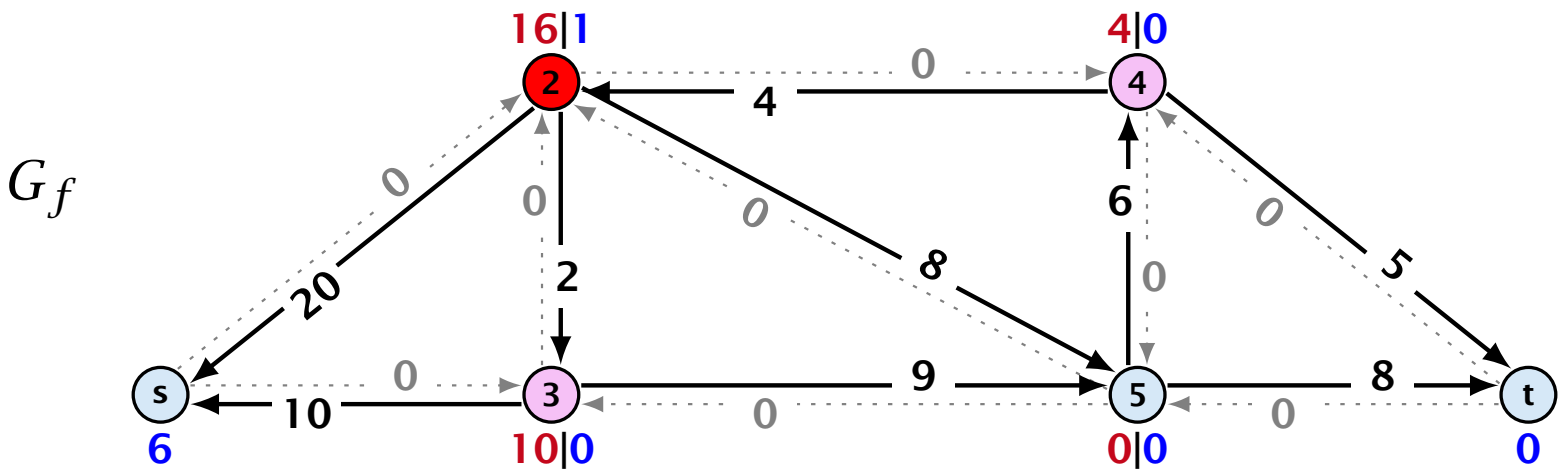
Preflow Push Algorithm

push



Preflow Push Algorithm

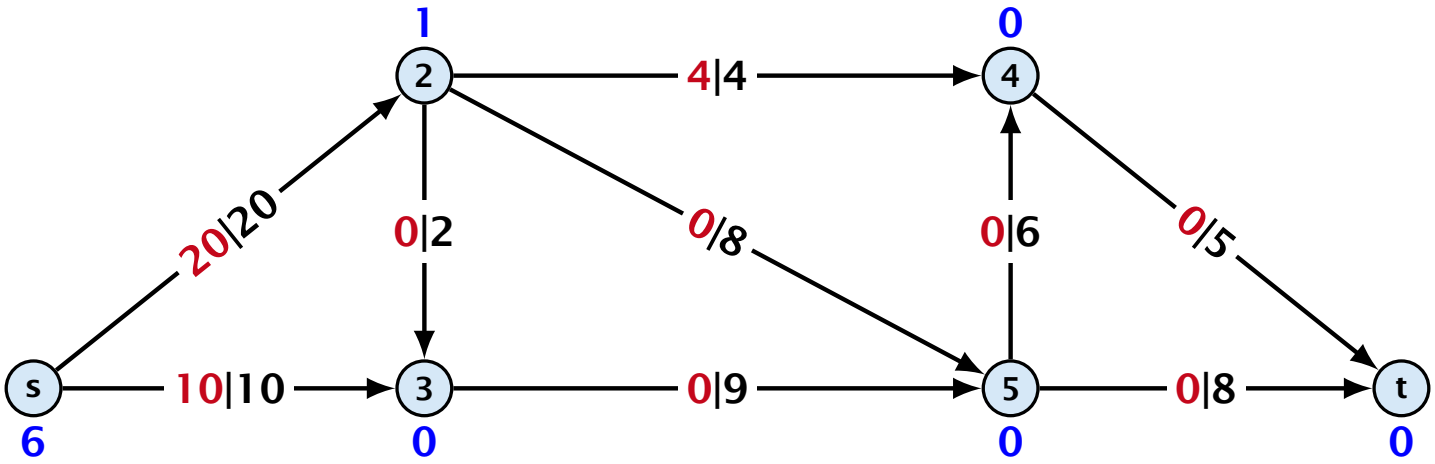




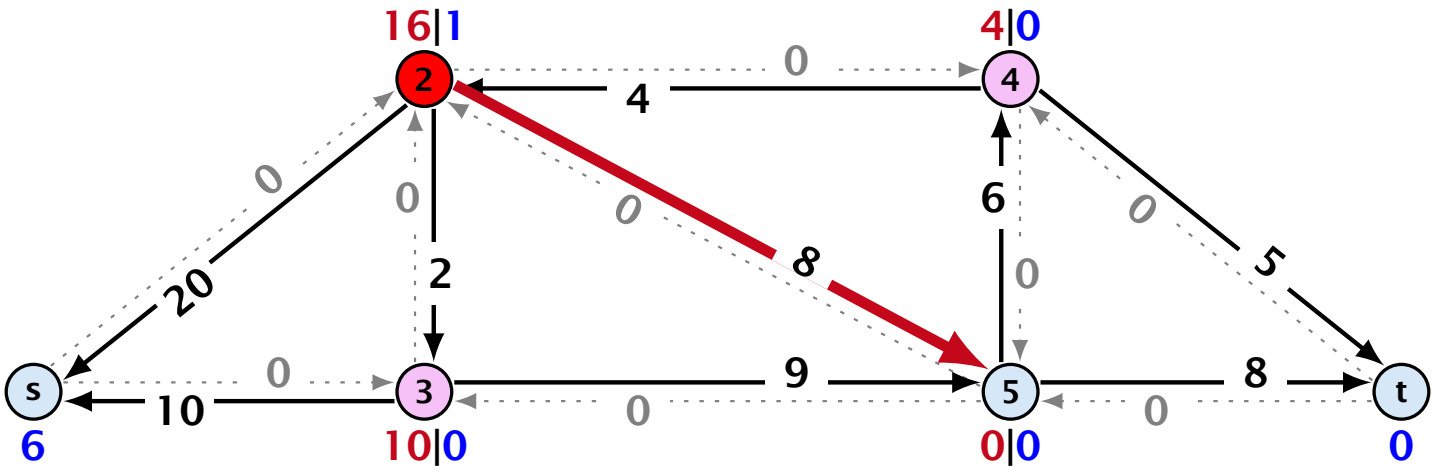
Preflow Push Algorithm

push

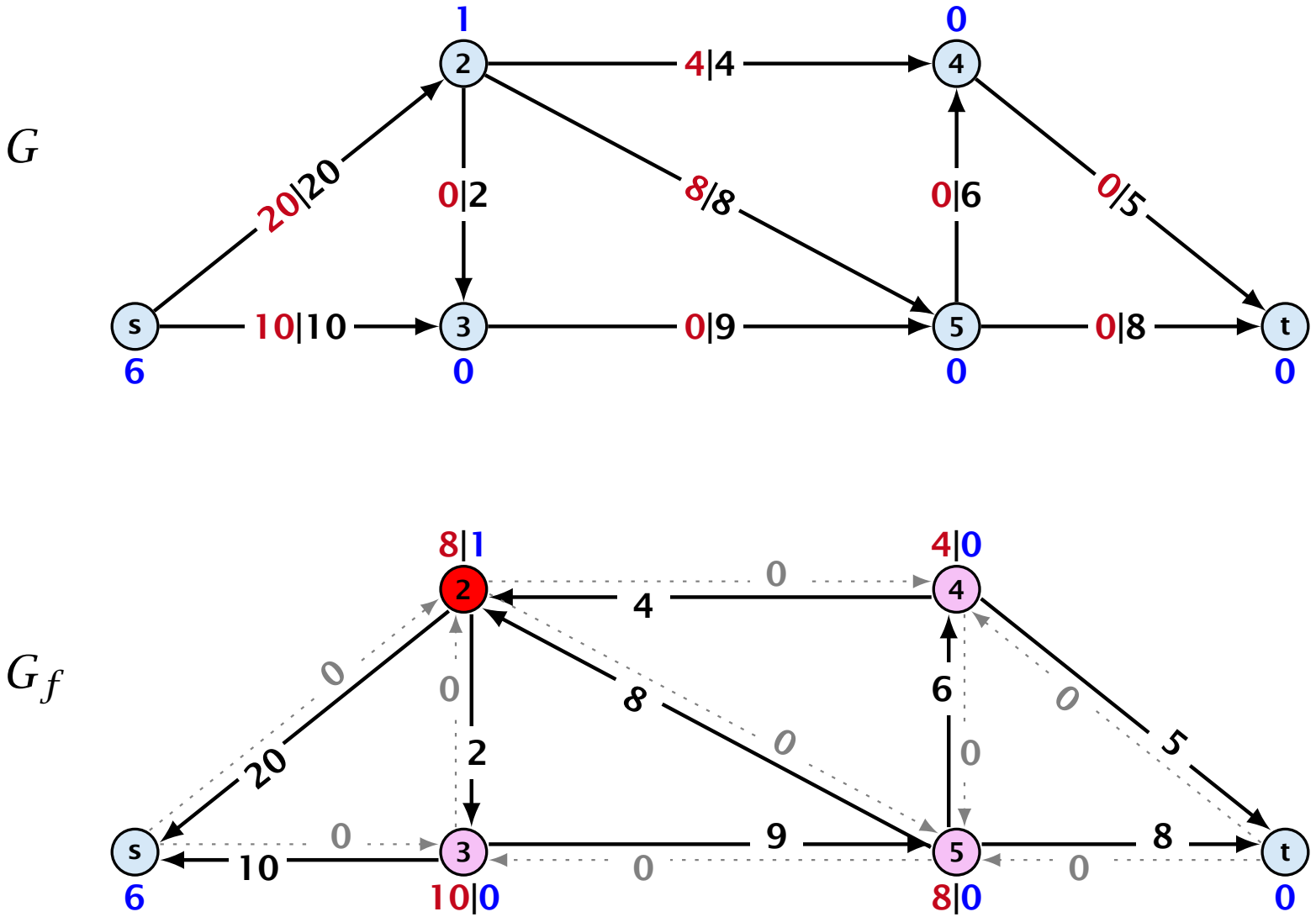
G



G_f



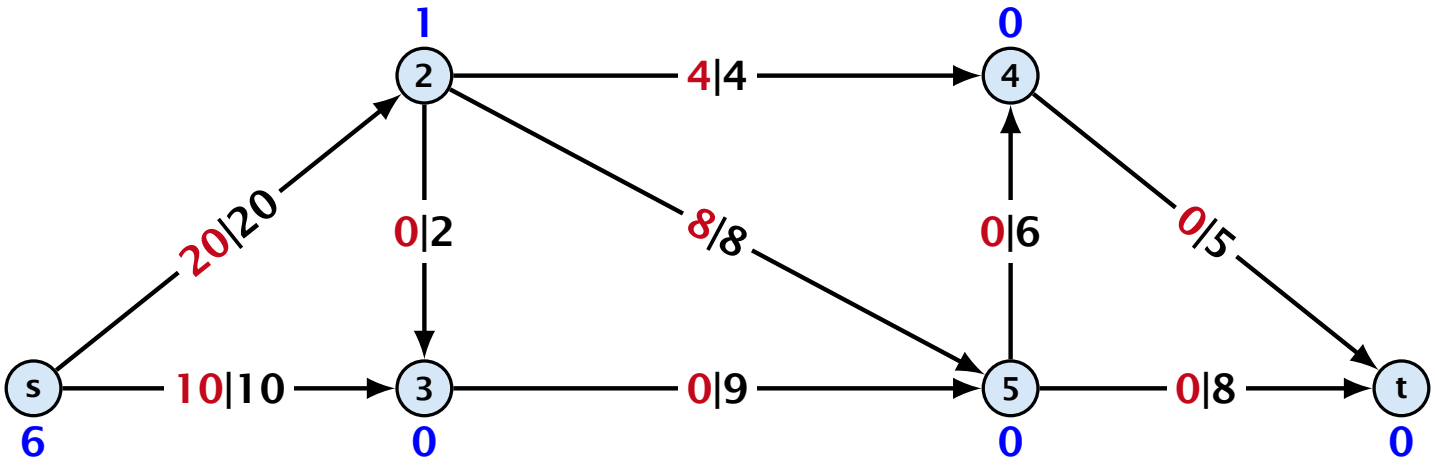
Preflow Push Algorithm



Preflow Push Algorithm

push

G



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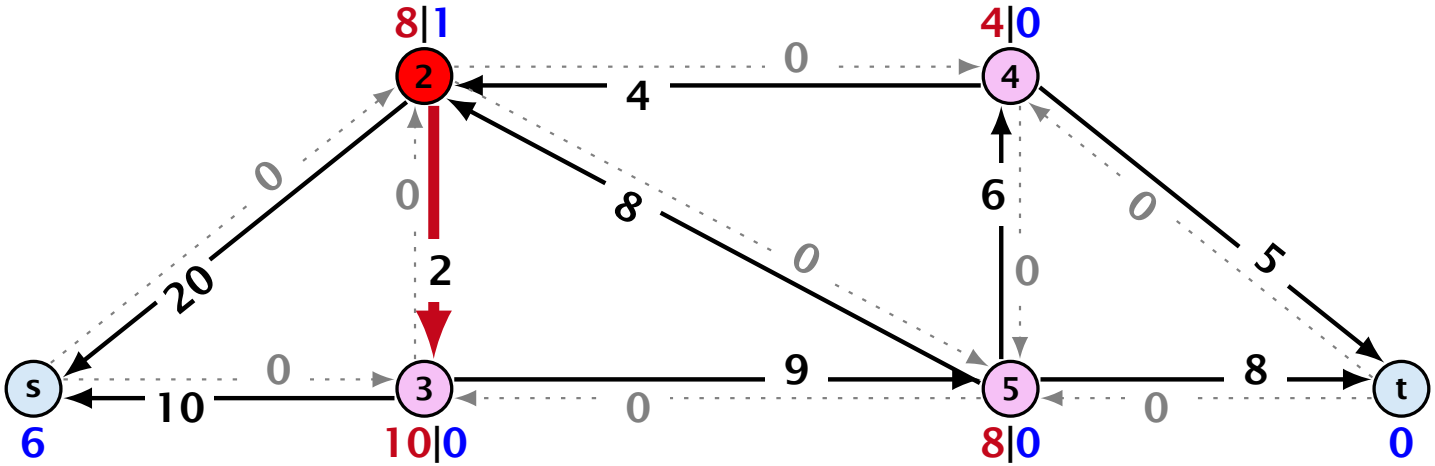
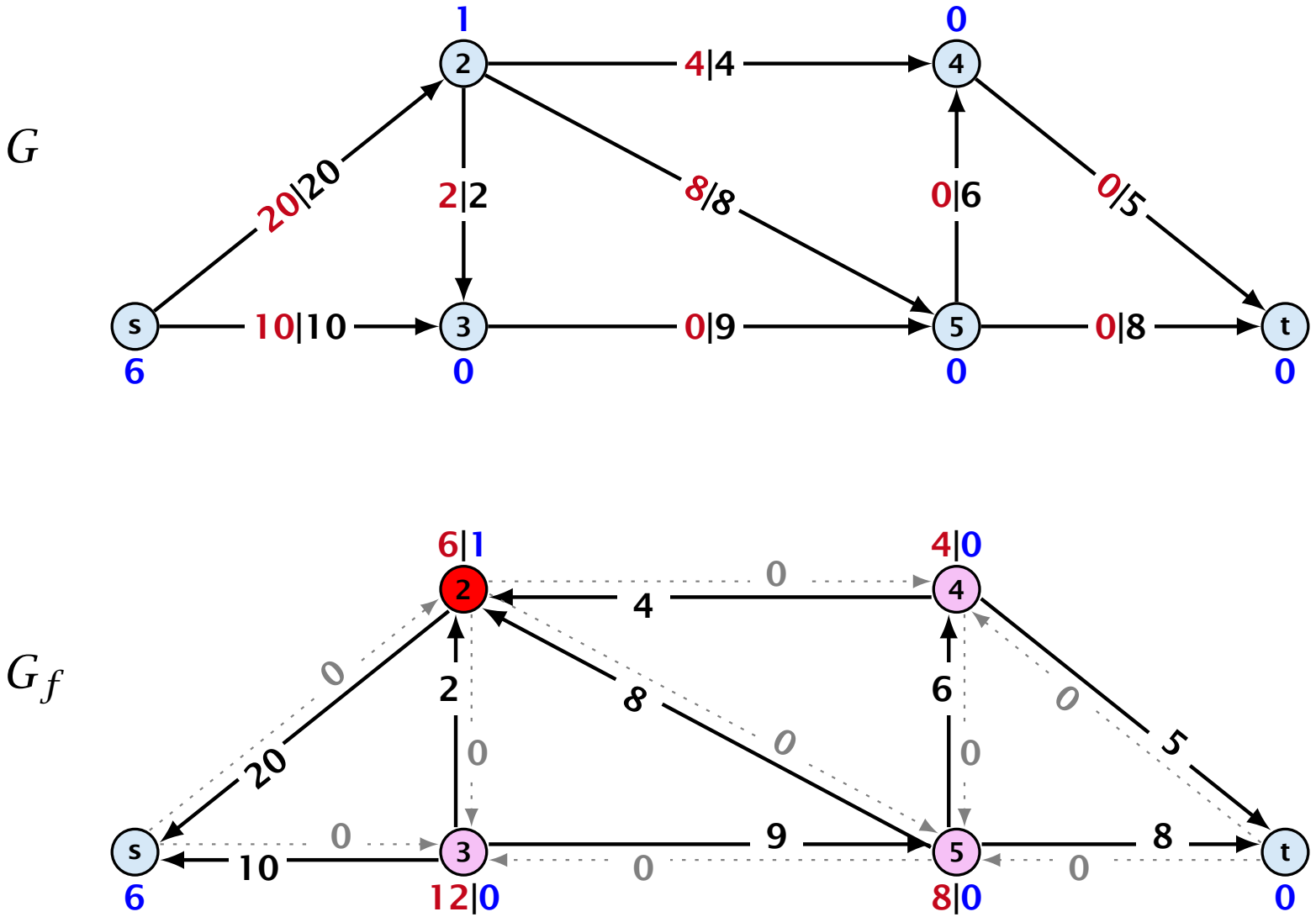


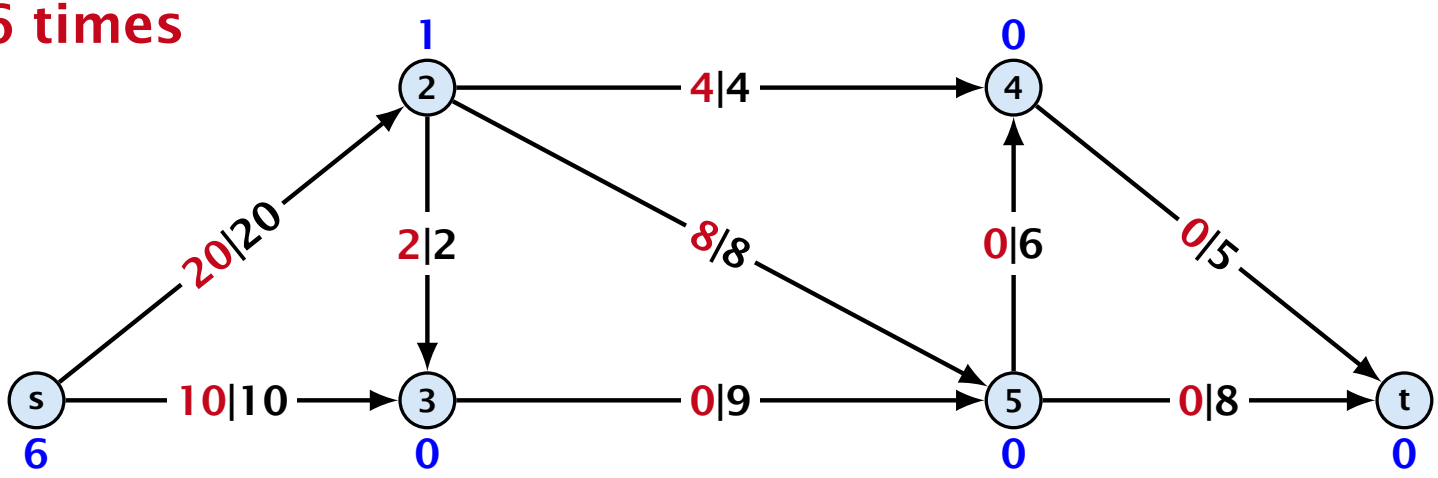
Preflow Push Algorithm



Preflow Push Algorithm

relabel 6 times

G



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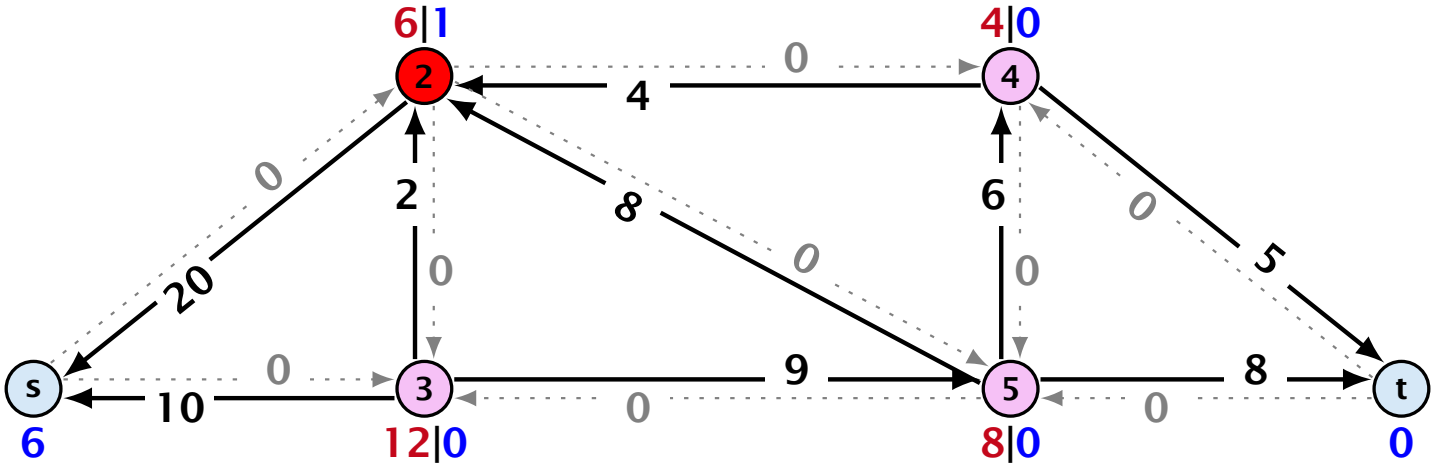
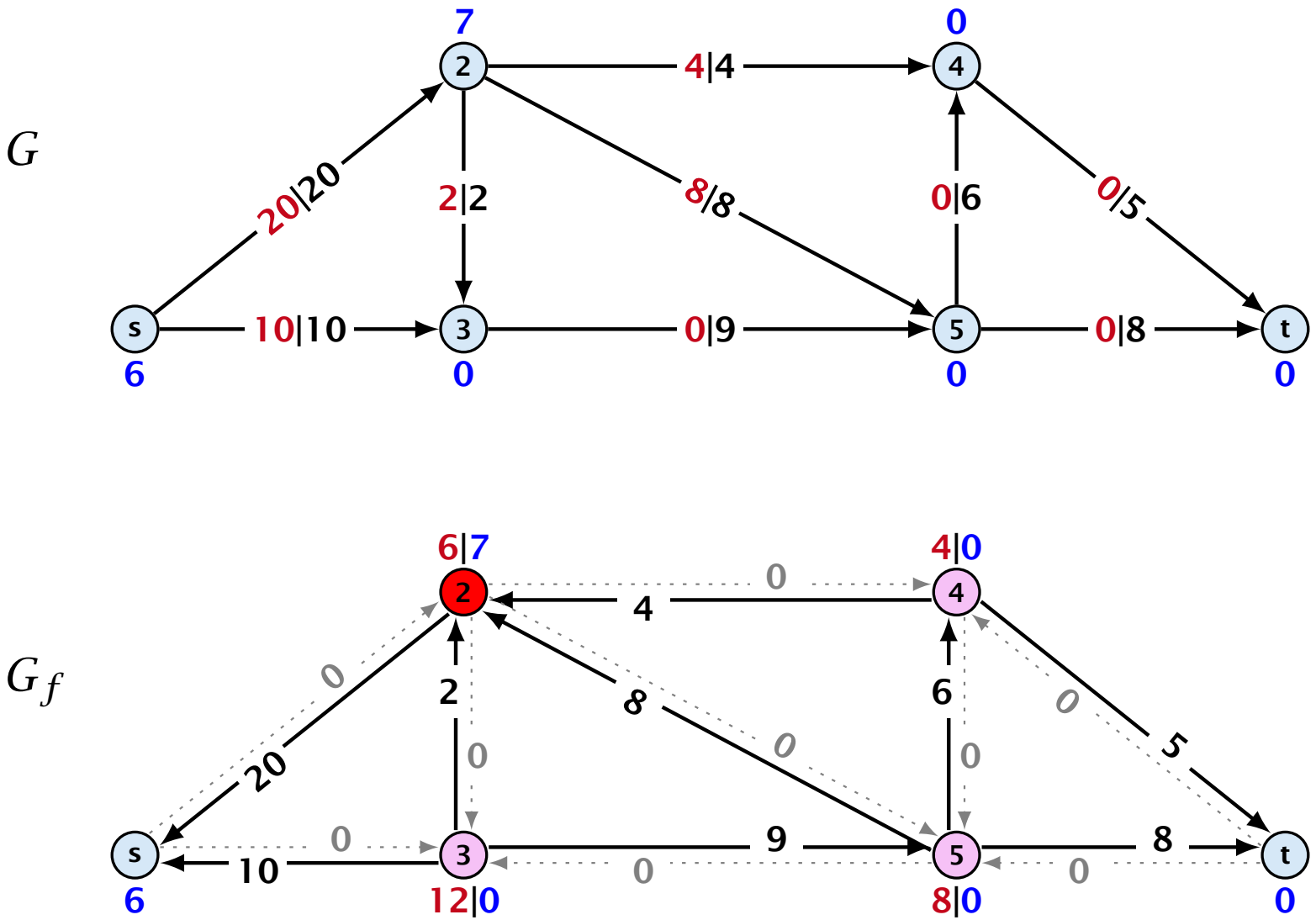
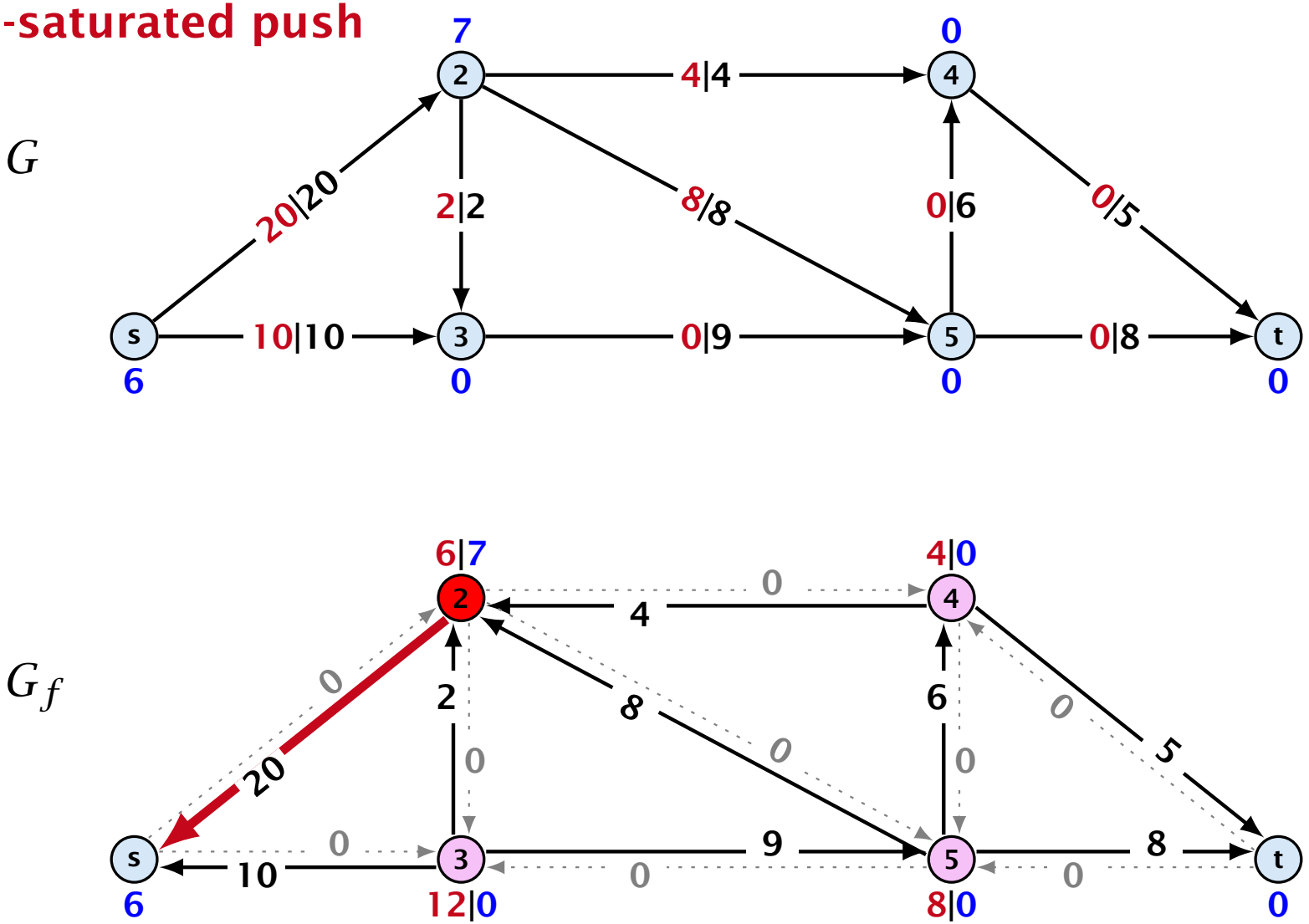


Preflow Push Algorithm

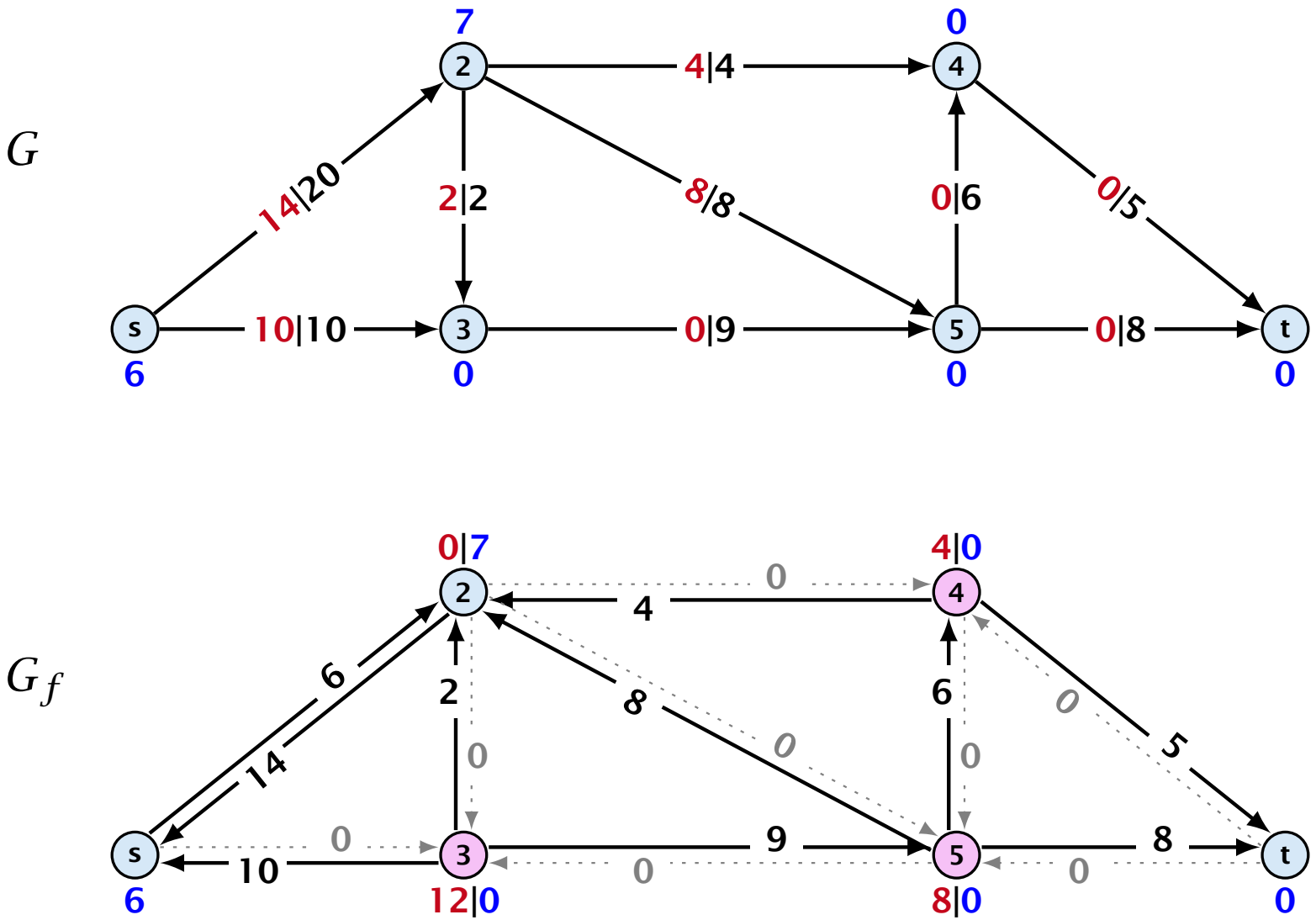


Preflow Push Algorithm

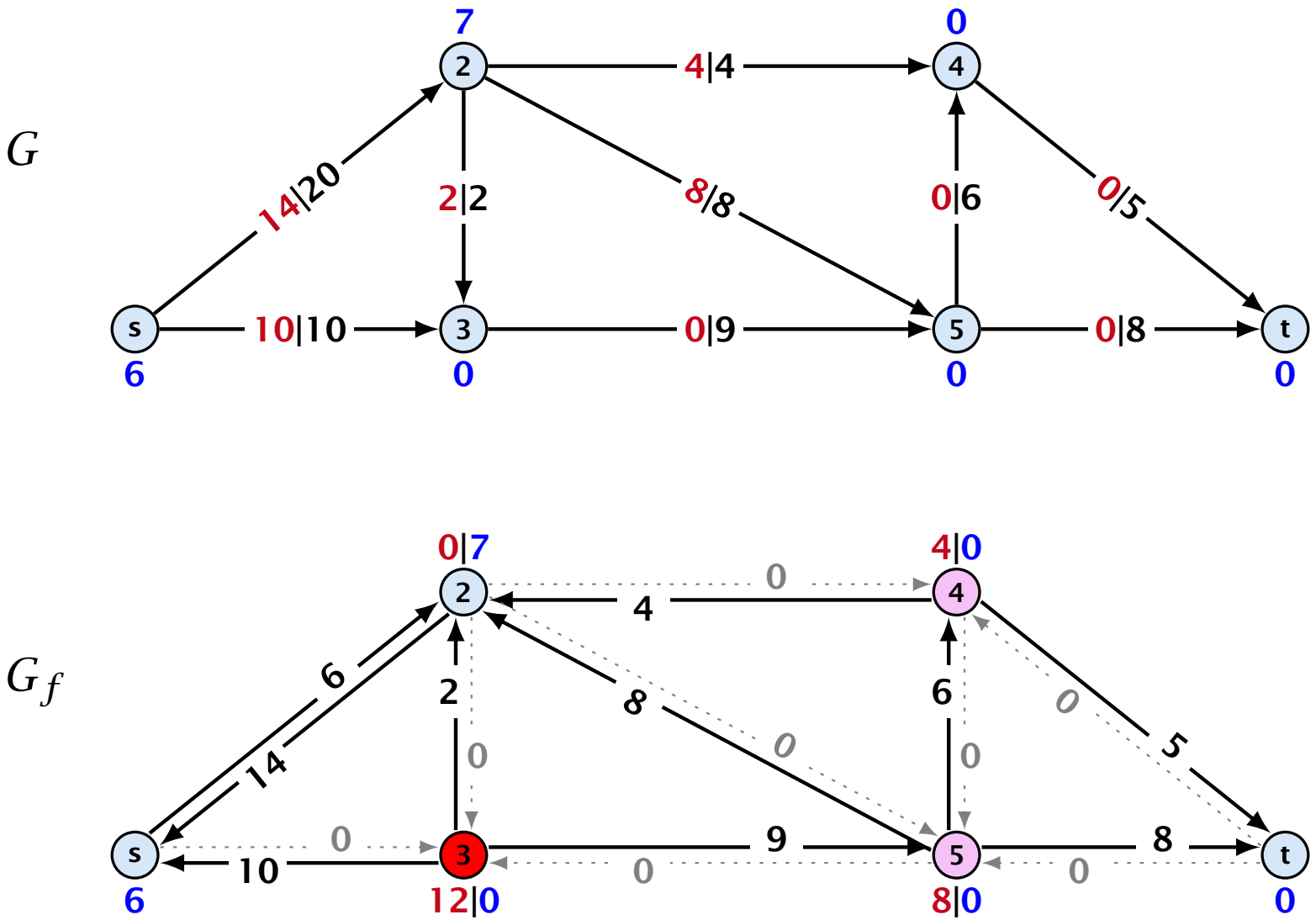
non-saturated push



Preflow Push Algorithm

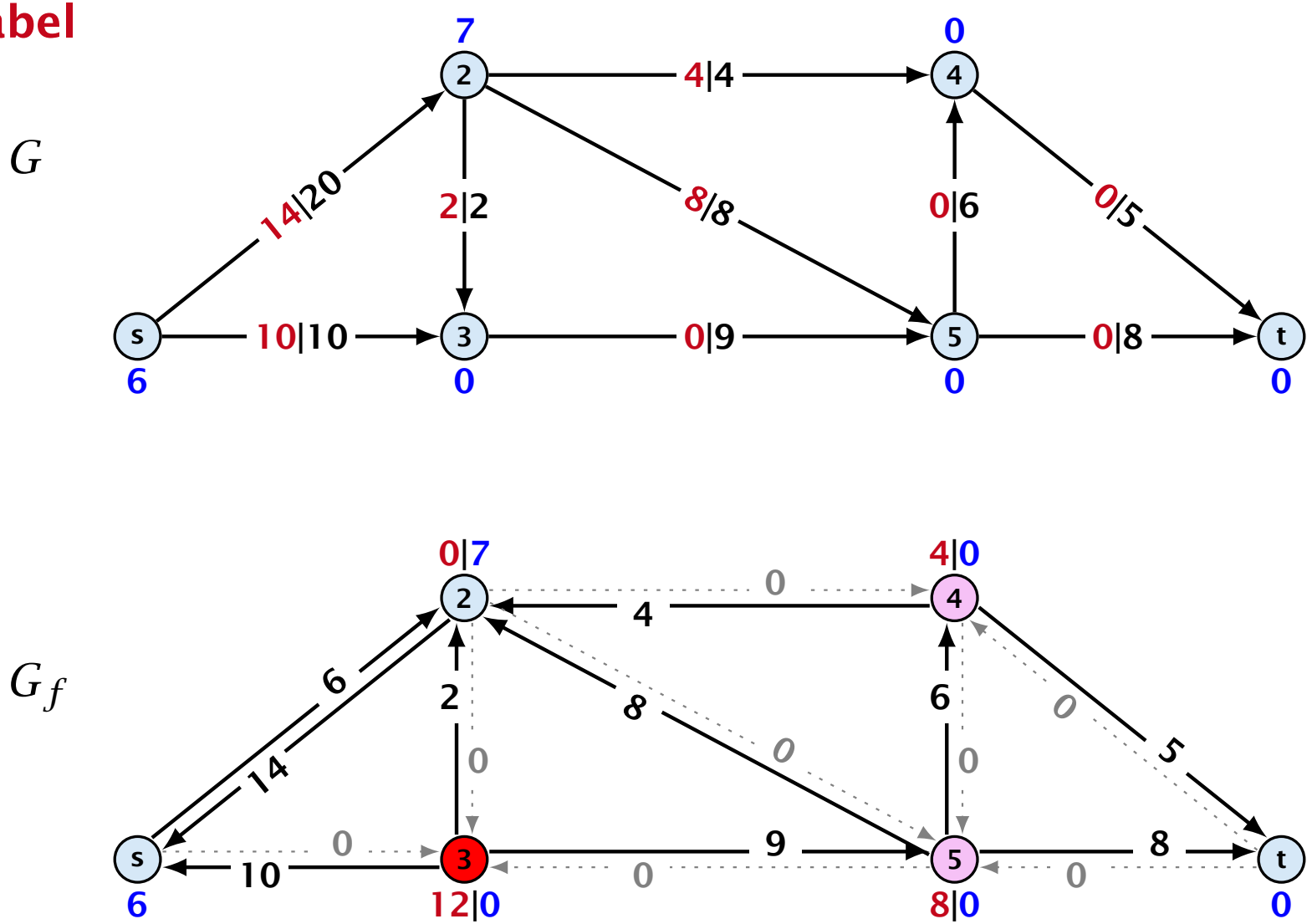


Preflow Push Algorithm

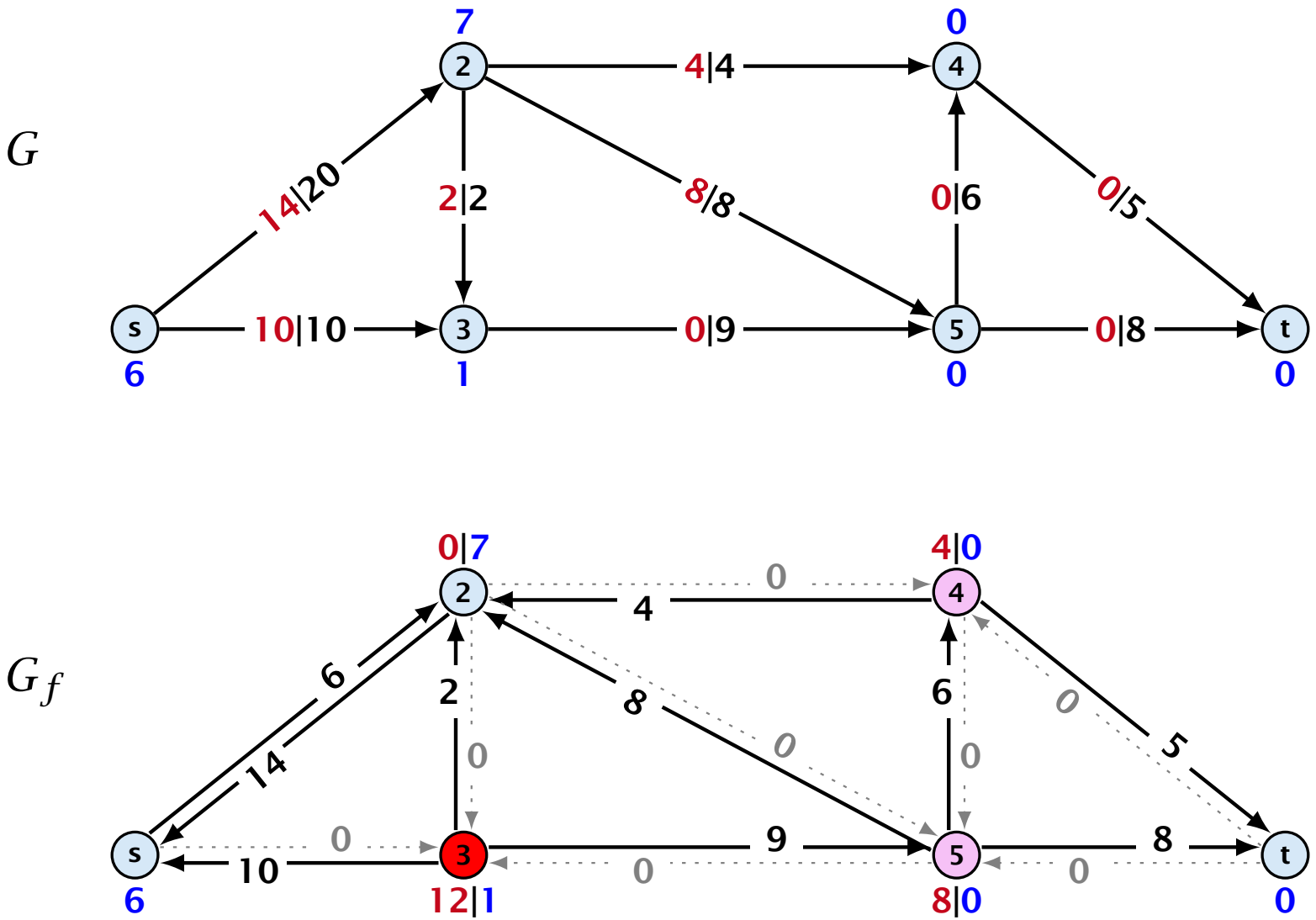


Preflow Push Algorithm

relabel



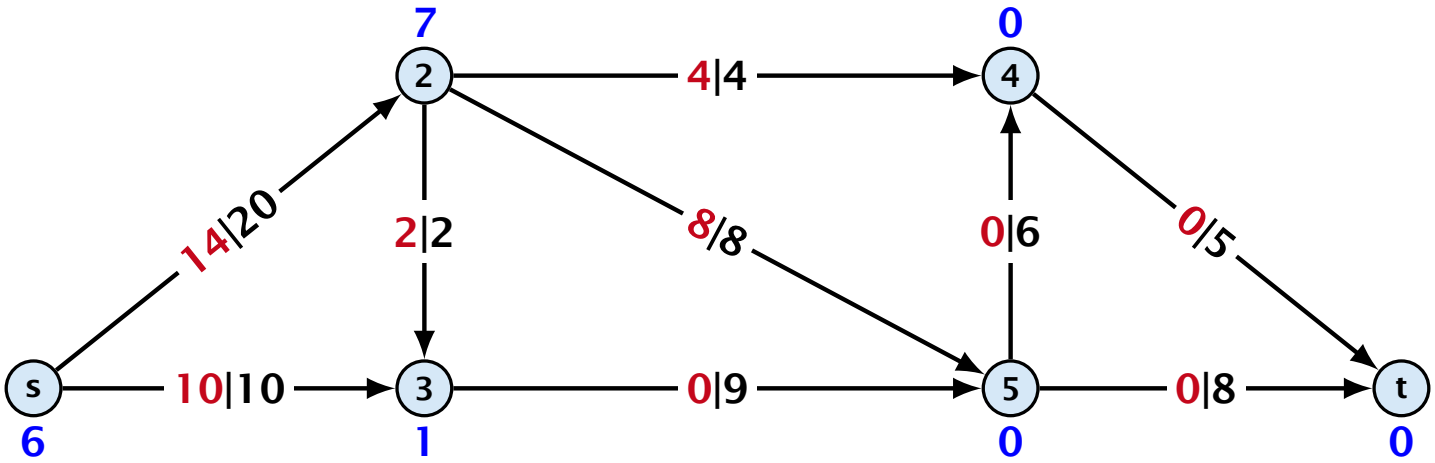
Preflow Push Algorithm



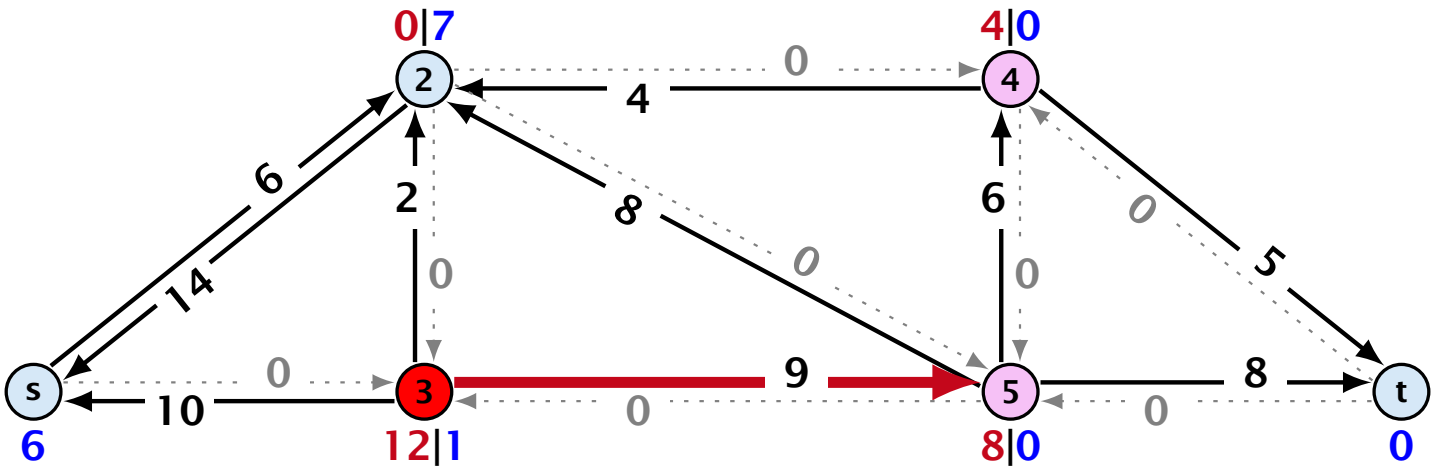
Preflow Push Algorithm

push

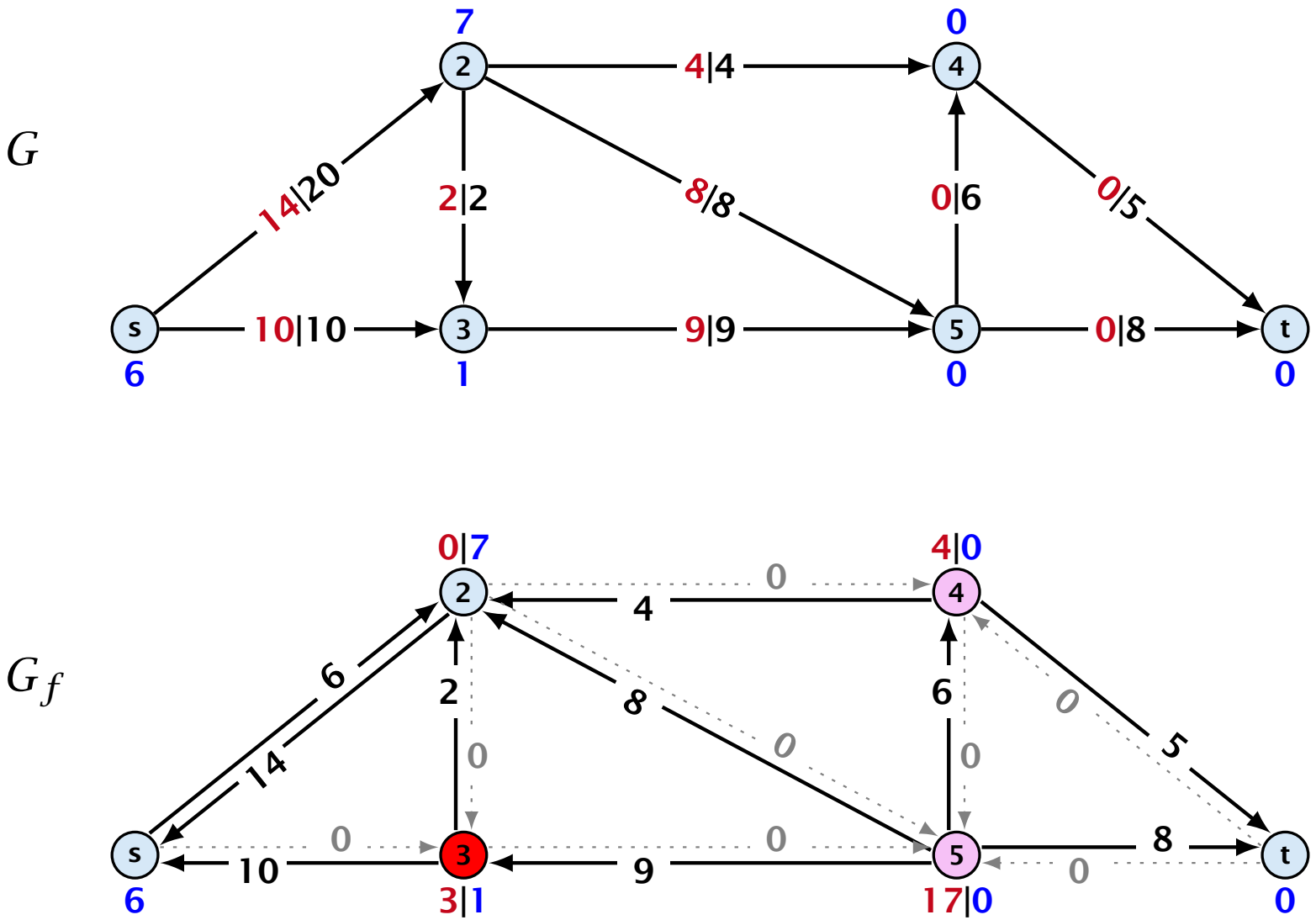
G



G_f



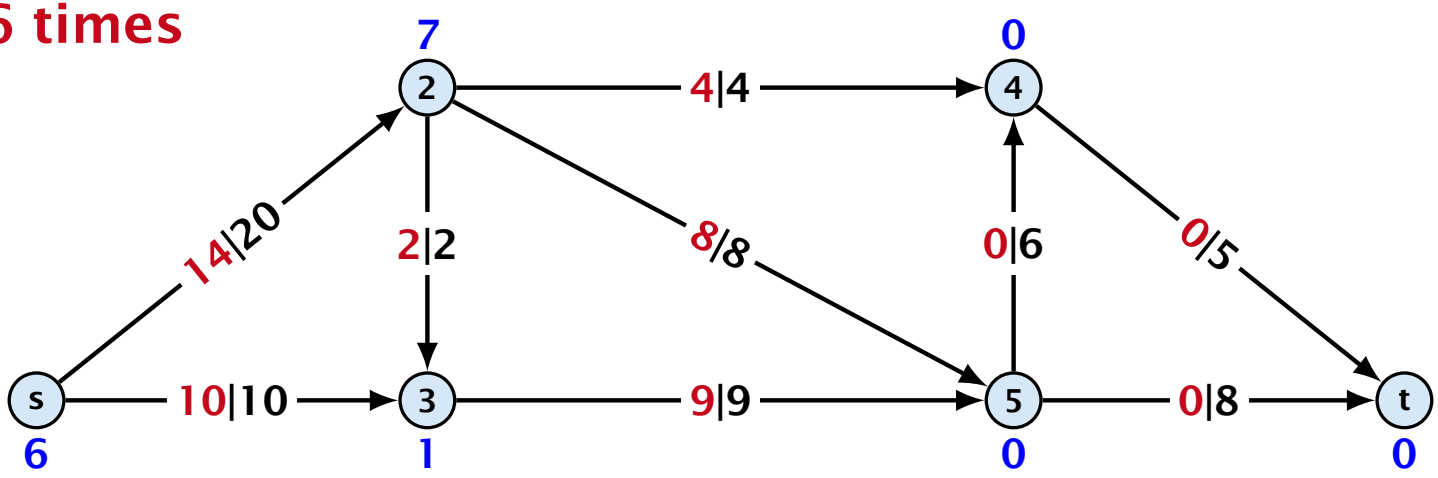
Preflow Push Algorithm



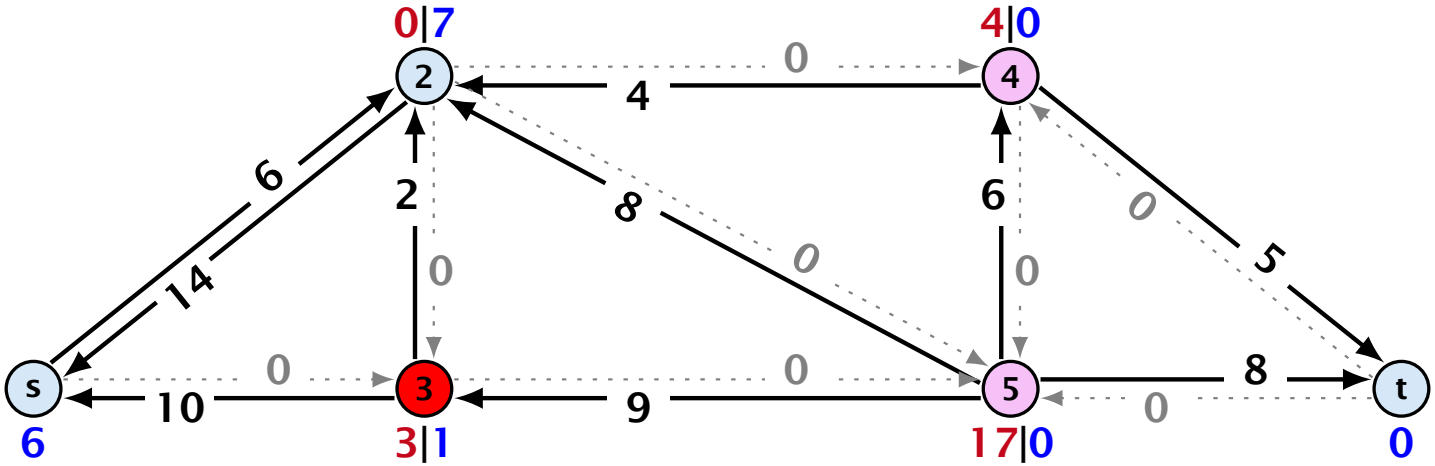
Preflow Push Algorithm

relabel 6 times

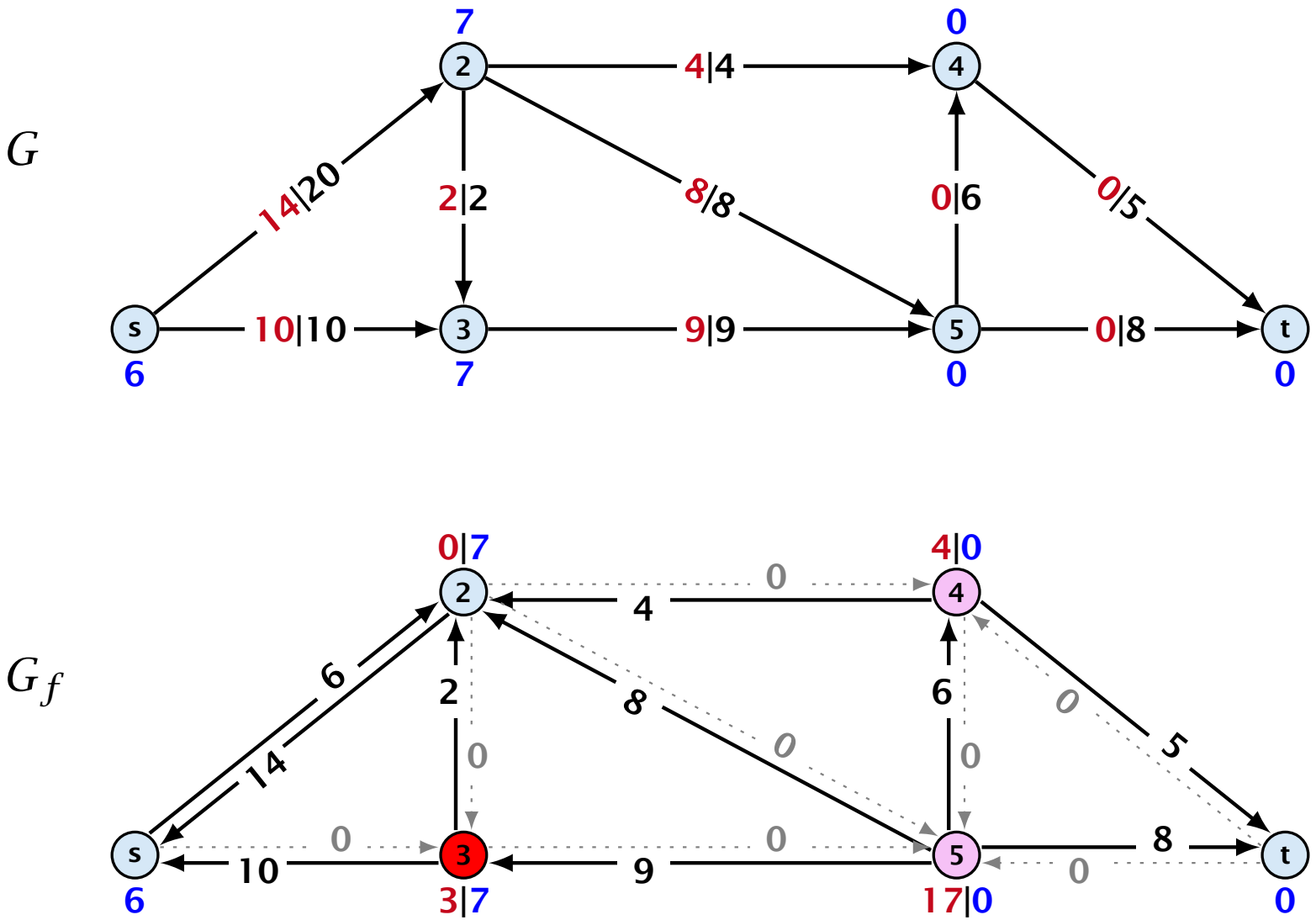
G



G_f

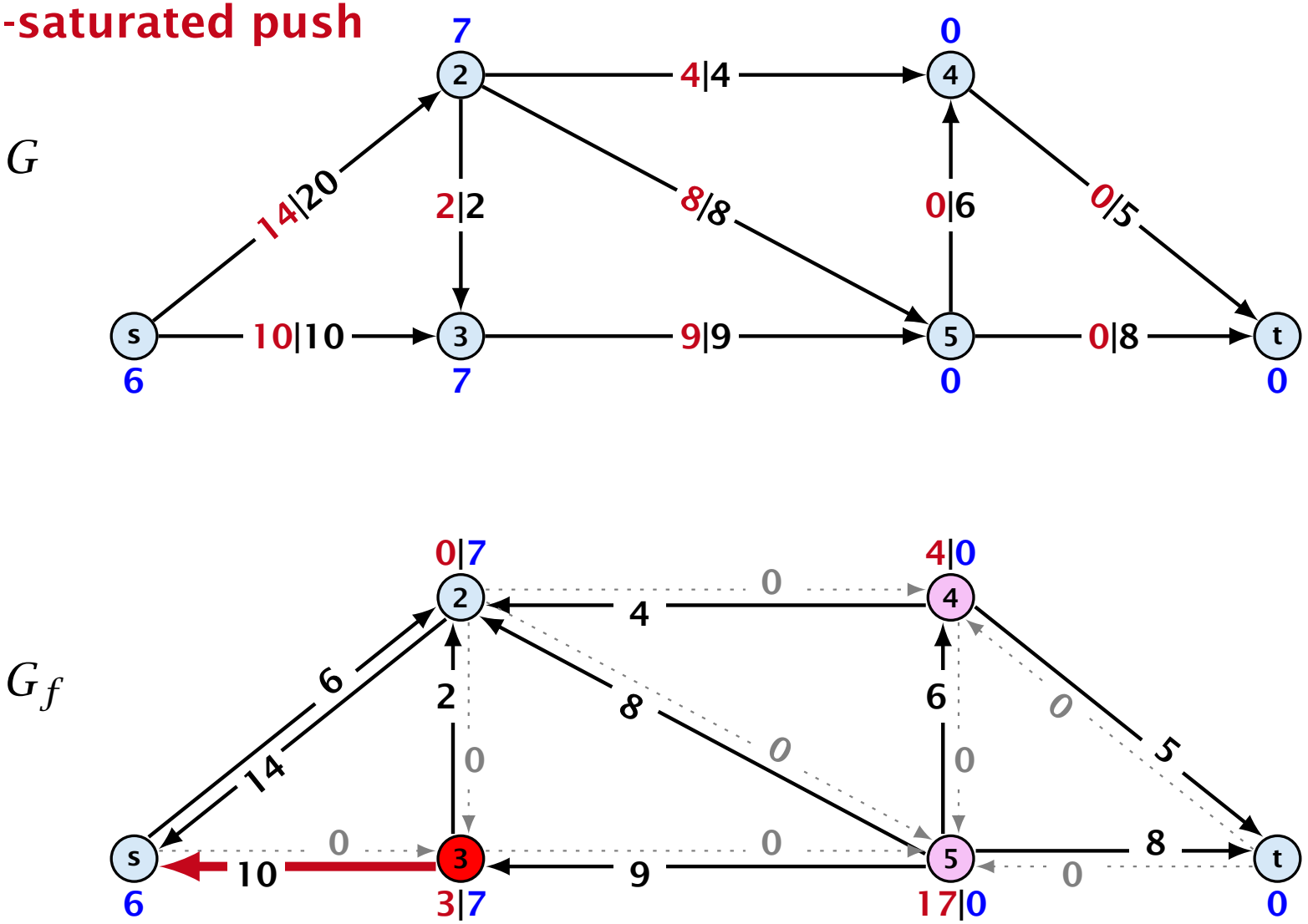


Preflow Push Algorithm

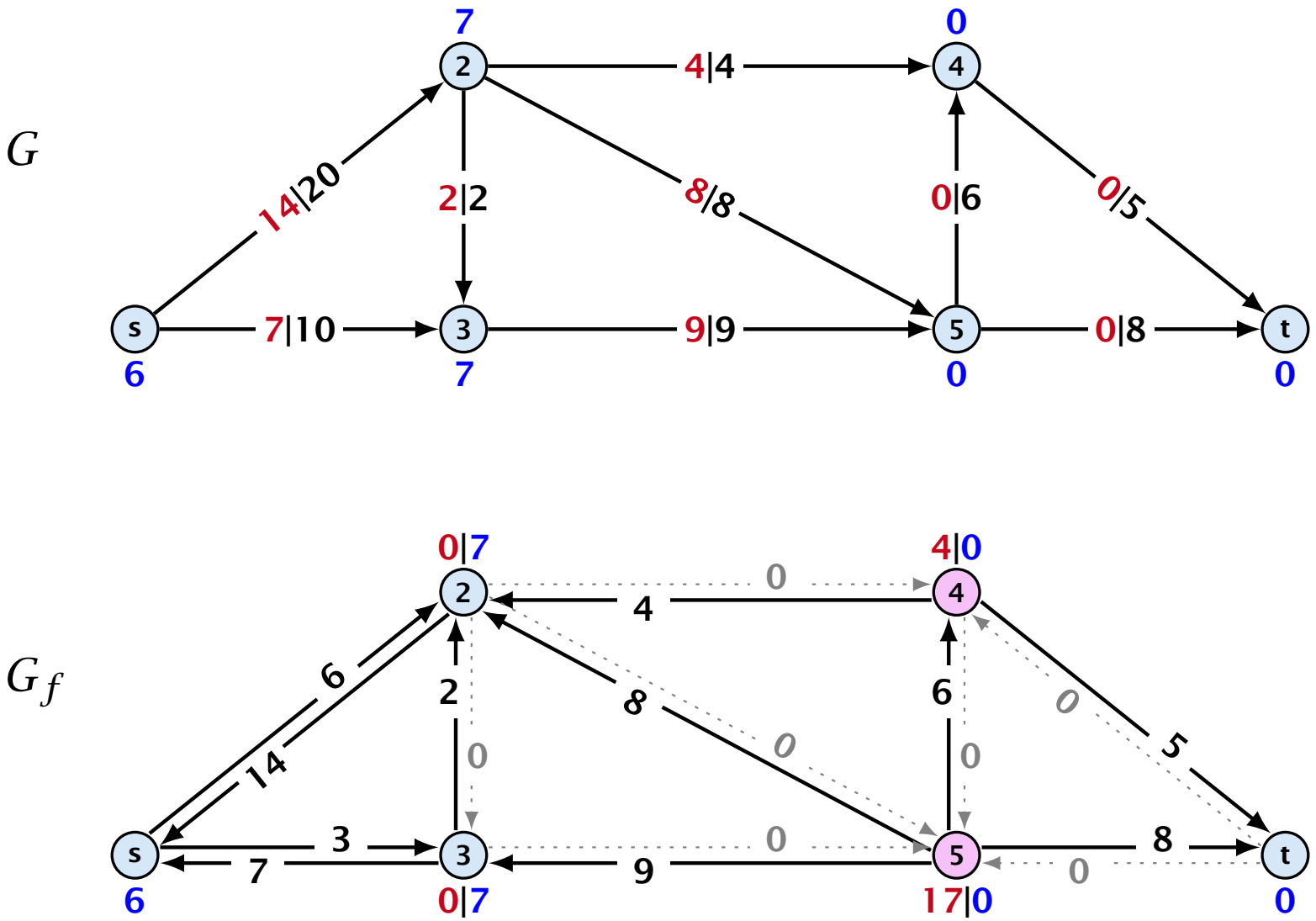


Preflow Push Algorithm

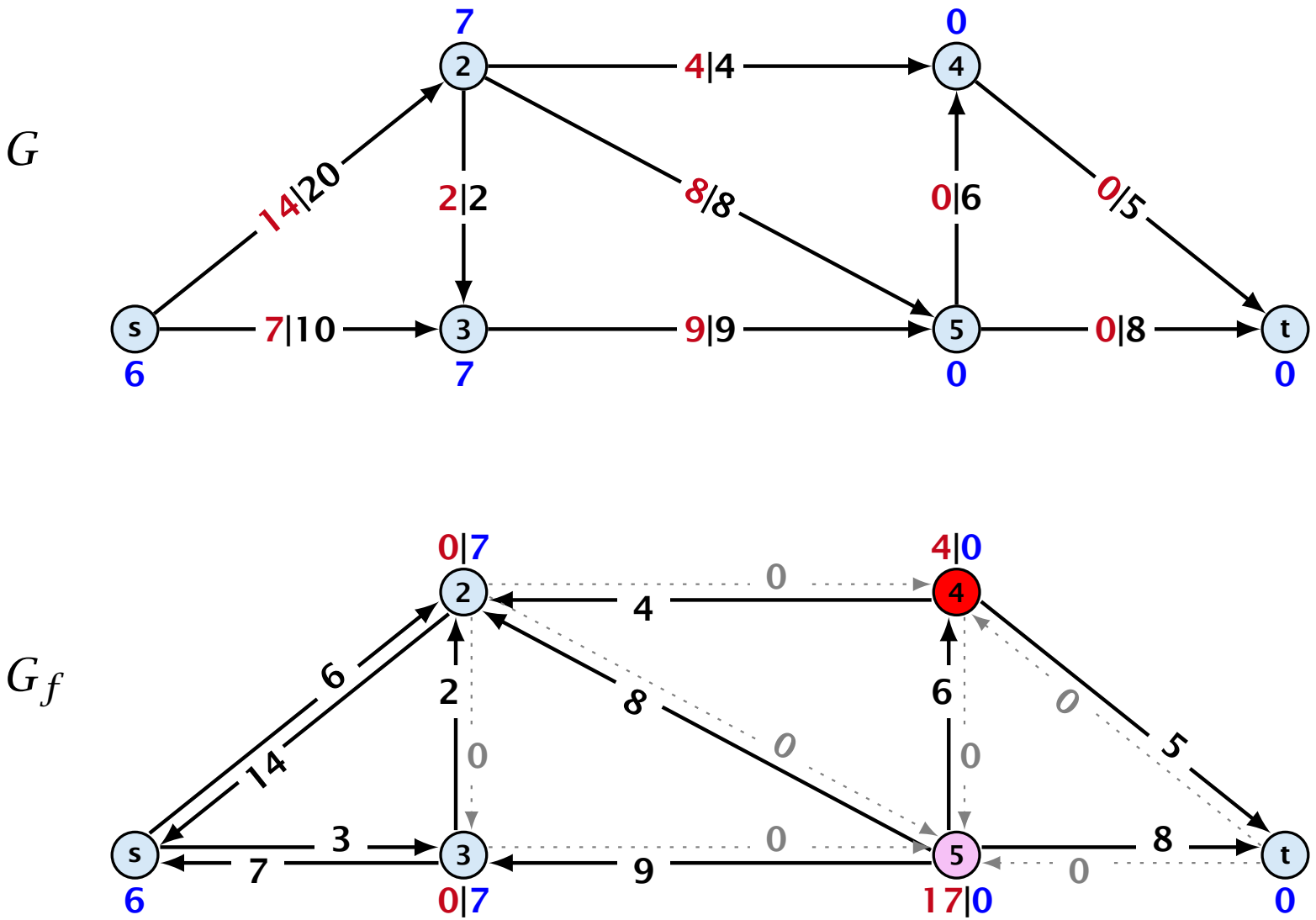
non-saturated push



Preflow Push Algorithm

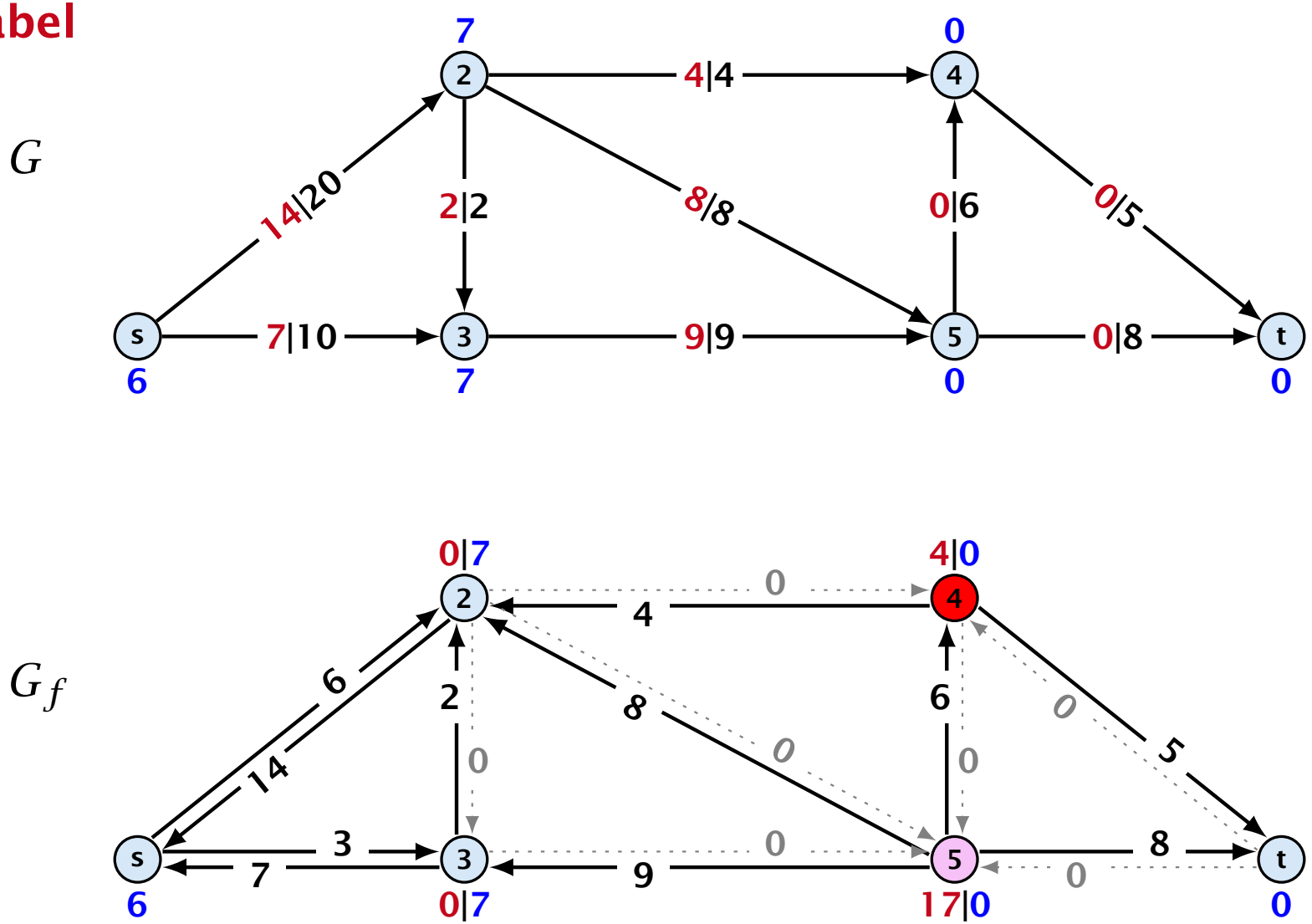


Preflow Push Algorithm

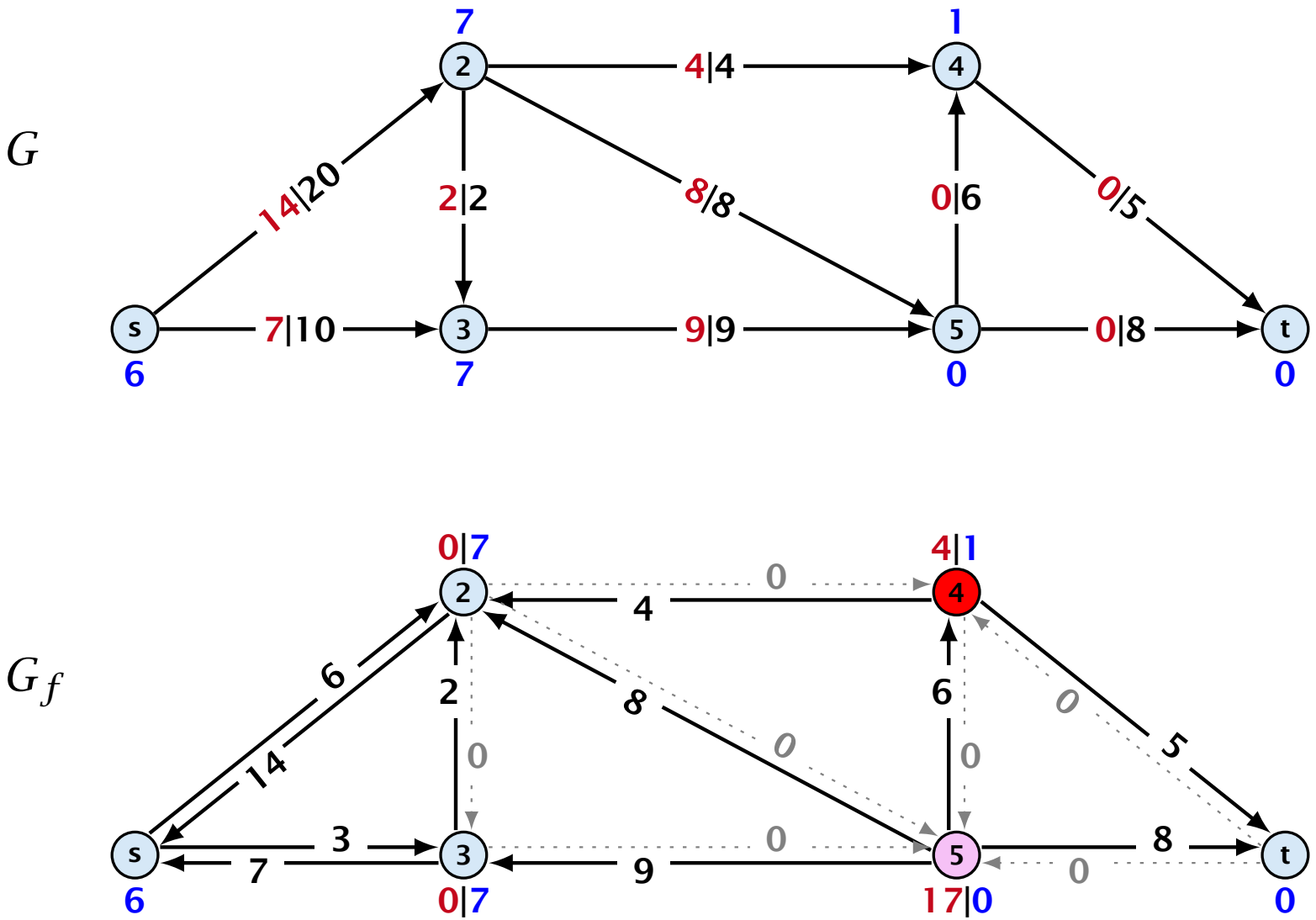


Preflow Push Algorithm

relabel

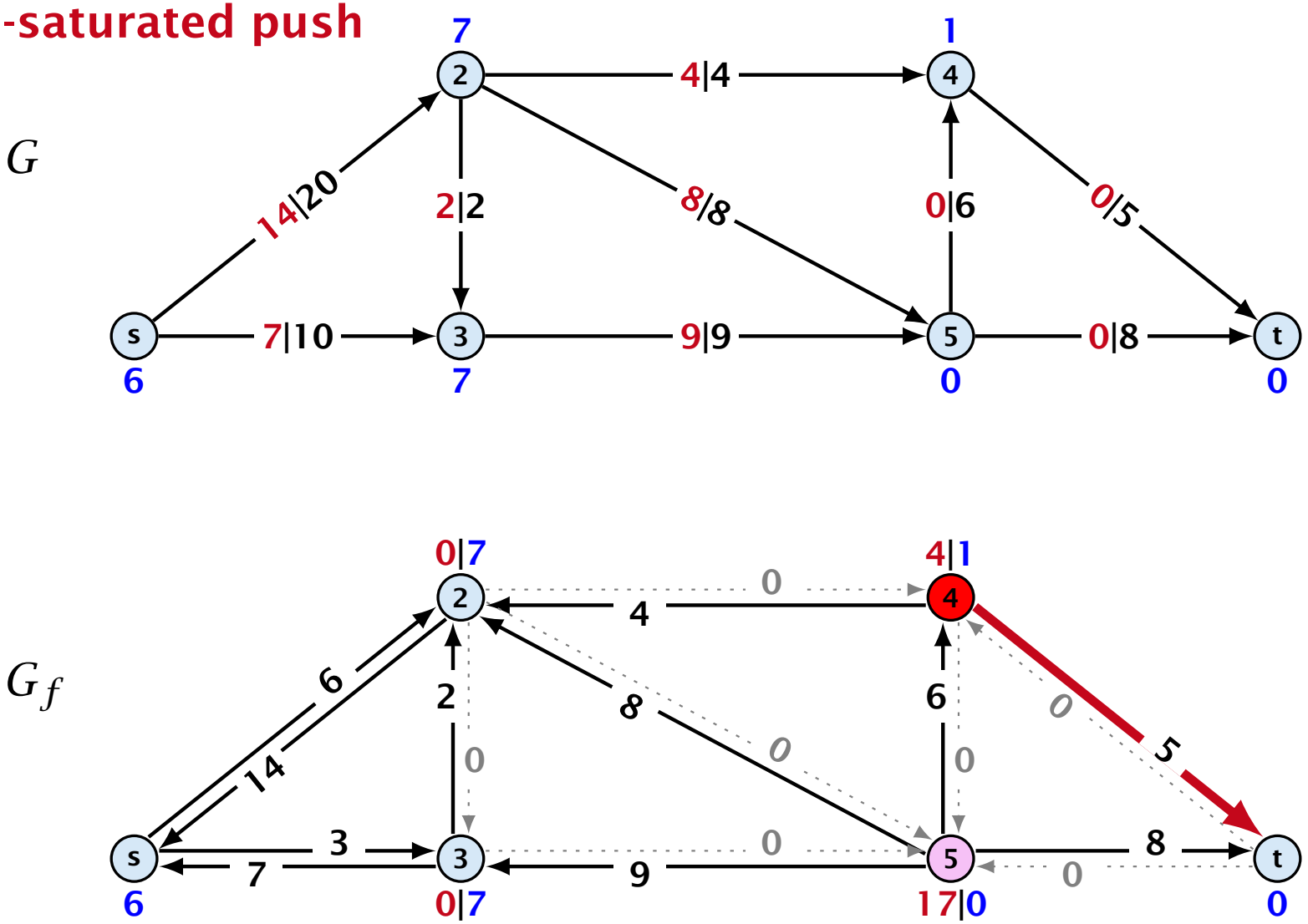


Preflow Push Algorithm

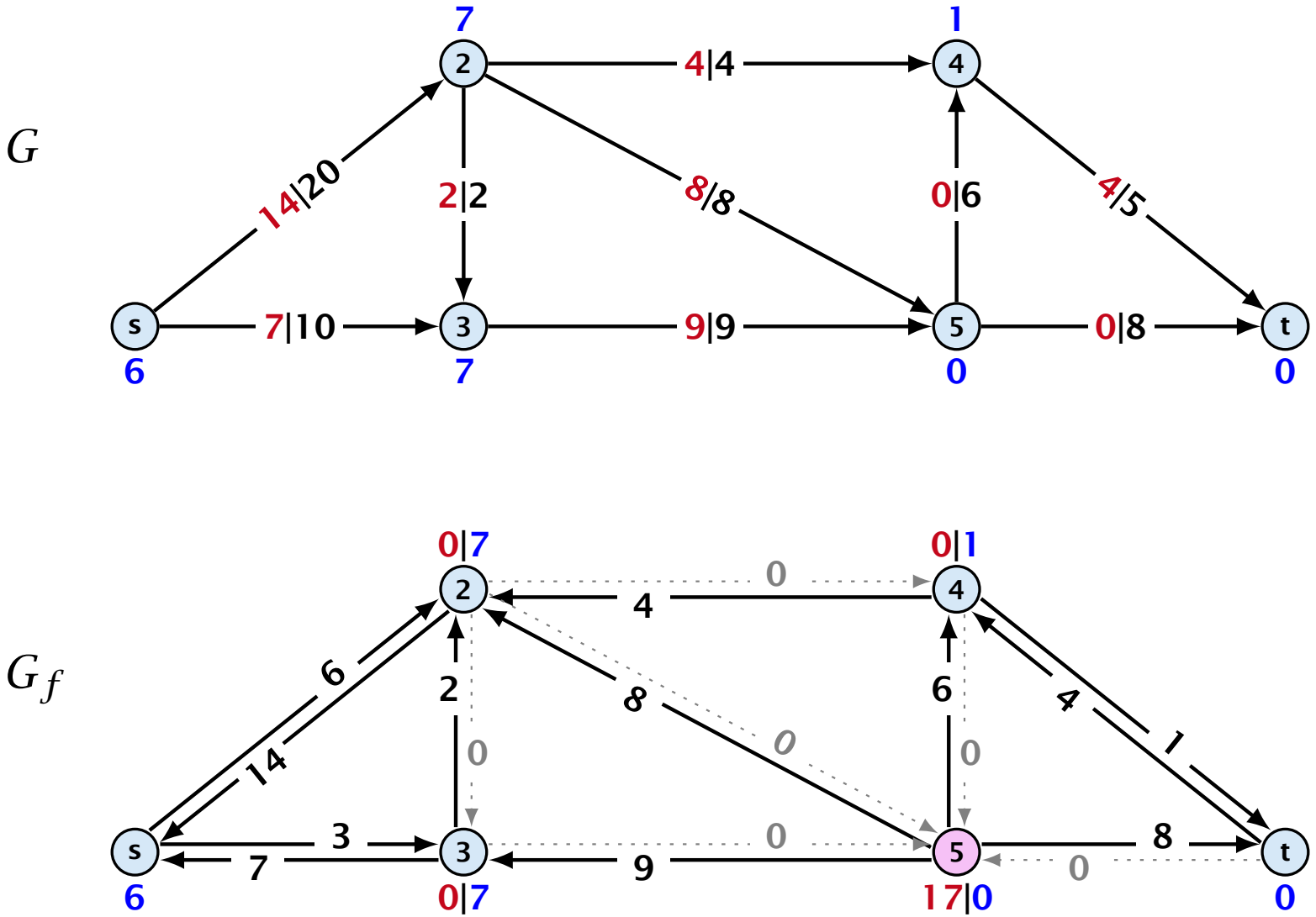


Preflow Push Algorithm

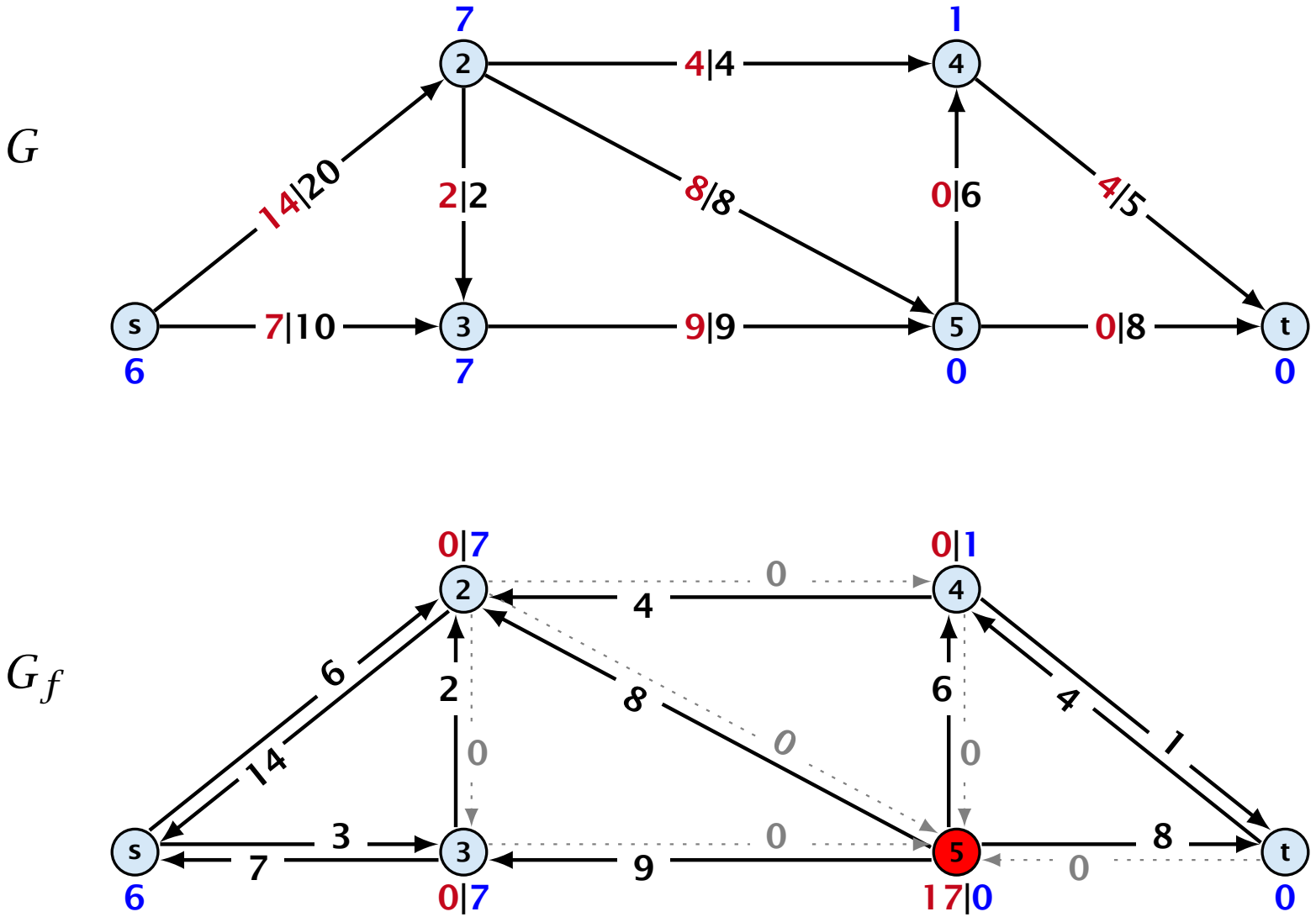
non-saturated push



Preflow Push Algorithm

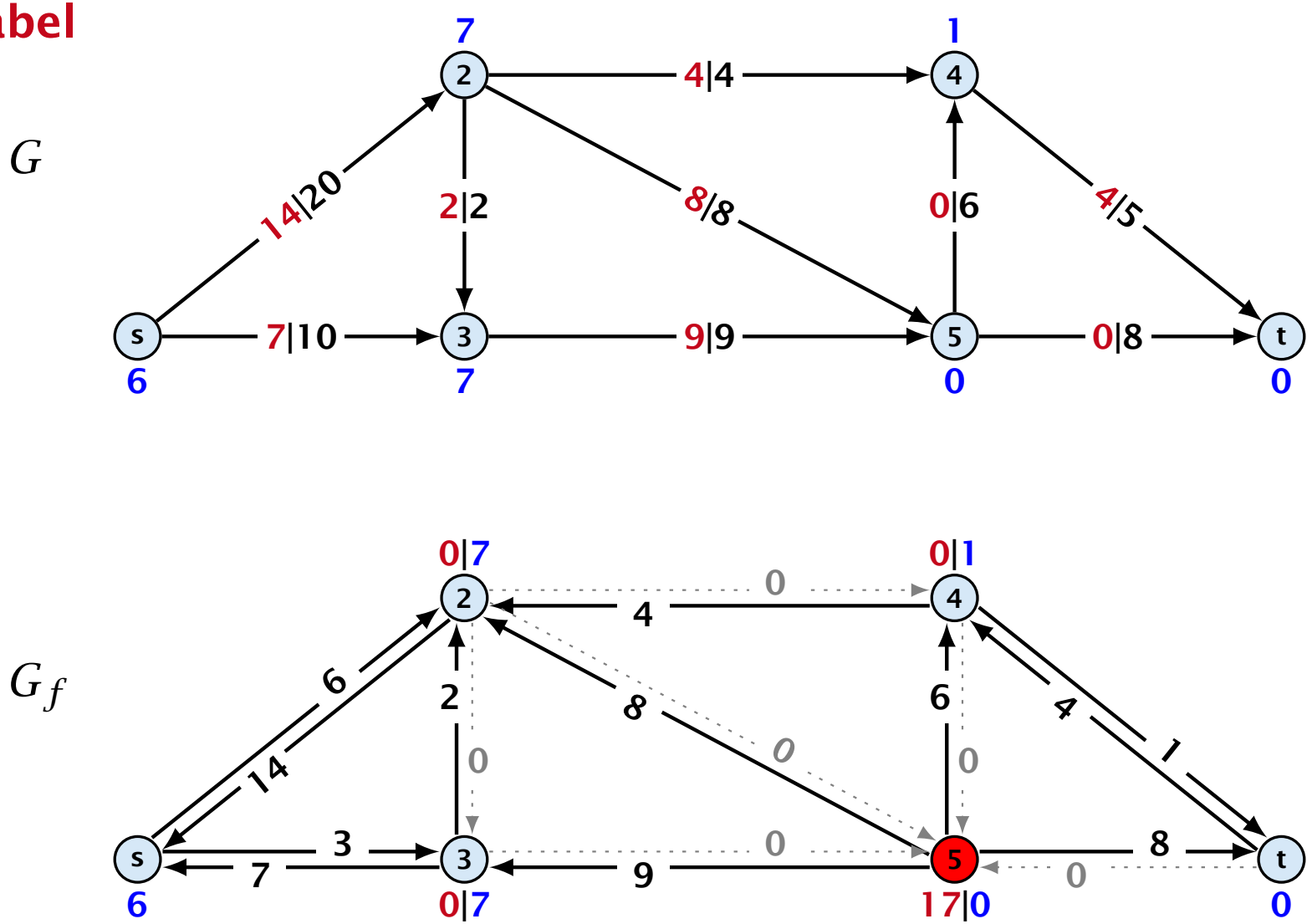


Preflow Push Algorithm

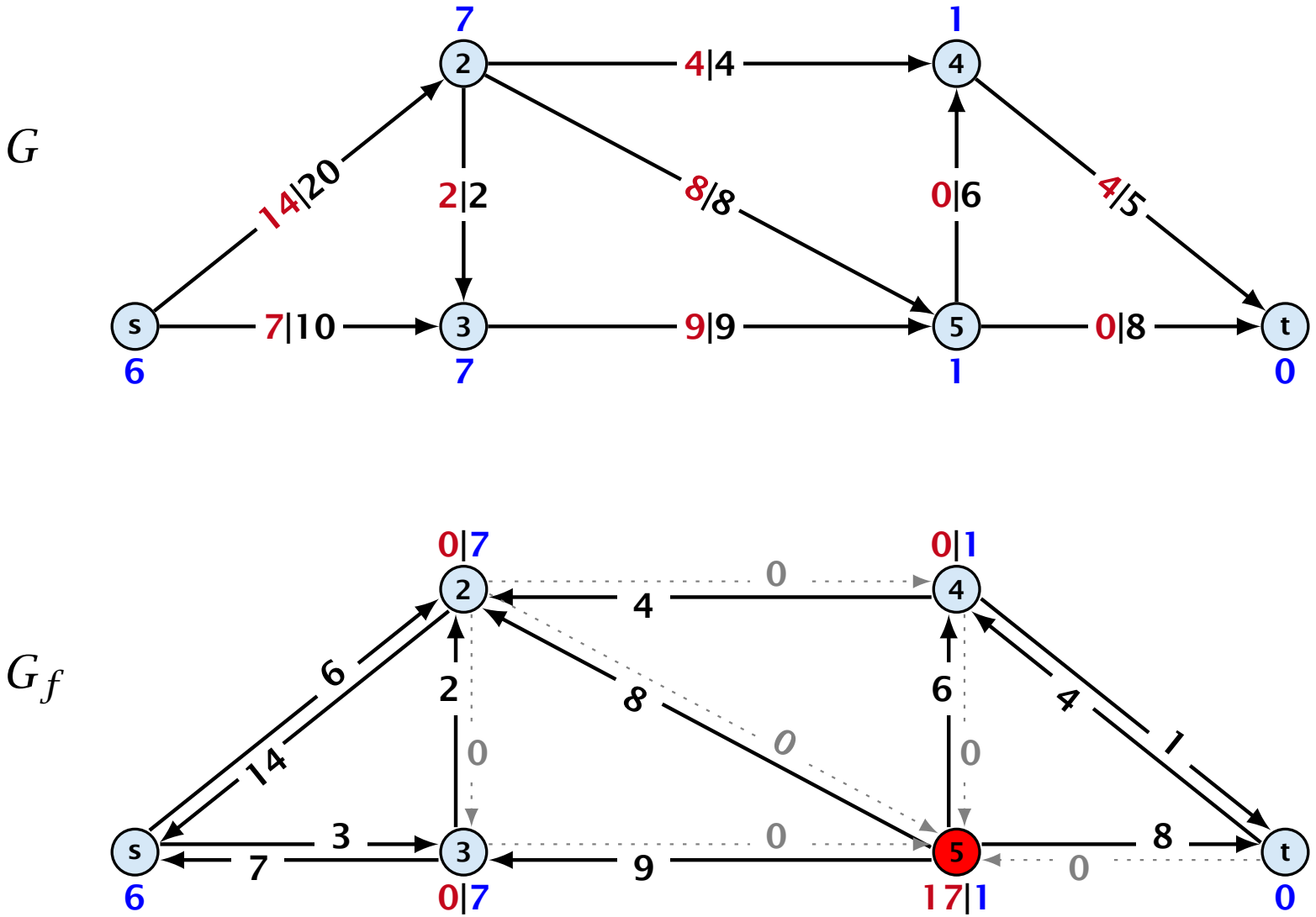


Preflow Push Algorithm

relabel

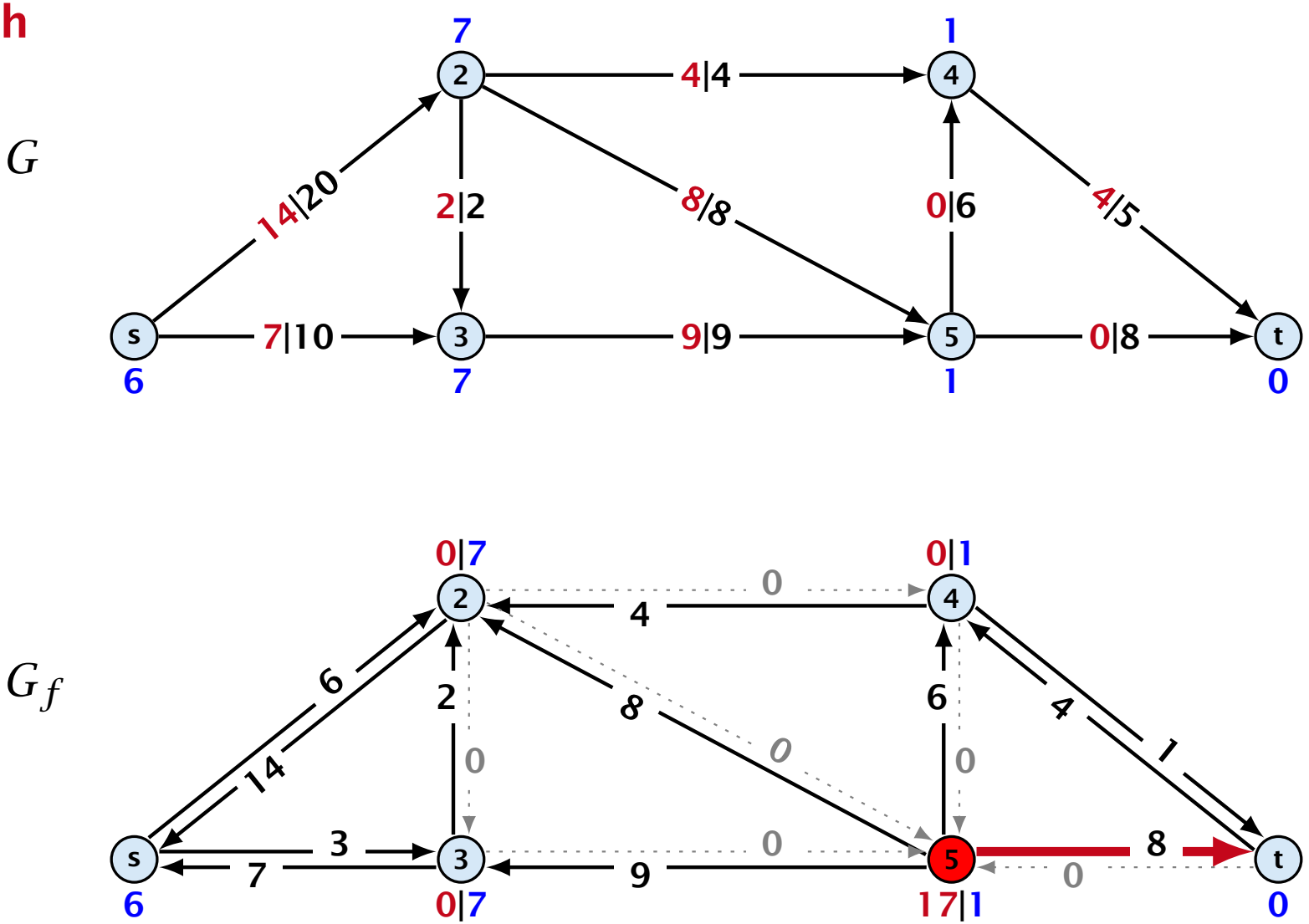


Preflow Push Algorithm

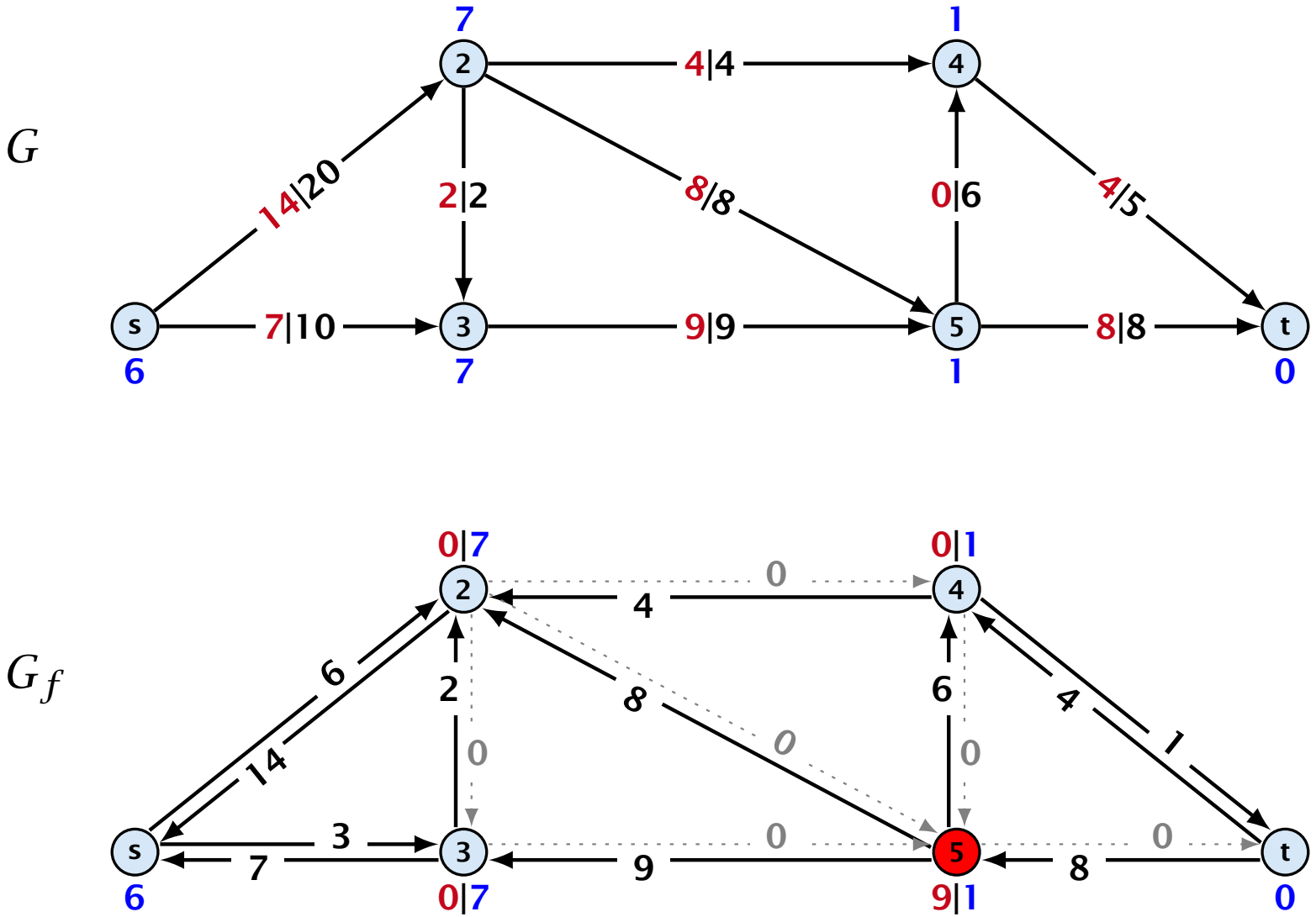


Preflow Push Algorithm

push



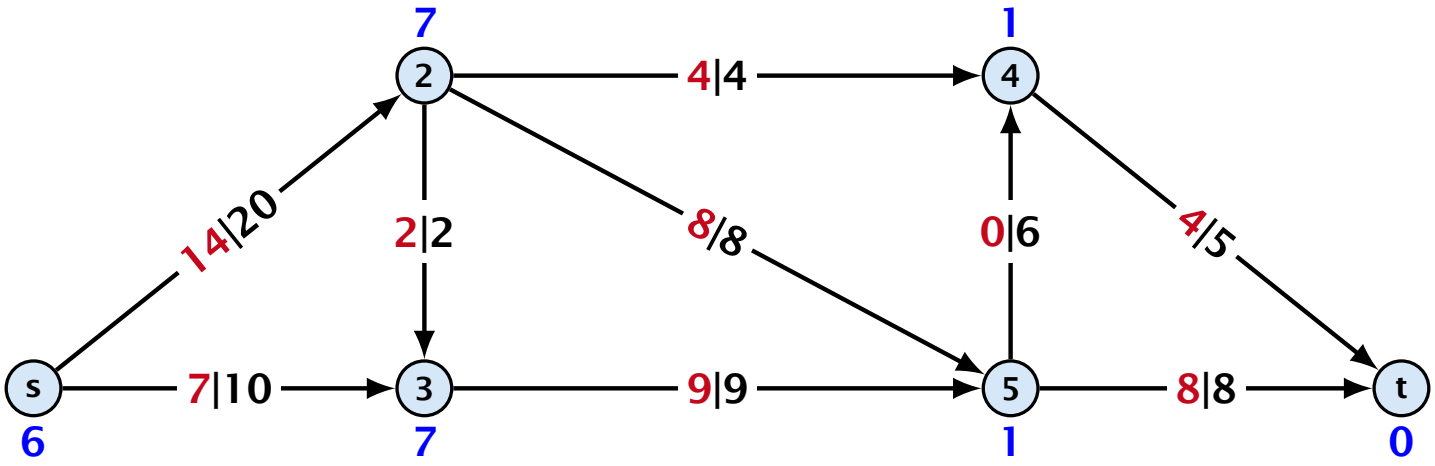
Preflow Push Algorithm



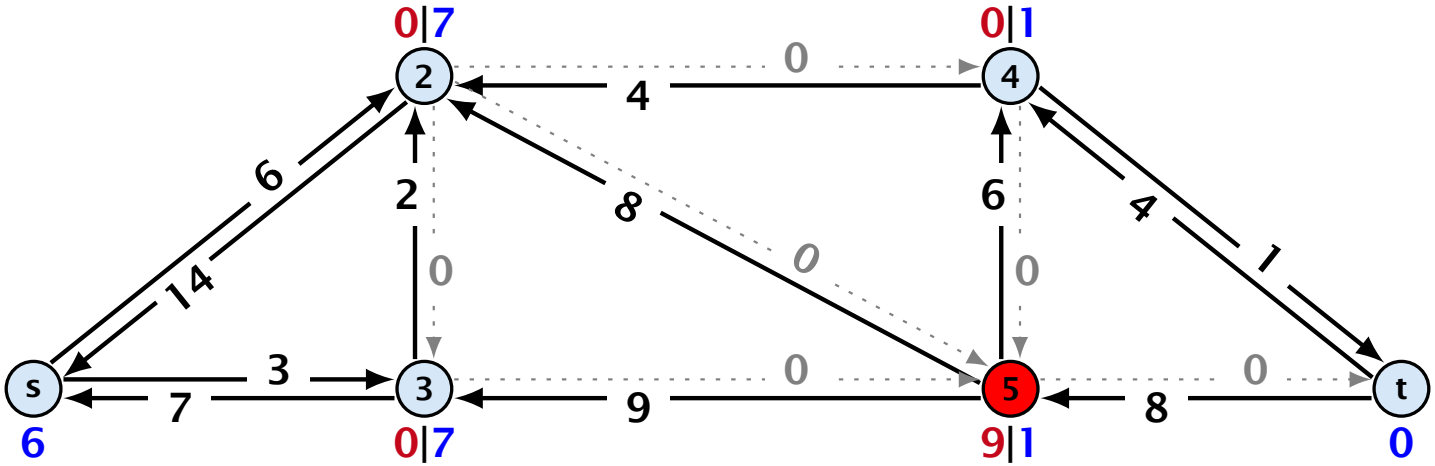
Preflow Push Algorithm

relabel

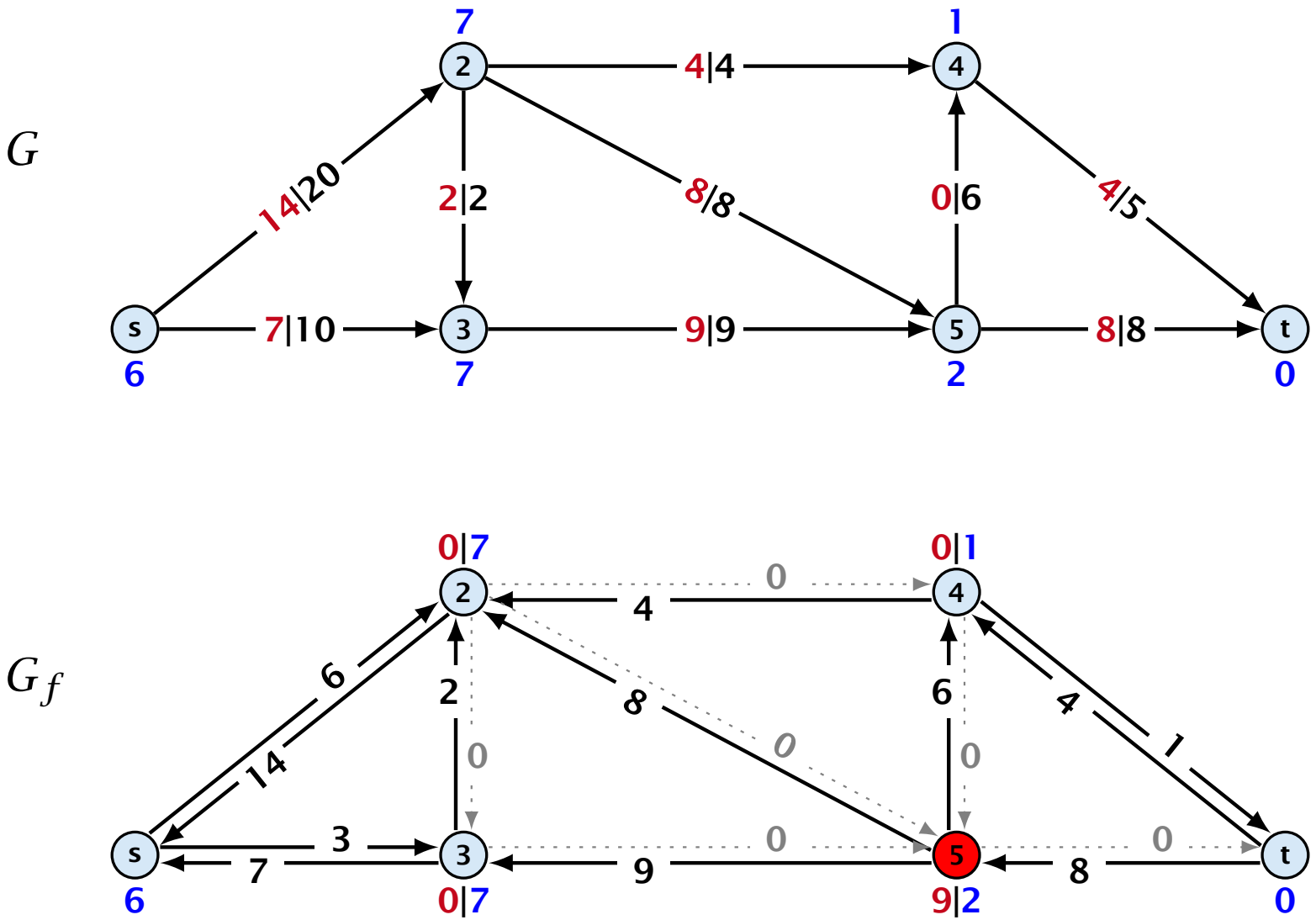
G



G_f



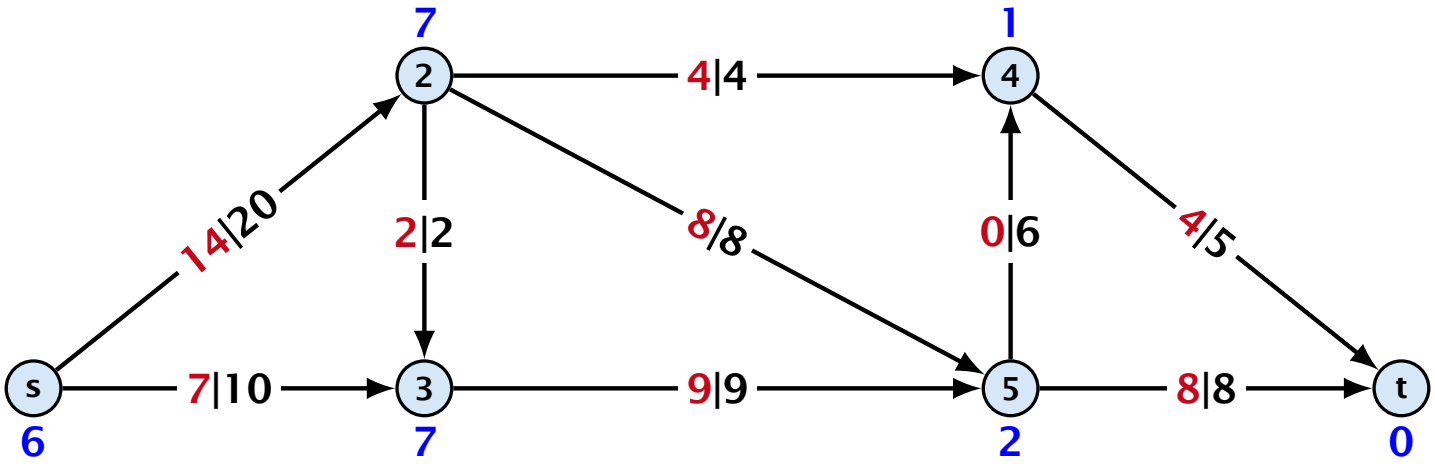
Preflow Push Algorithm



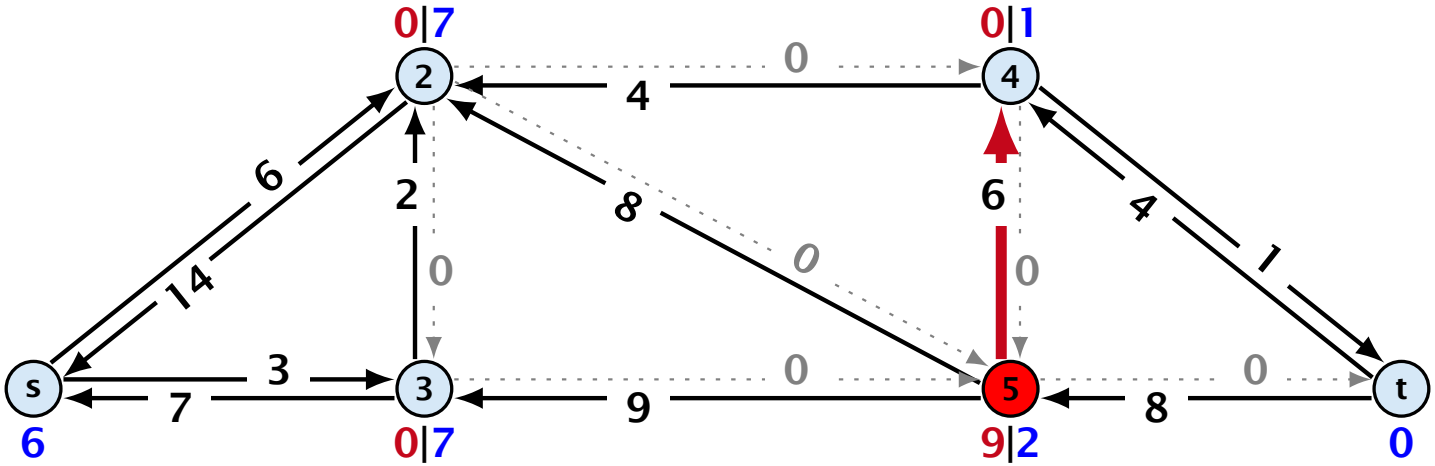
Preflow Push Algorithm

push

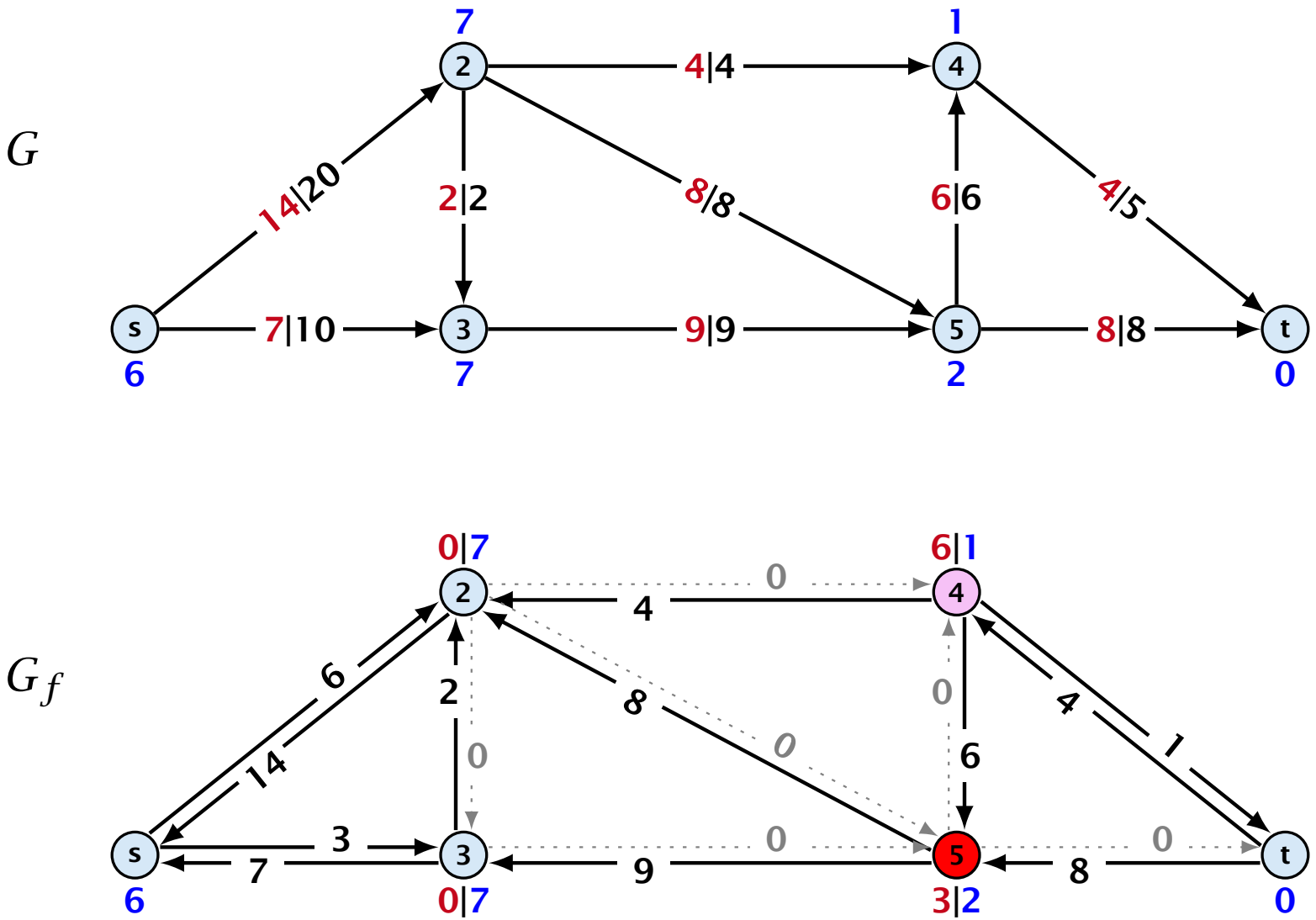
G



G_f



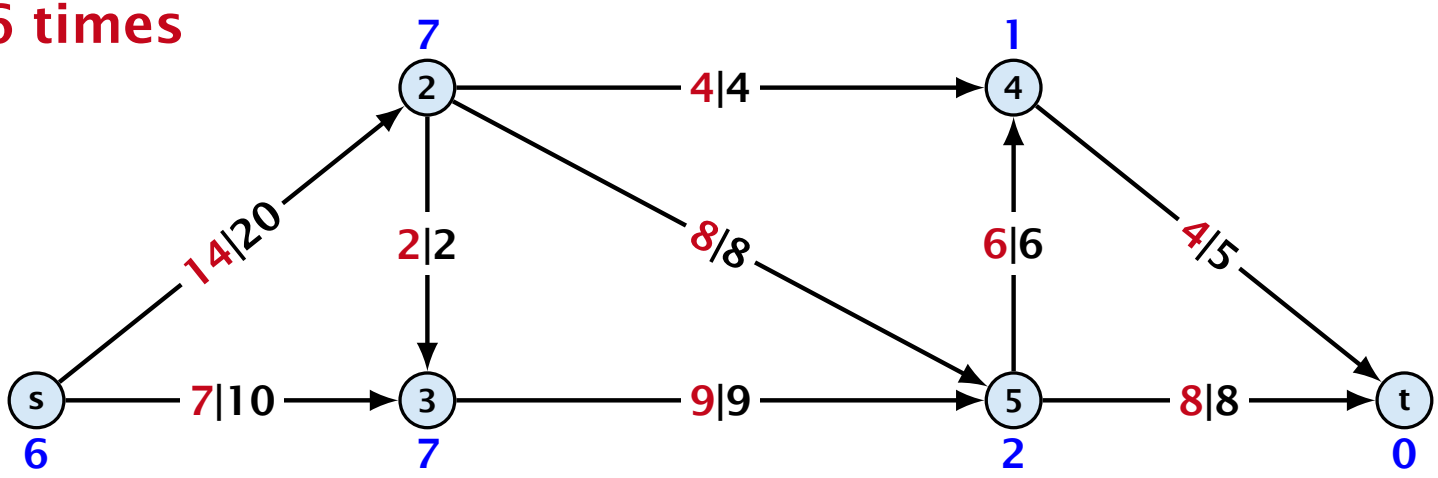
Preflow Push Algorithm



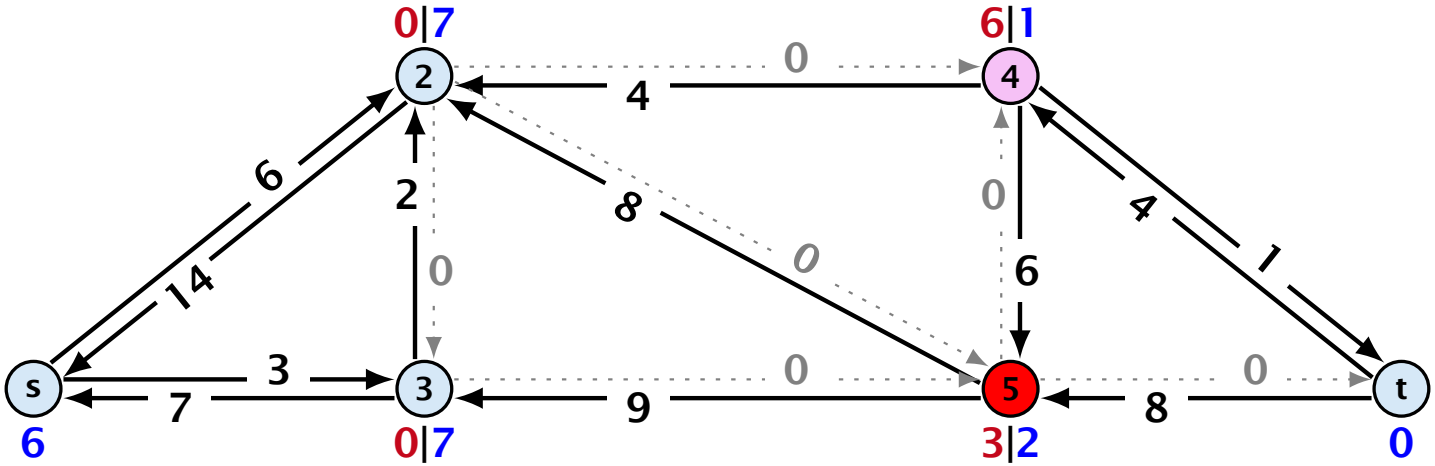
Preflow Push Algorithm

relabel 6 times

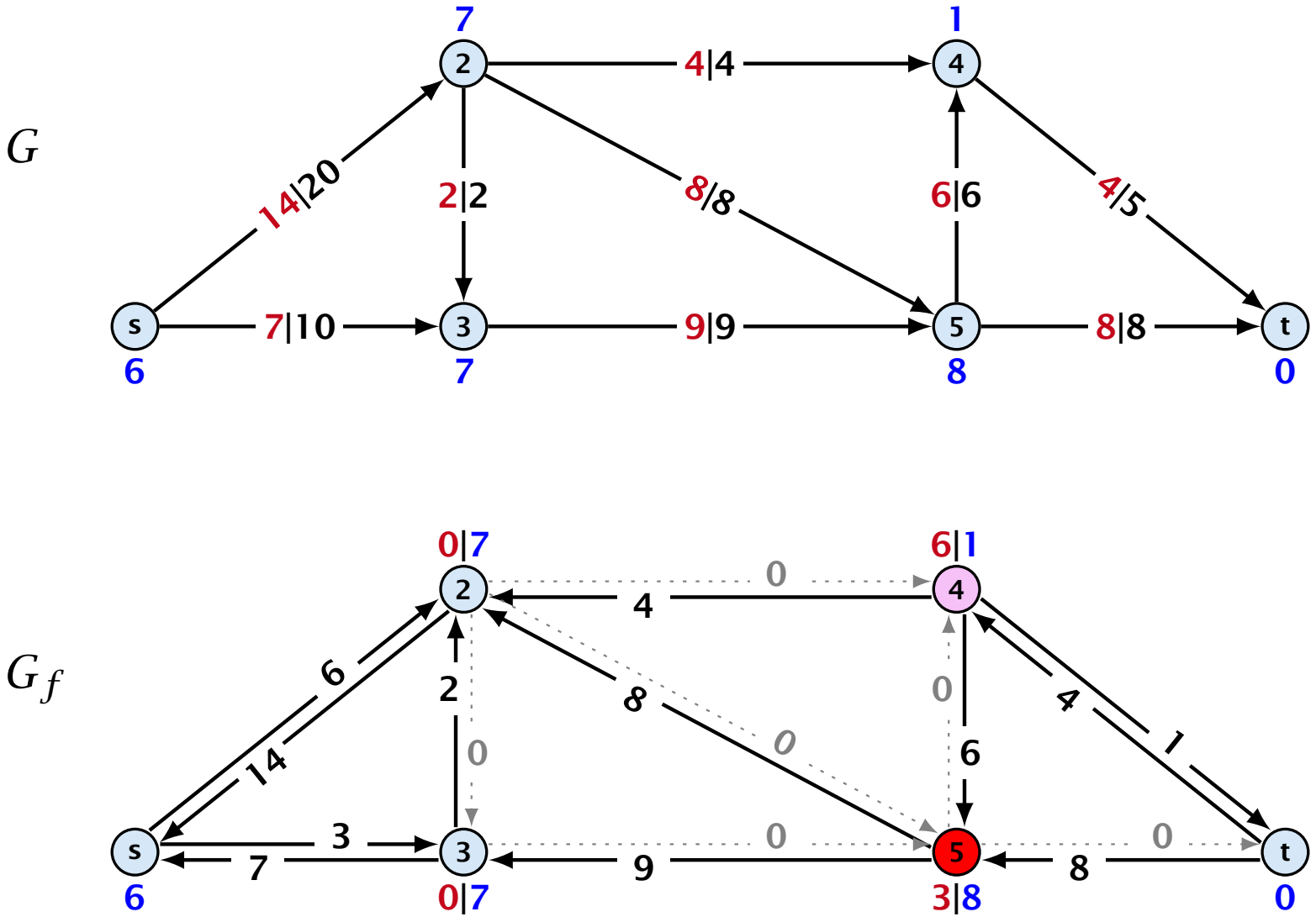
G



G_f

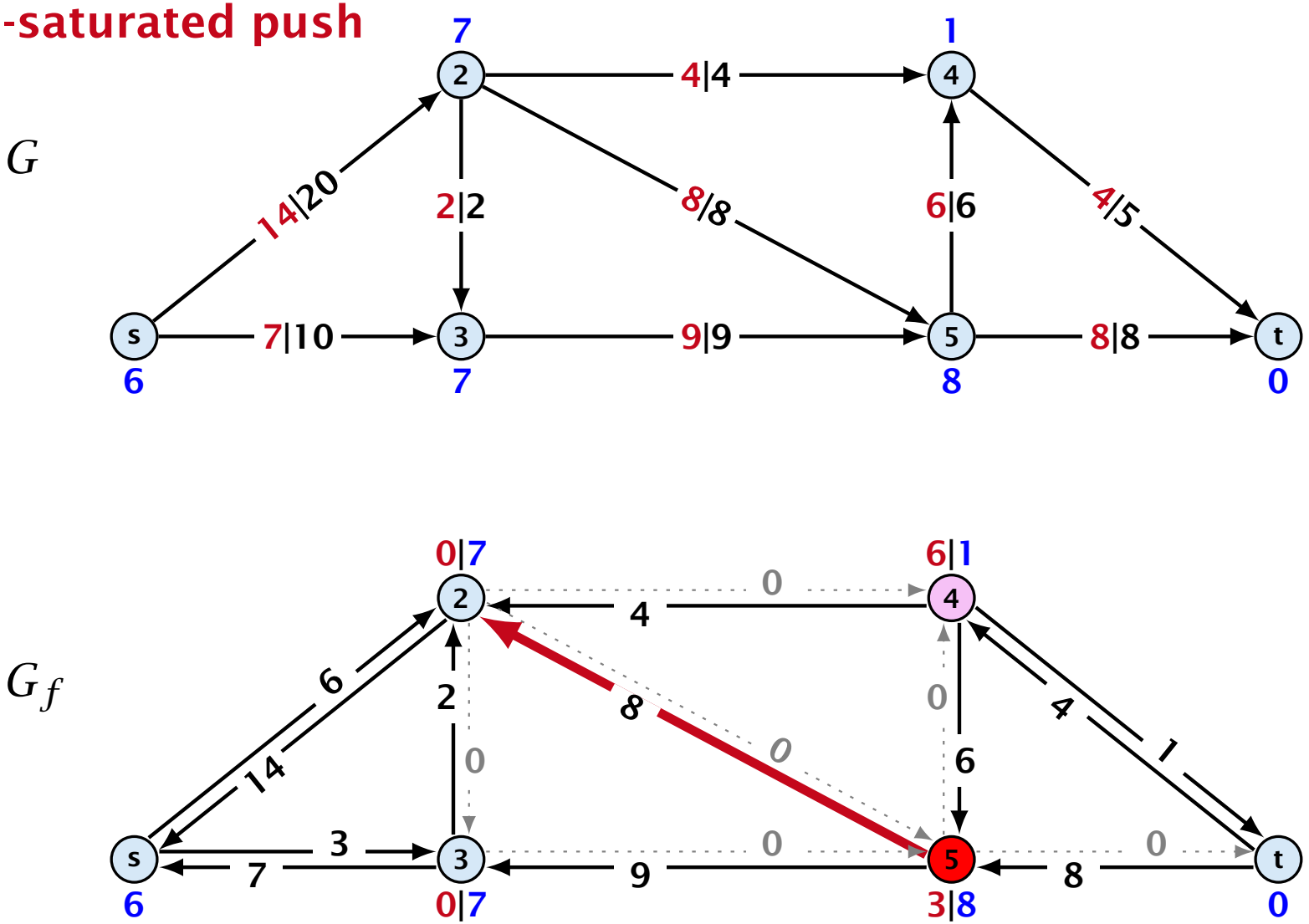


Preflow Push Algorithm

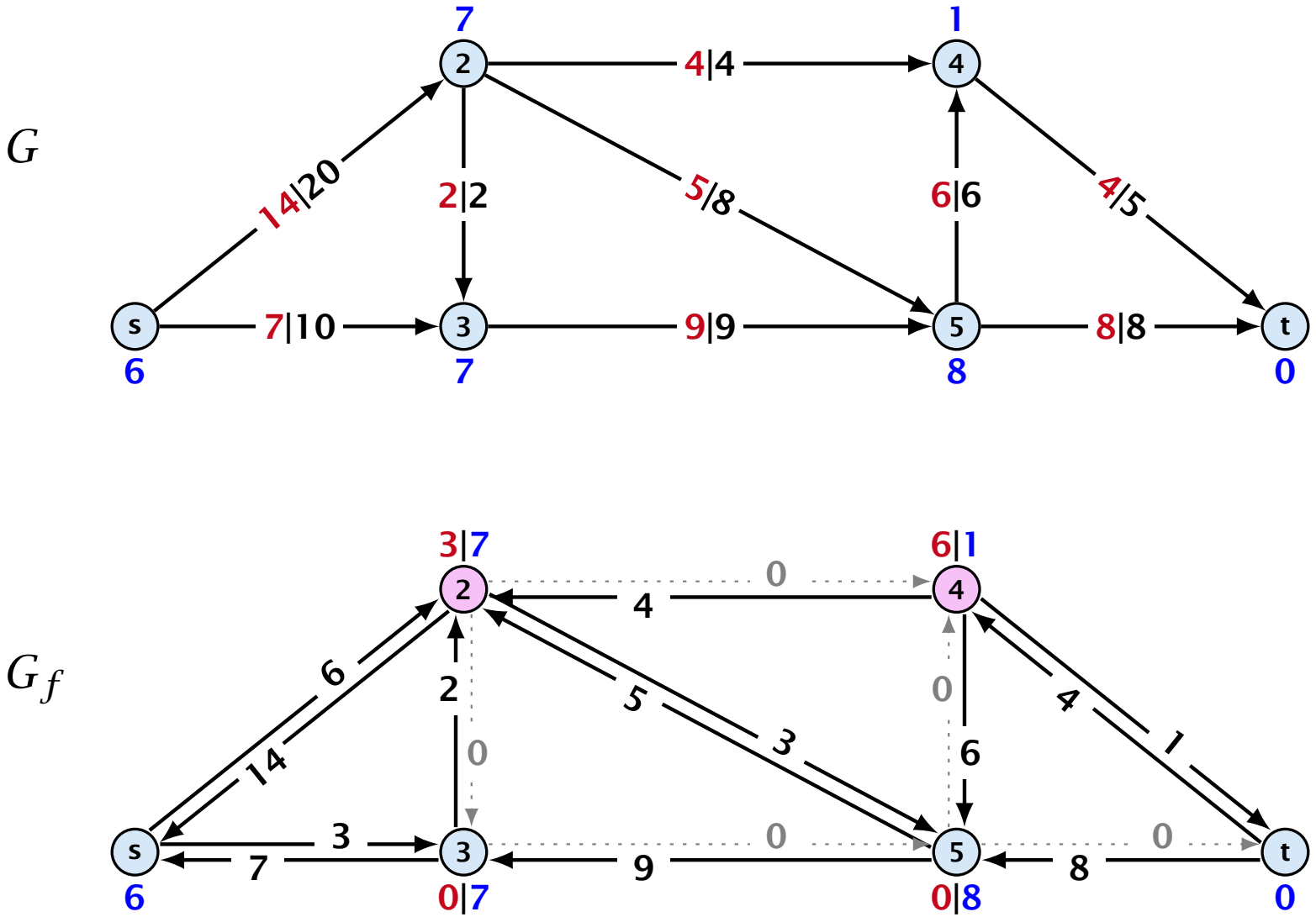


Preflow Push Algorithm

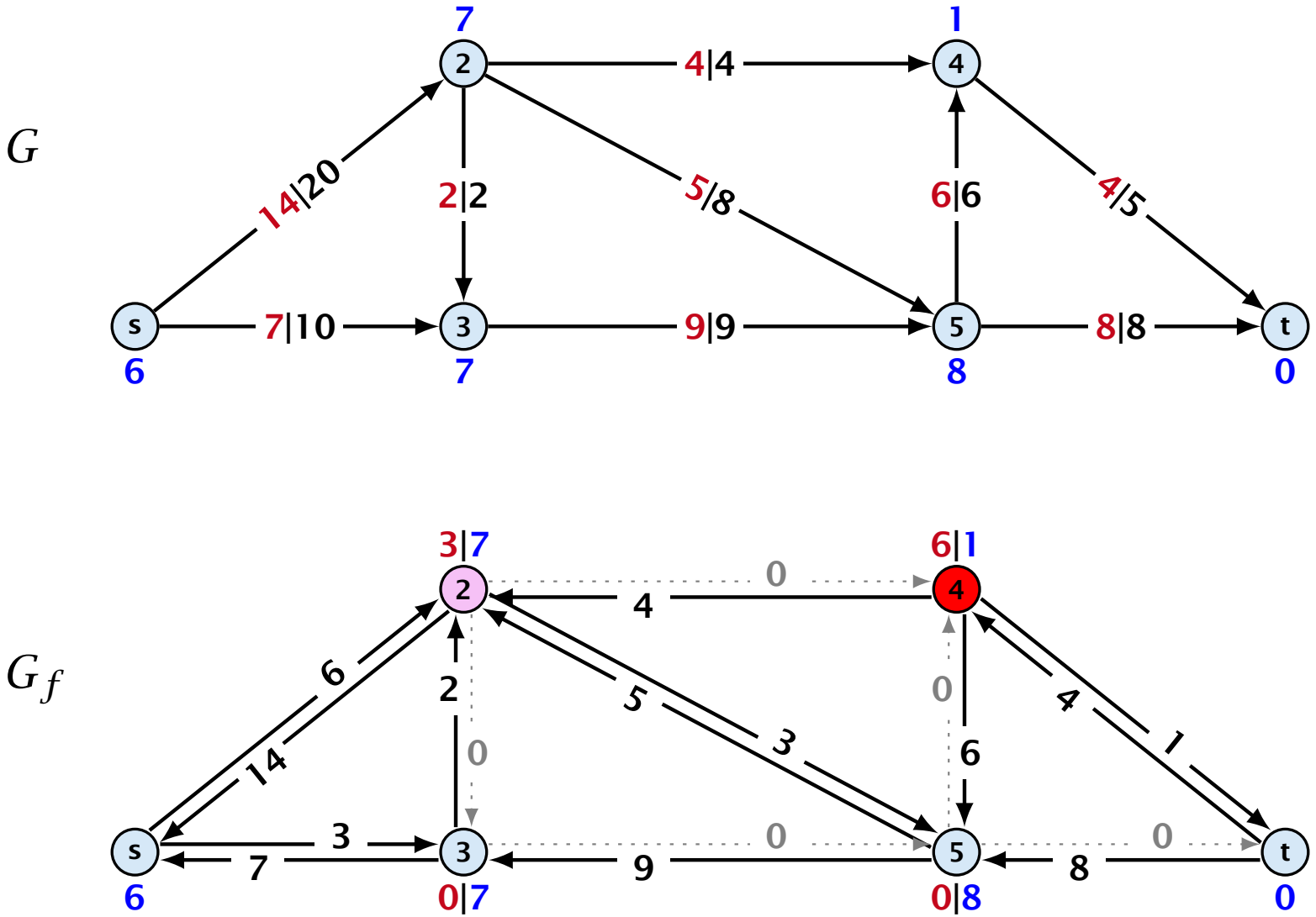
non-saturated push



Preflow Push Algorithm



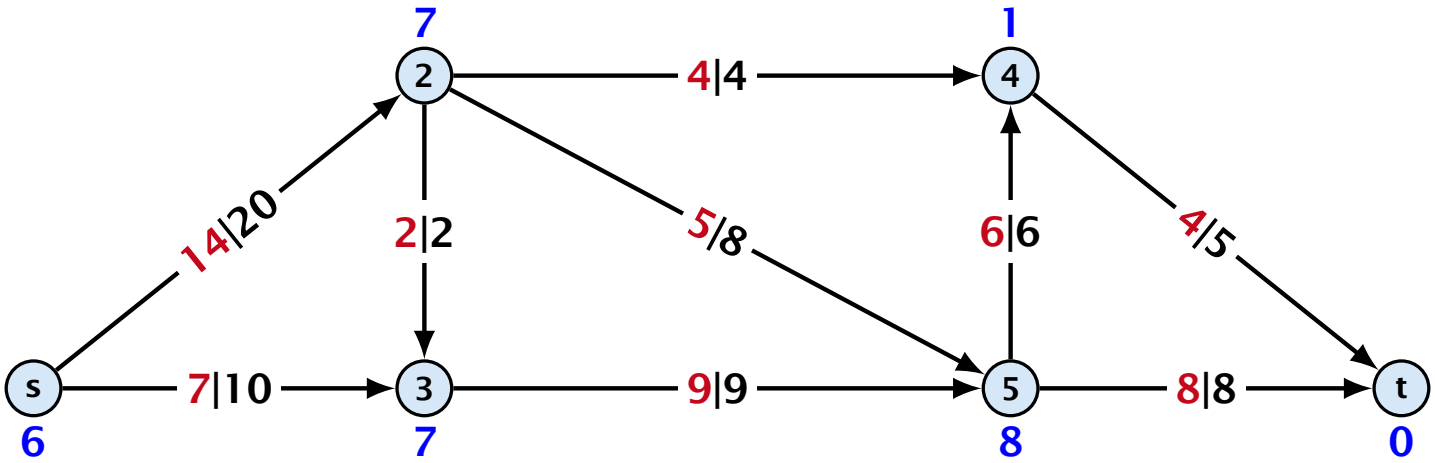
Preflow Push Algorithm



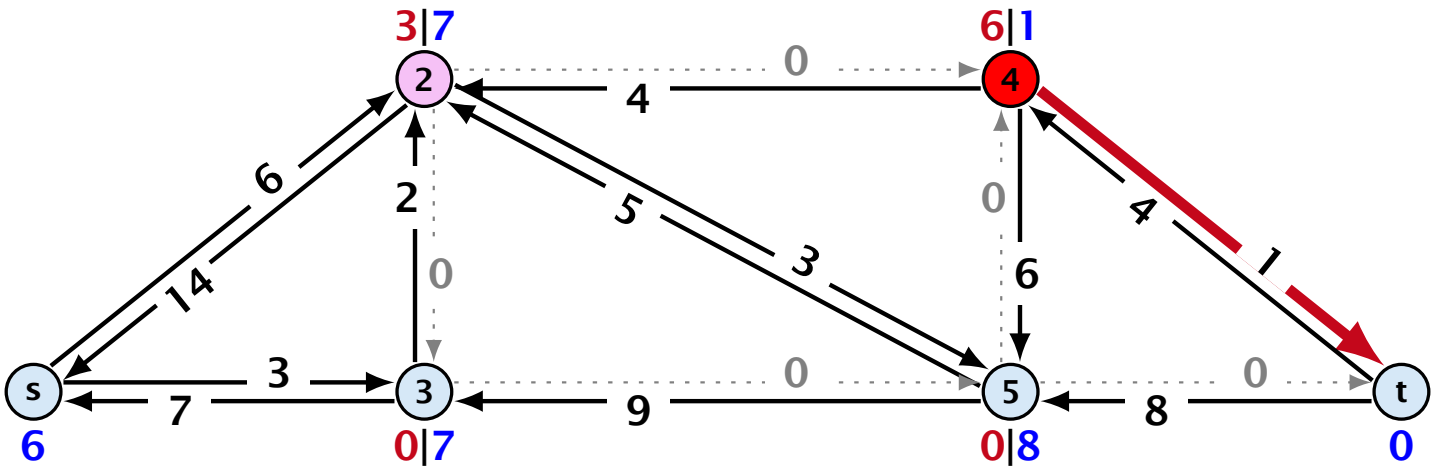
Preflow Push Algorithm

push

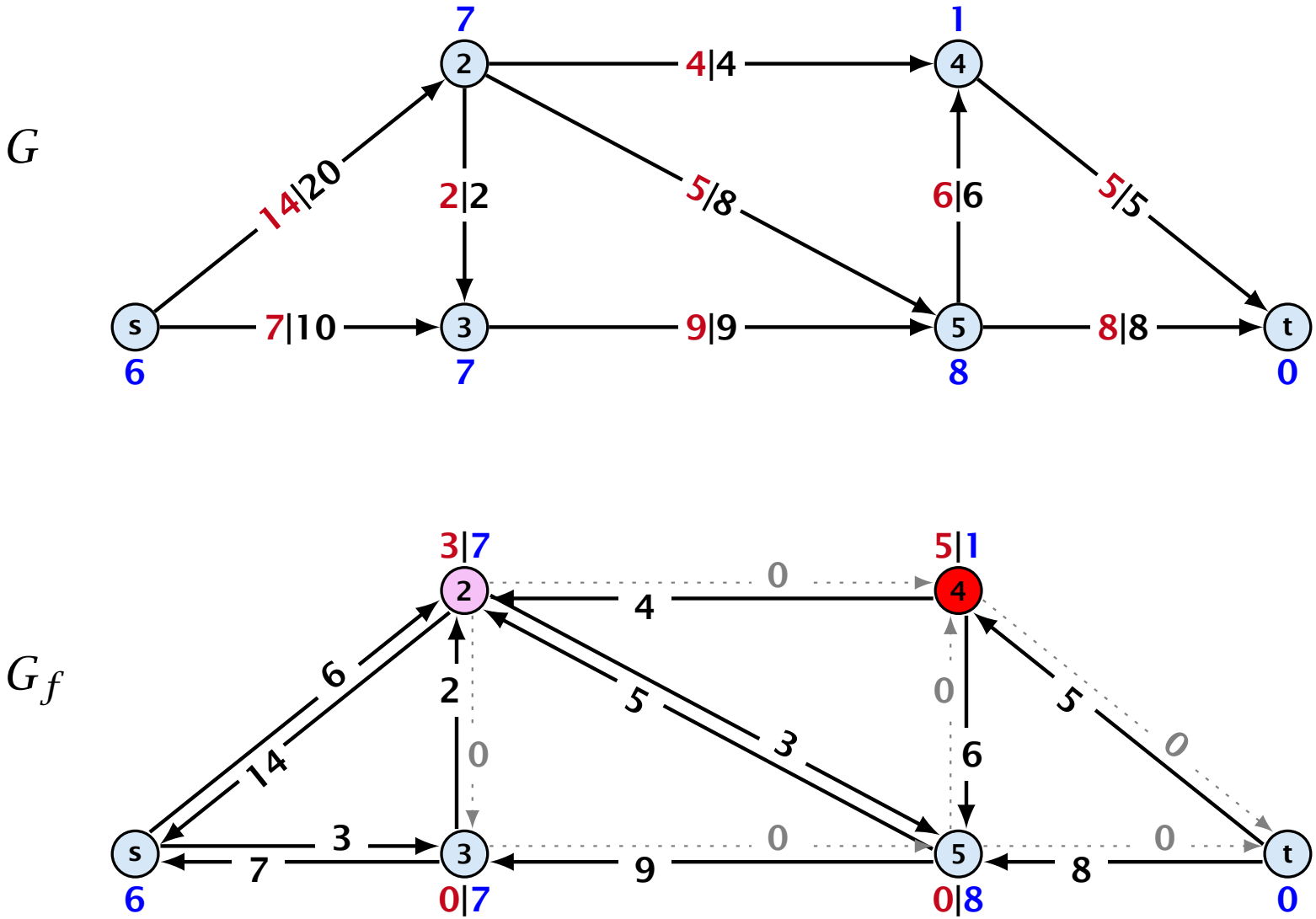
G



G_f

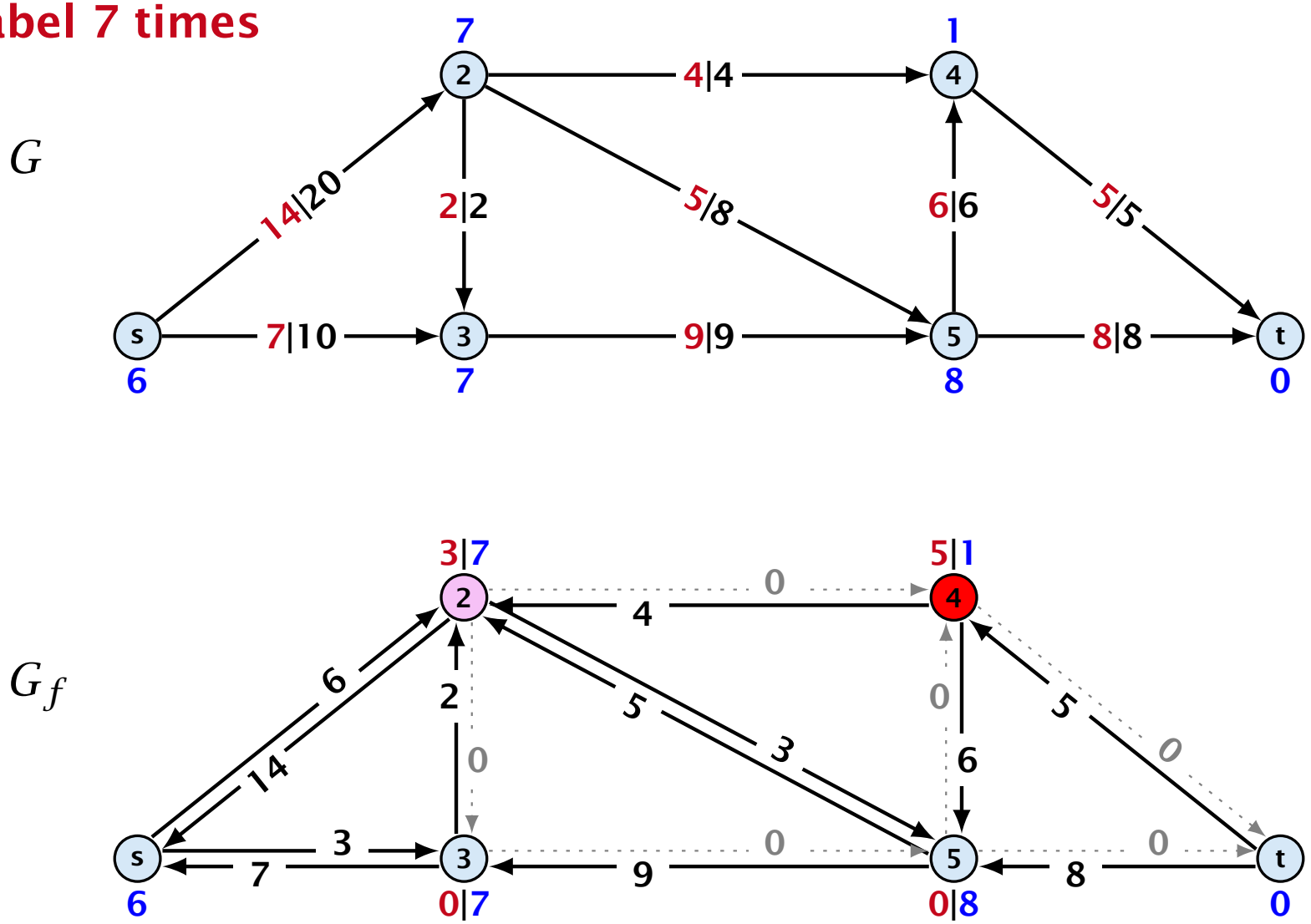


Preflow Push Algorithm

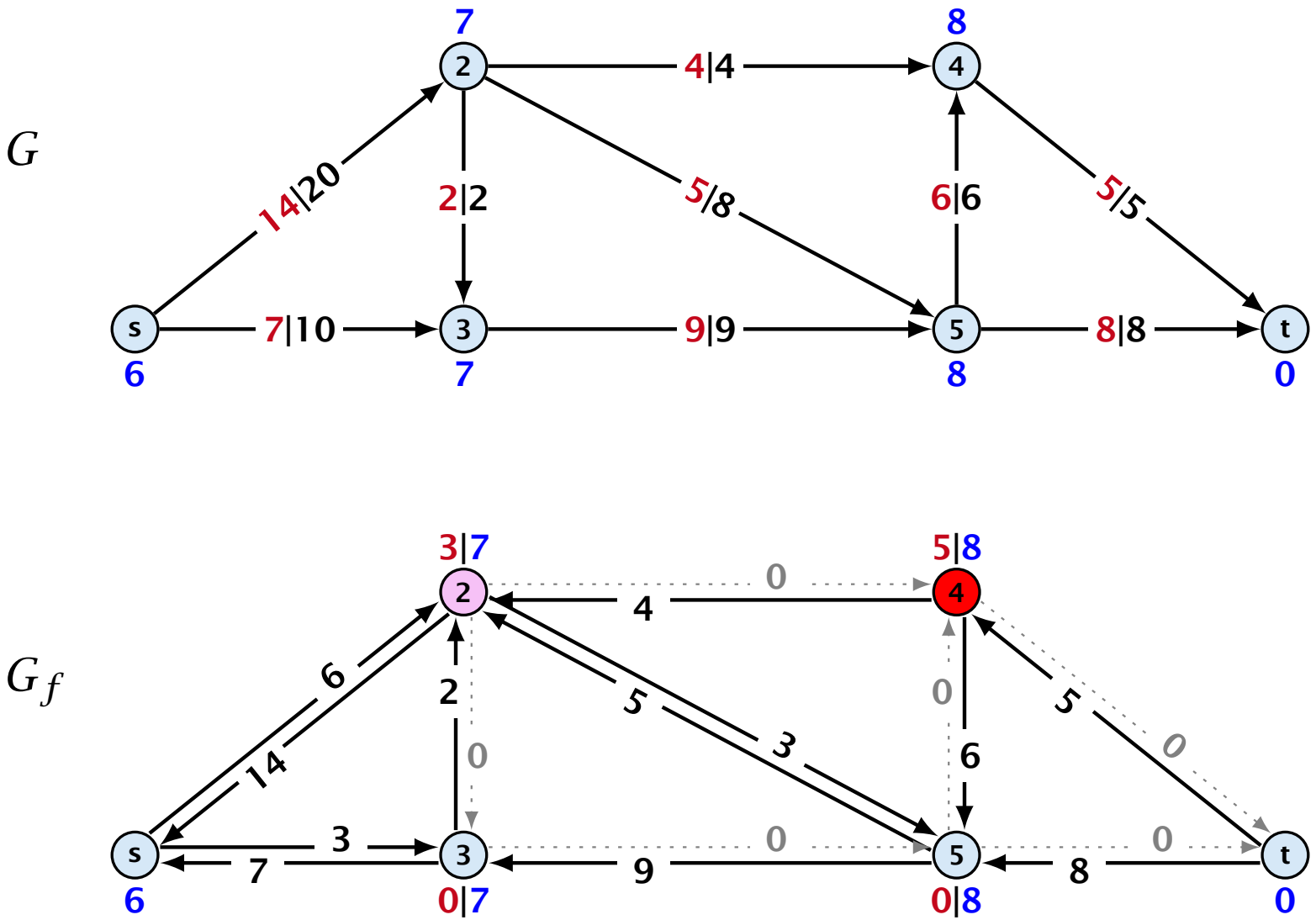


Preflow Push Algorithm

relabel 7 times

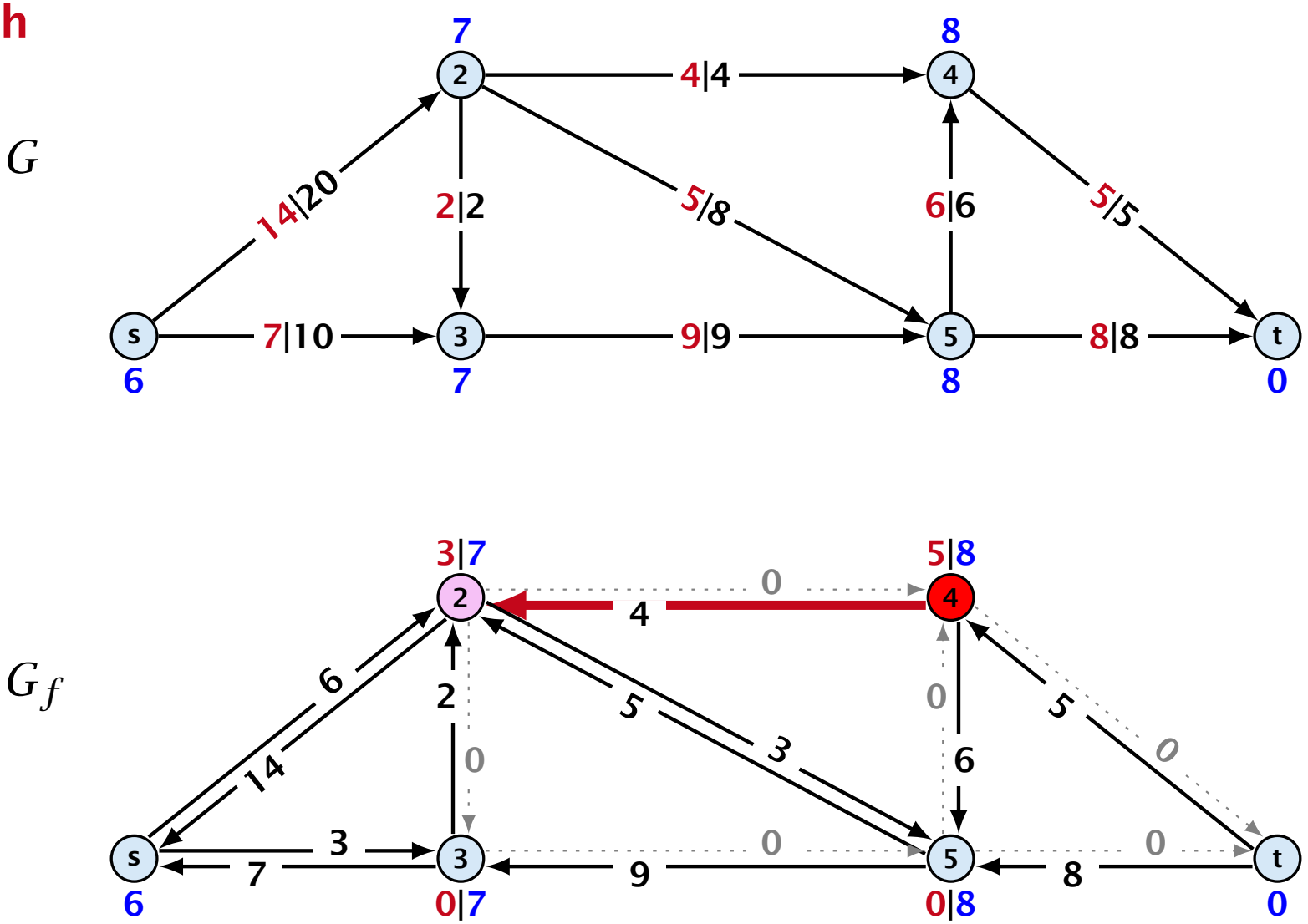


Preflow Push Algorithm

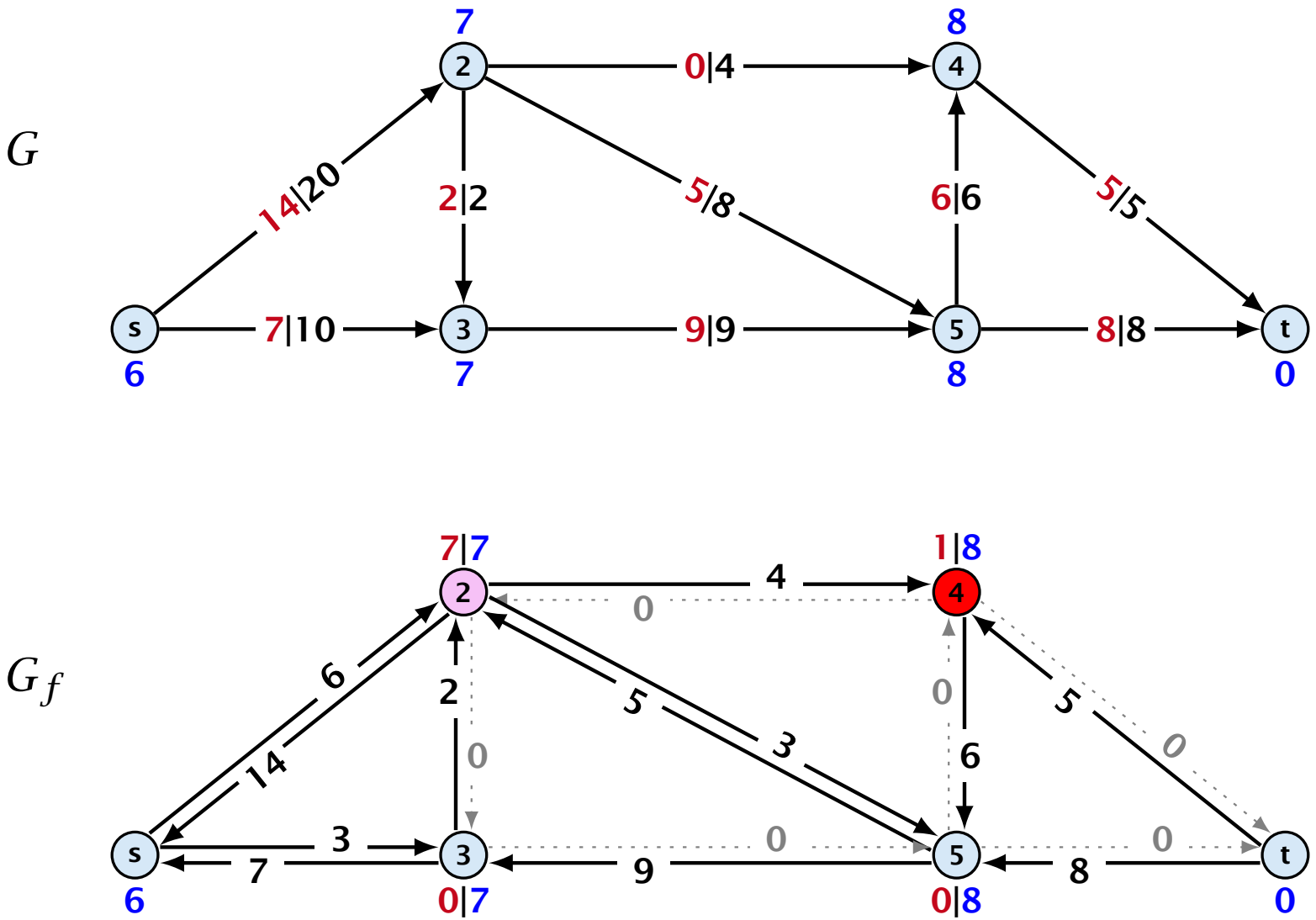


Preflow Push Algorithm

push

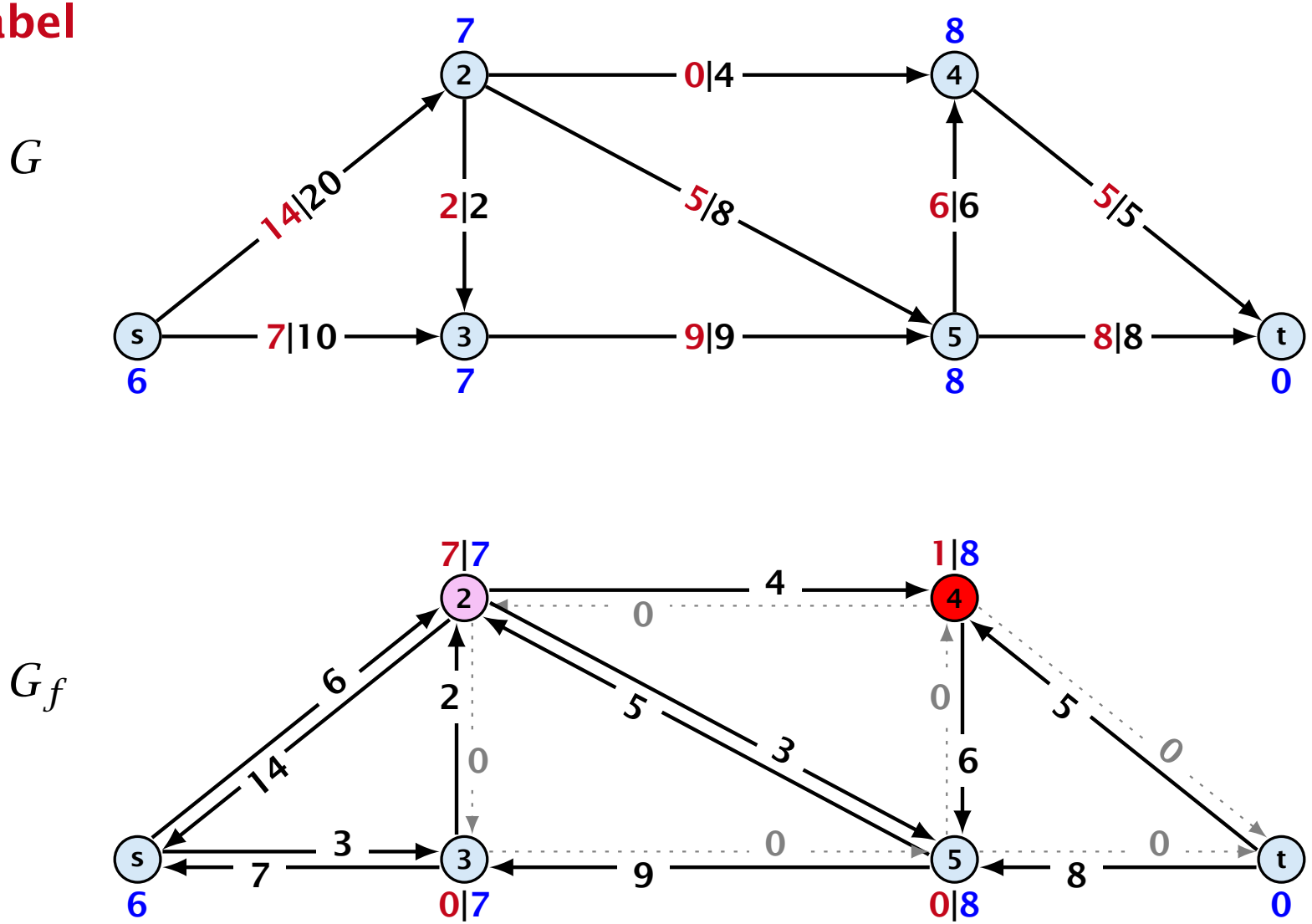


Preflow Push Algorithm

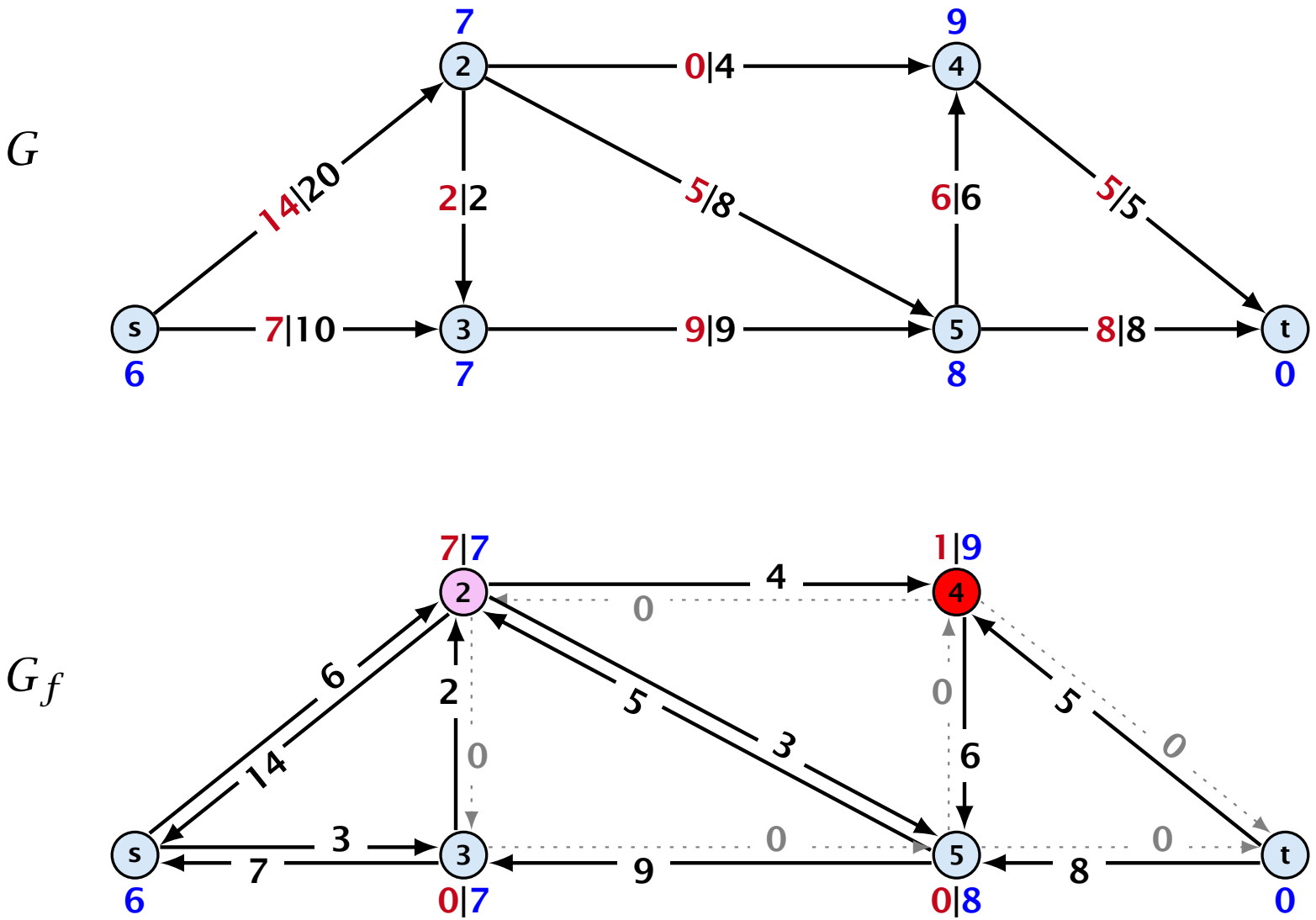


Preflow Push Algorithm

relabel

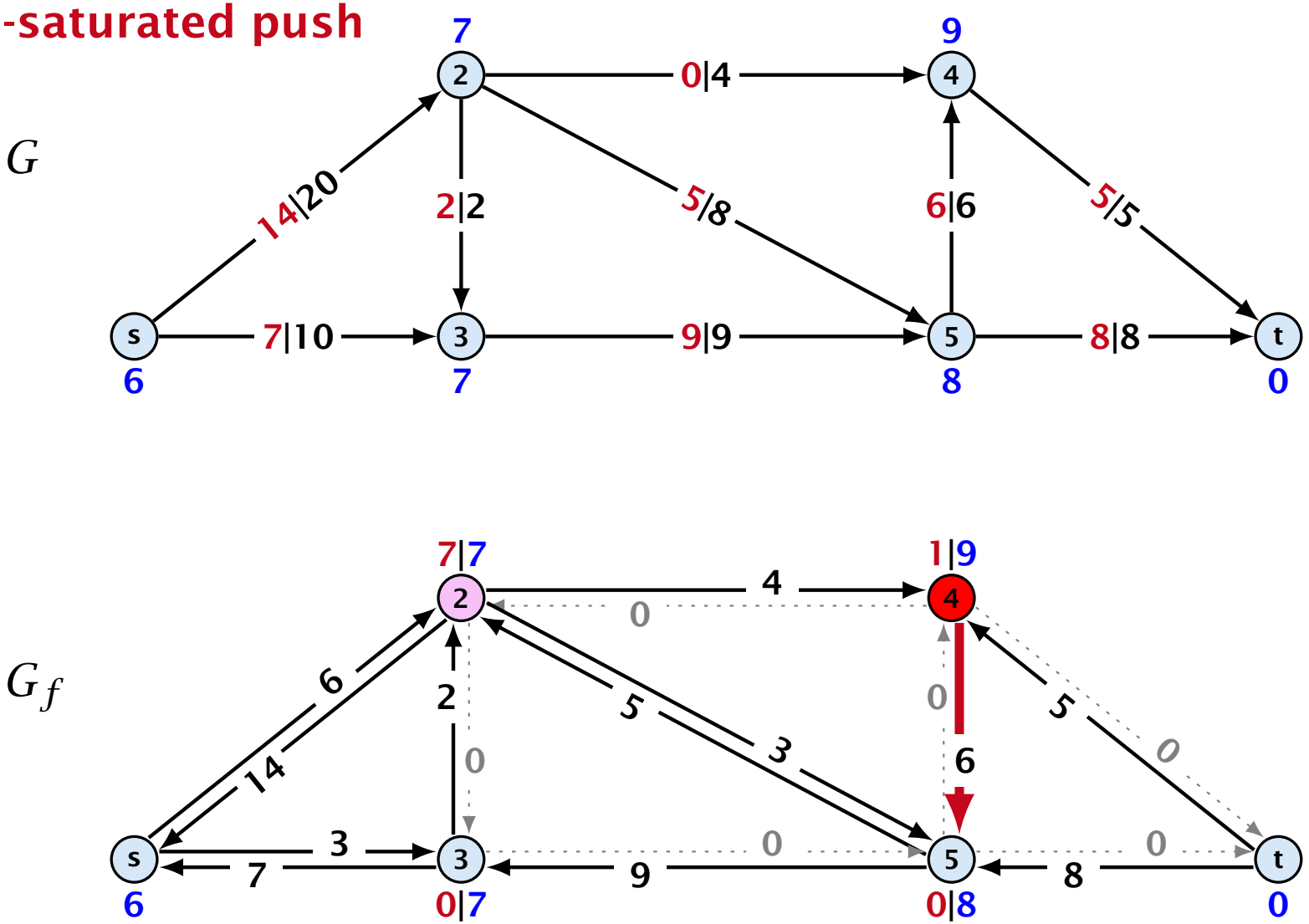


Preflow Push Algorithm



Preflow Push Algorithm

non-saturated push



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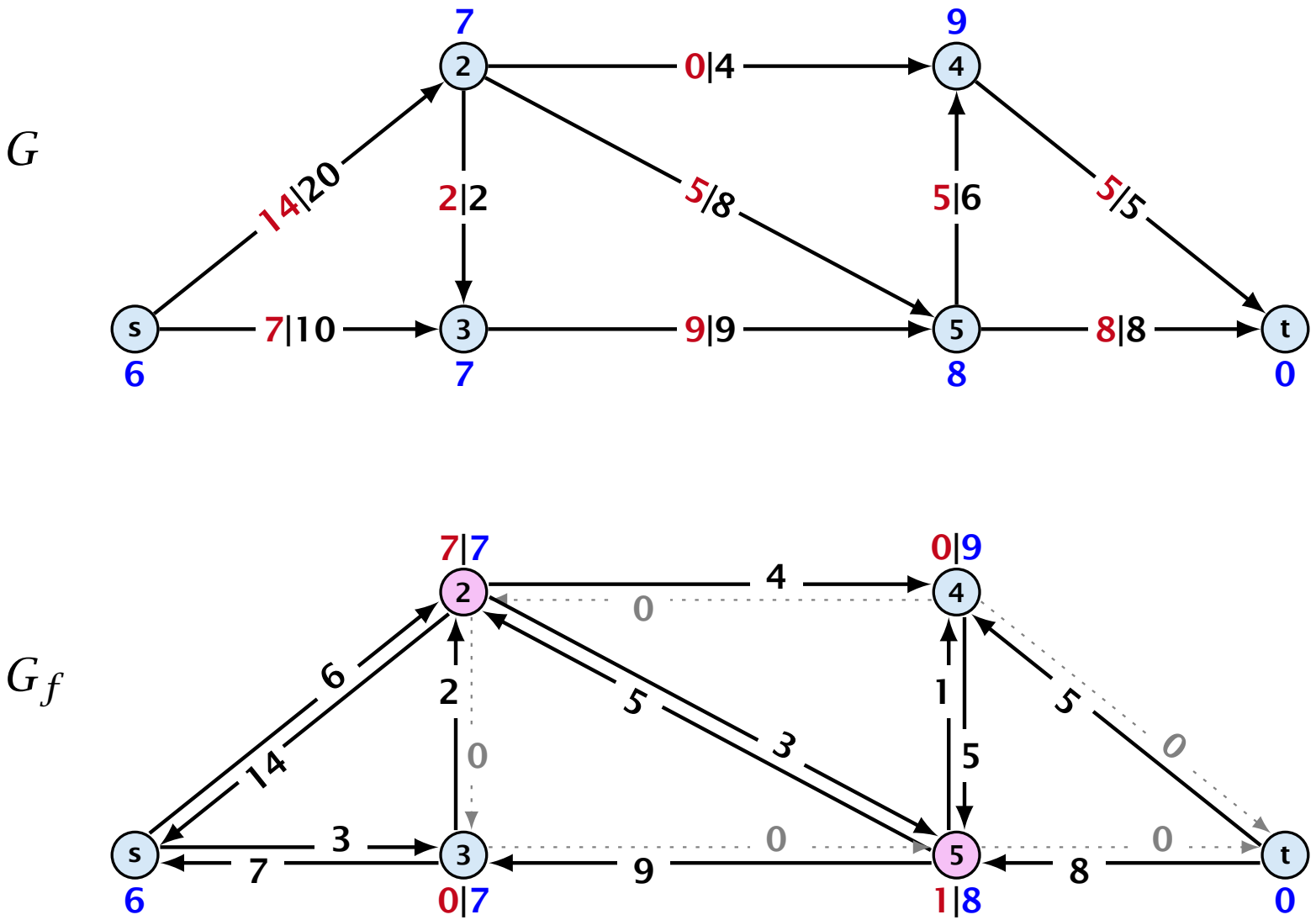

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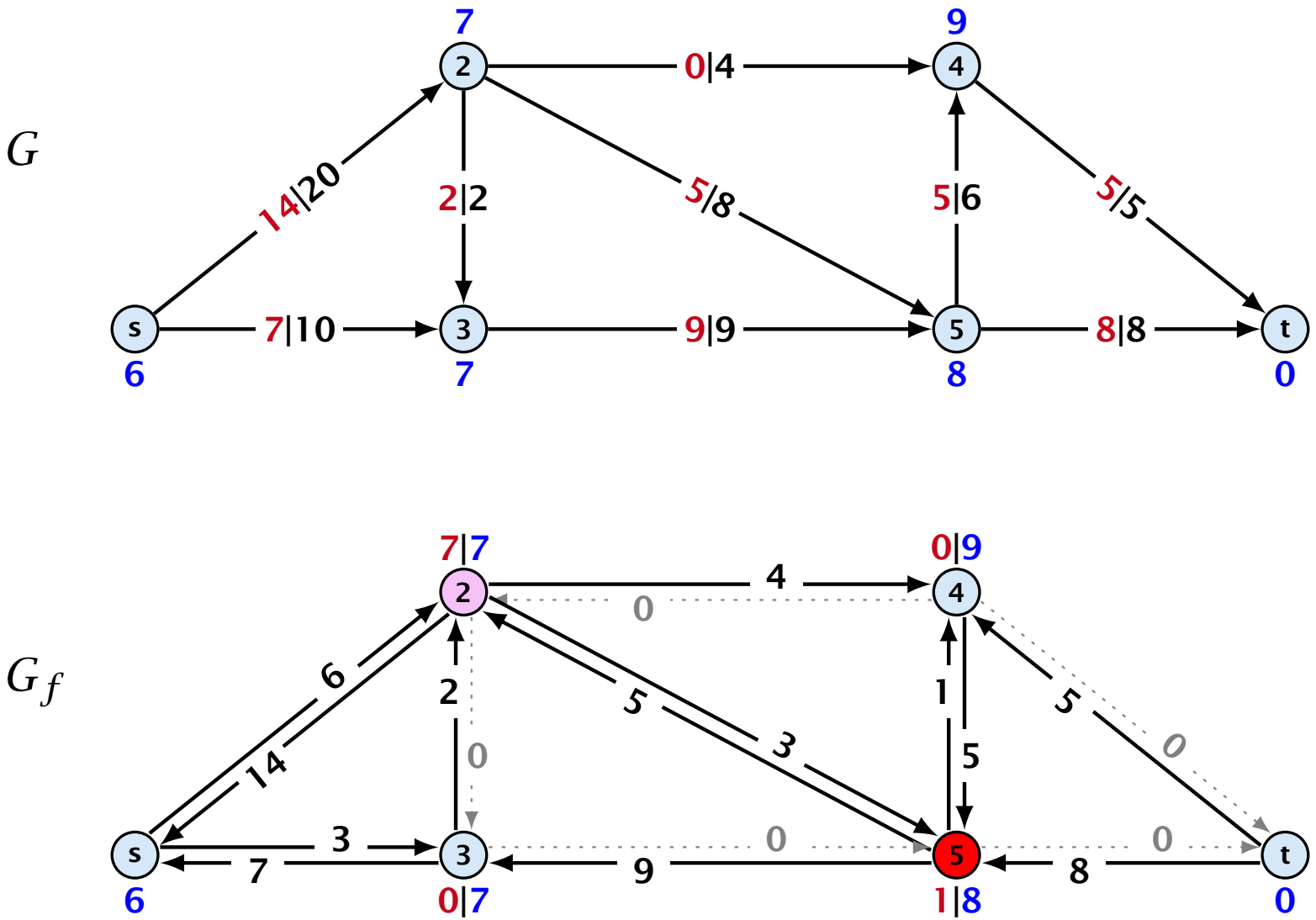

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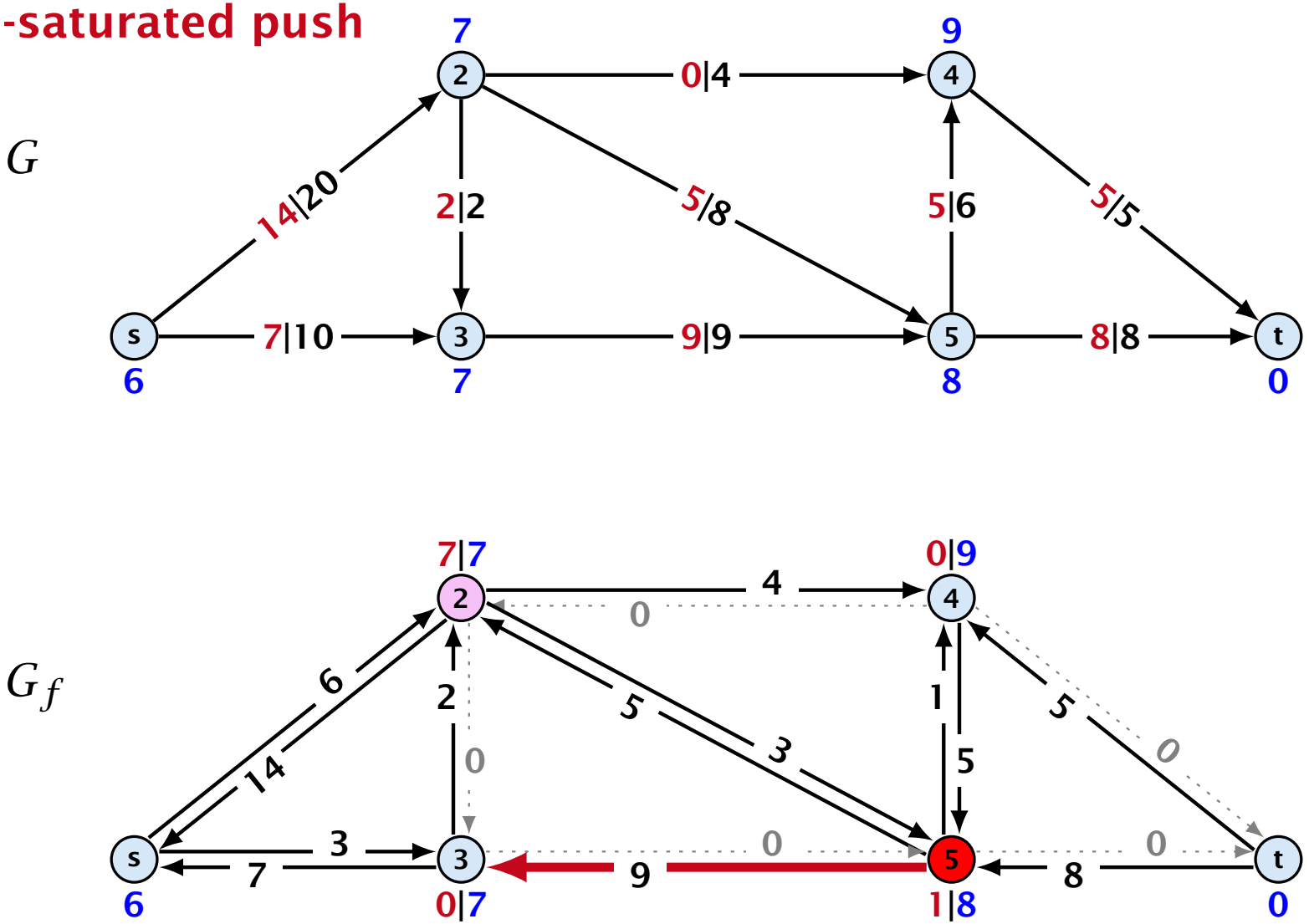

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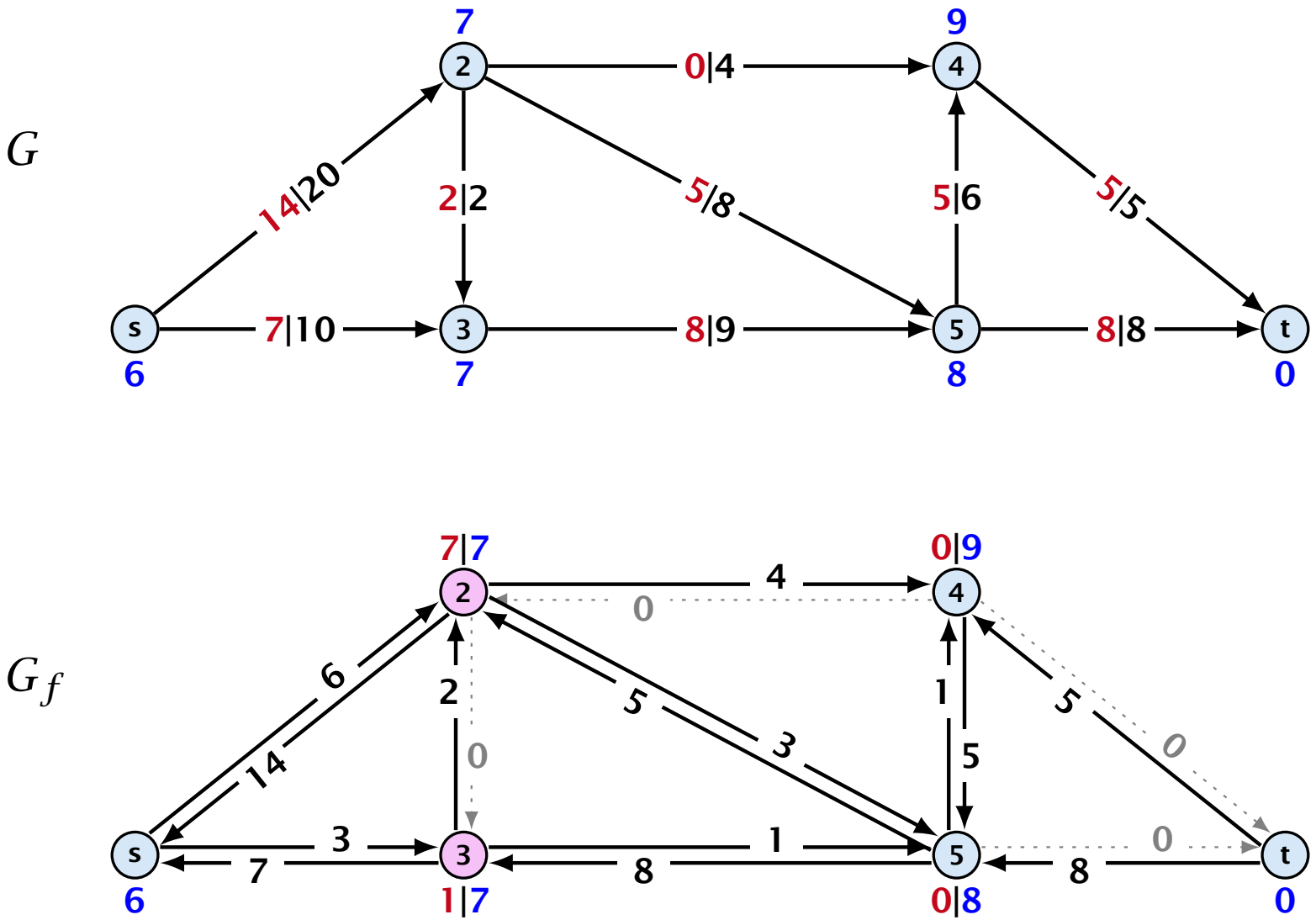

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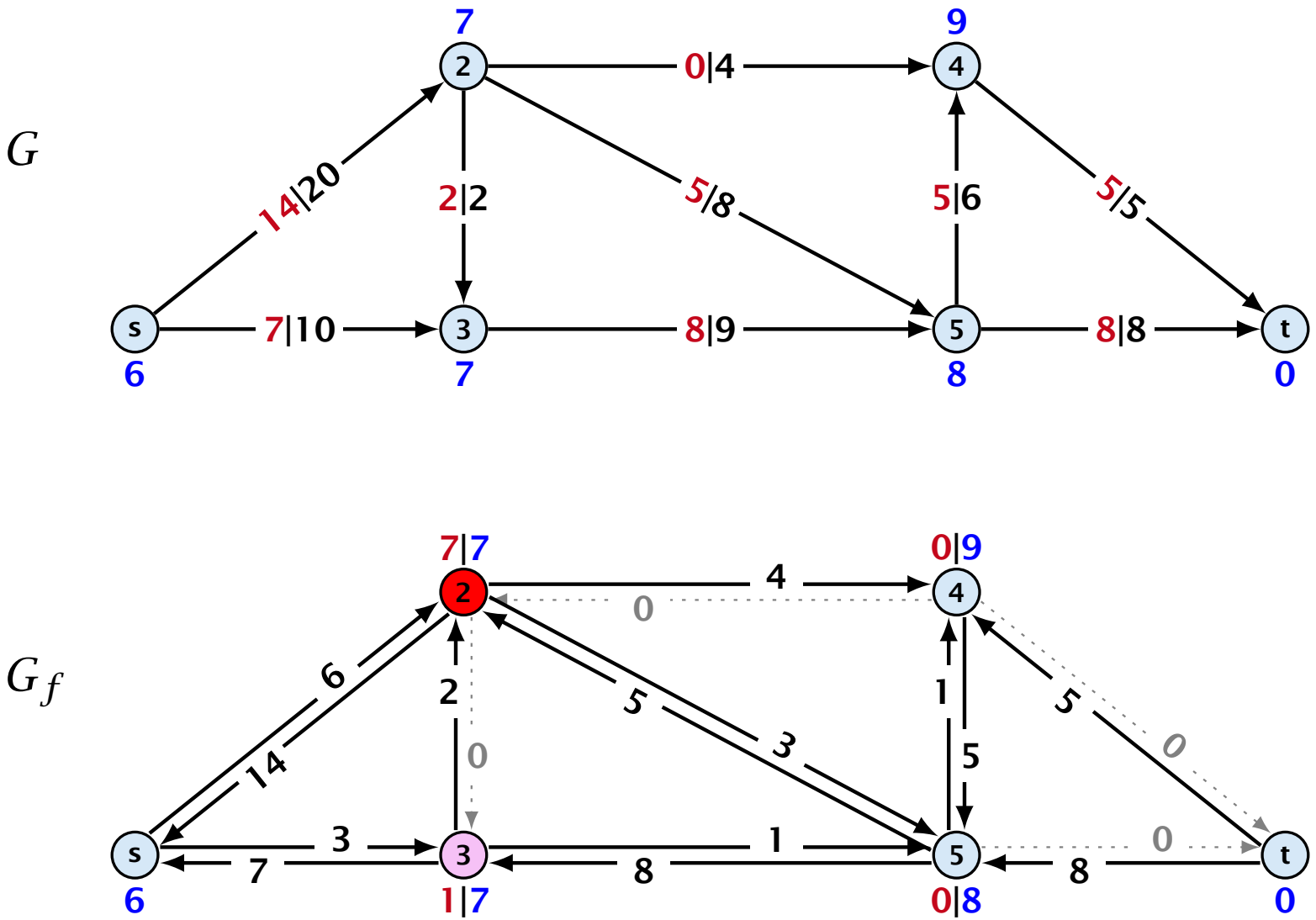

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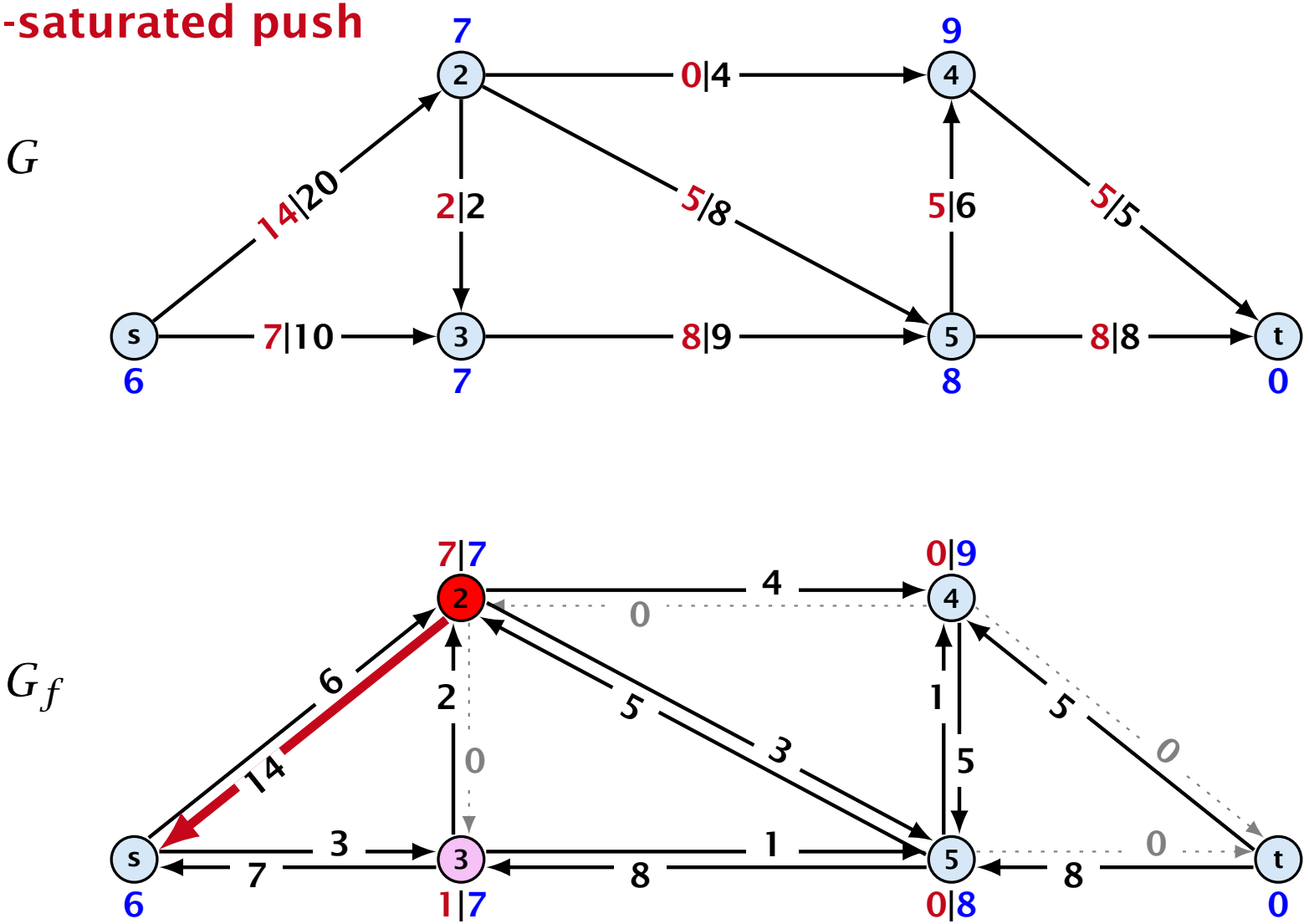

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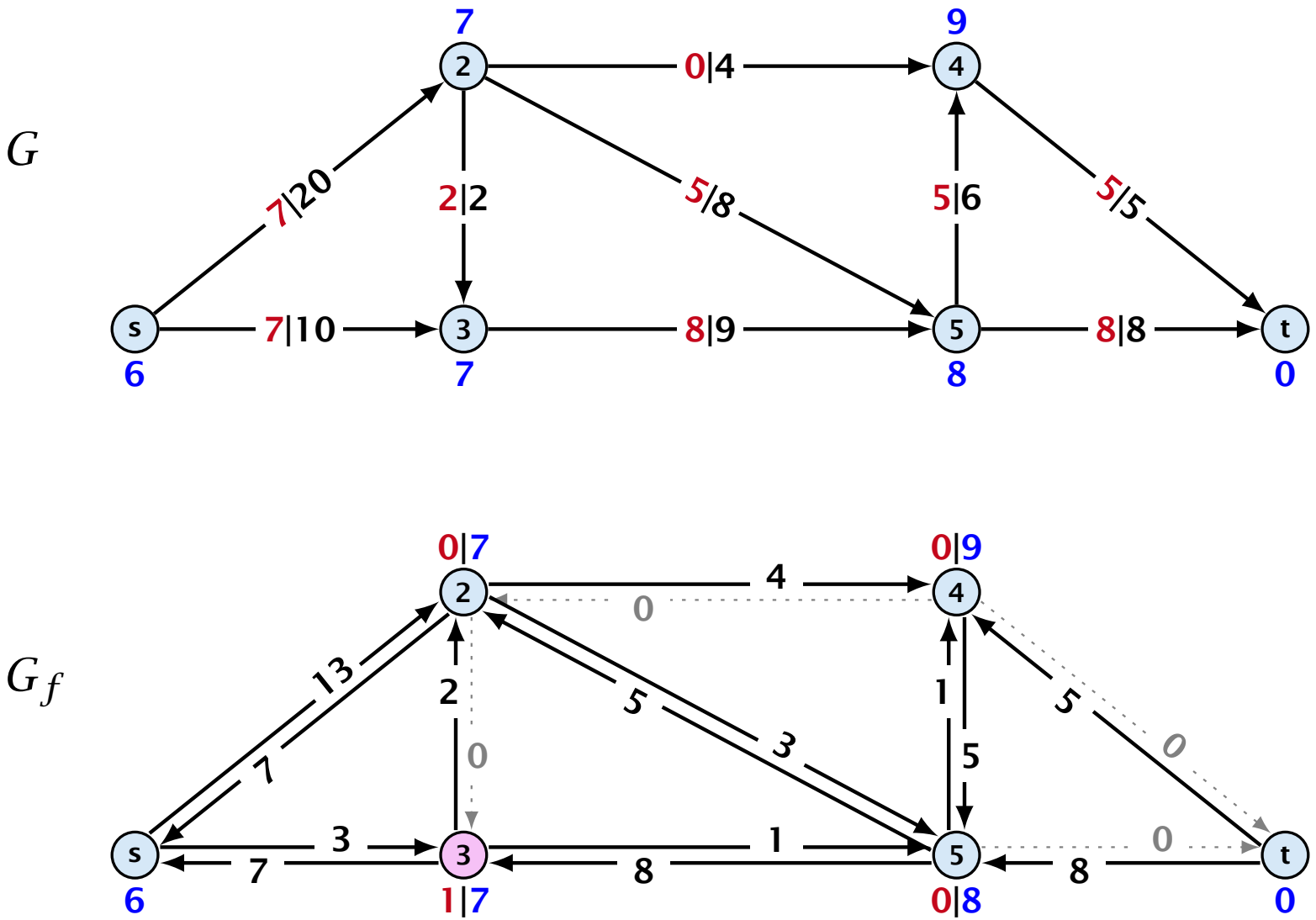

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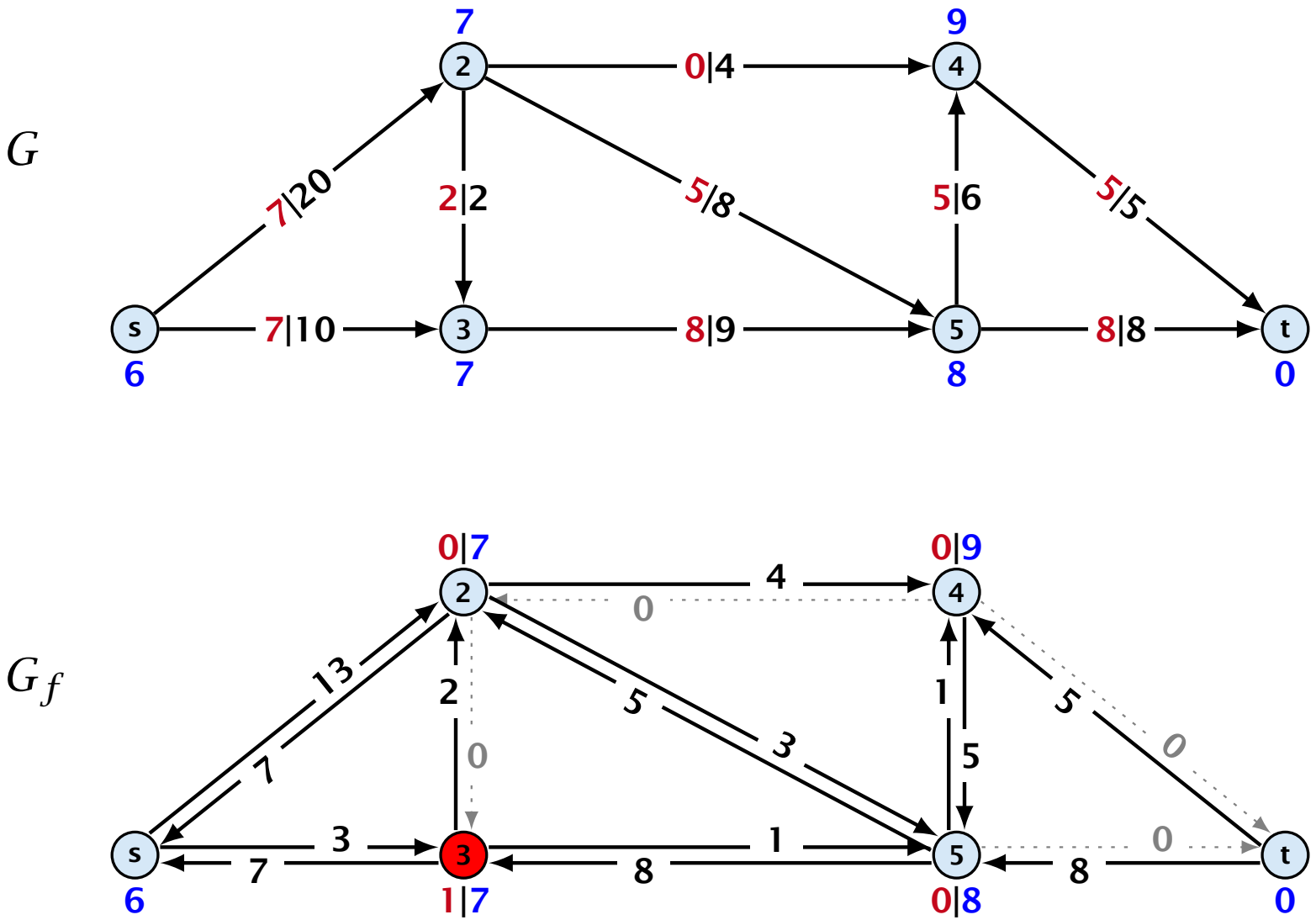
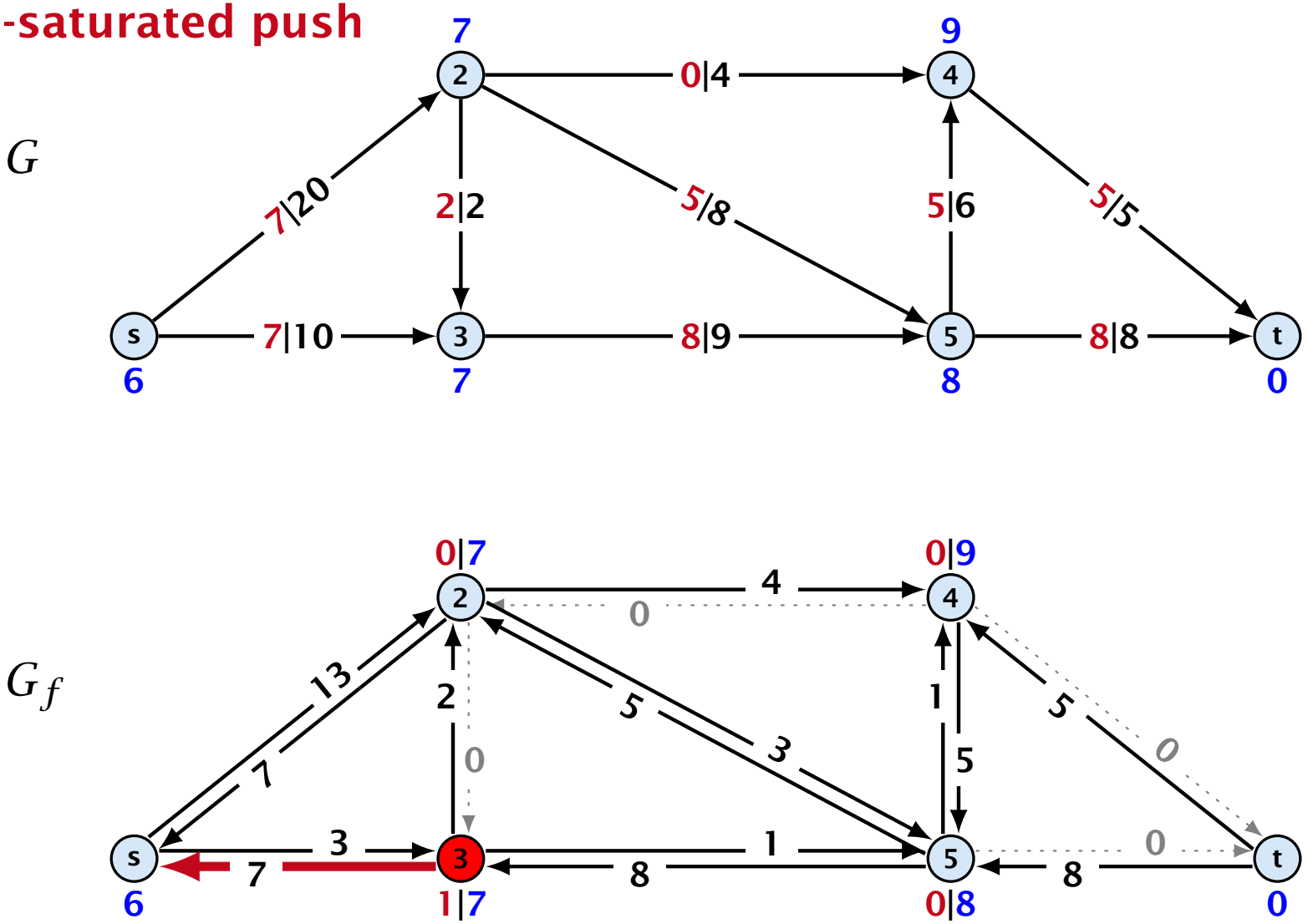


Preflow Push Algorithm

non-saturated push



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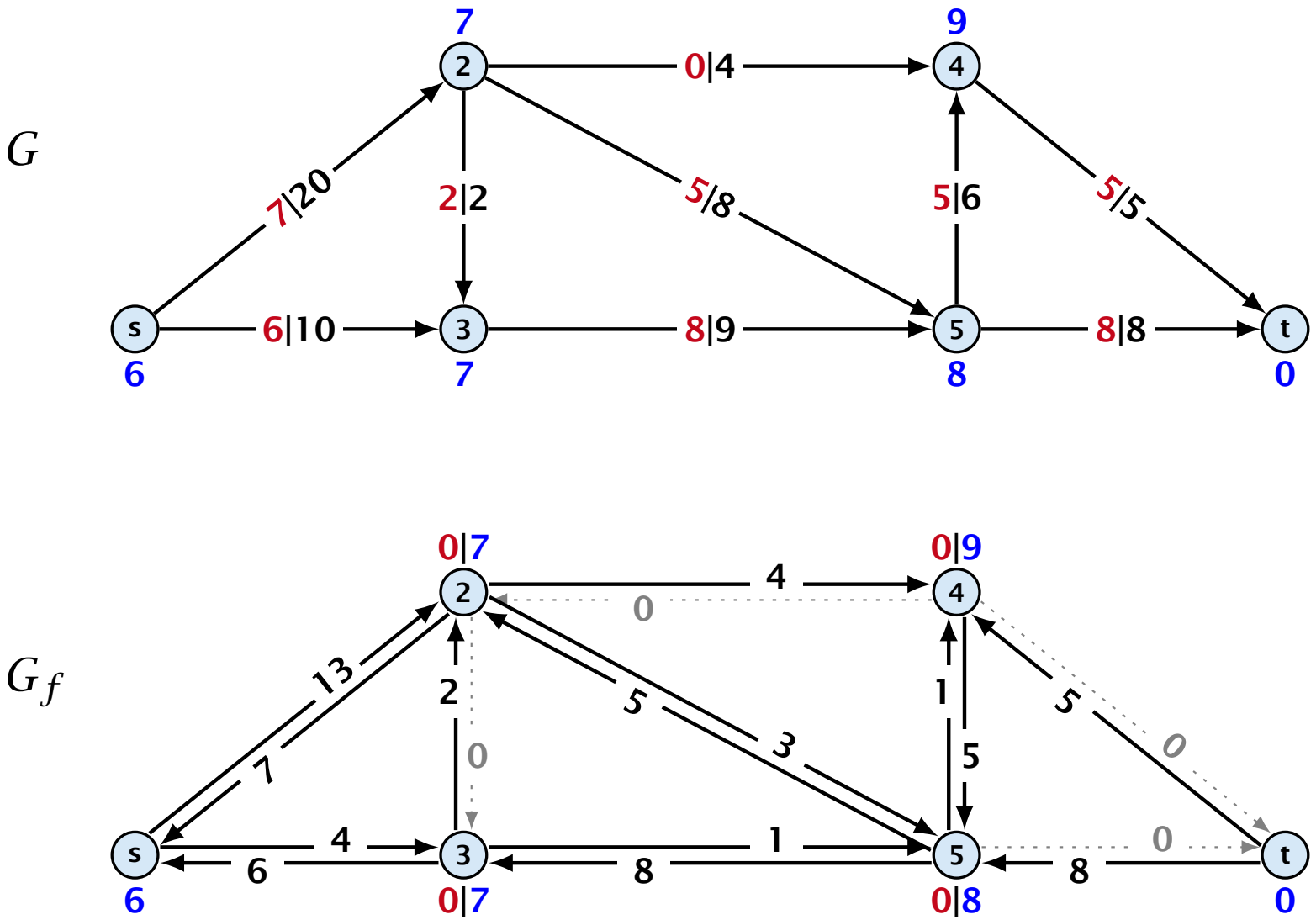

Analysis

Lemma 72

An active node has a path to s in the residual graph.

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- Let A denote the set of nodes that can reach s, and let B denote the remaining nodes. Note that $s \in A$.

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- In the following we show that a node $b \in B$ has excess flow $f(b) = 0$ which gives the lemma.
- In the residual graph there are no edges into A, and, hence, no edges leaving A/entering B can carry any flow.
- Let $f(B) = \sum_{v \in B} f(v)$ be the excess flow of all nodes in B.

Let $f : E \to \mathbb{R}_0^+$ be a preflow. We introduce the notation

$$f(x,y) = \begin{cases} 0 & (x,y) \notin E \\ f((x,y)) & (x,y) \in E \end{cases}$$

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Hence, the excess flow $f(b)$ must be 0 for every node $b \in B$.

Analysis

Lemma 73

The label of a node cannot become larger than $2n - 1$.

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Proof.

- When increasing the label at a node u there exists a path from u to s of length at most $n-1$. Along each edge of the path the height/label can at most drop by 1, and the label of the source is n.

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Proof.

- When increasing the label at a node u there exists a path from u to s of length at most $n-1$. Along each edge of the path the height/label can at most drop by 1, and the label of the source is n.

Lemma 74

There are only $\mathcal{O}(n^2)$ relabel operations.

Analysis

Lemma 75

The number of saturating pushes performed is at most $\mathcal{O}(mn)$.

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- Suppose that we just made a saturating push along (u, v).

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The number of saturating pushes performed is at most $\mathcal{O}(mn)$.

Proof.

- Suppose that we just made a saturating push along (u, v).
- Hence, the edge (u, v) is deleted from the residual graph.
- For the edge to appear again, a push from v to u is required.

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- Currently, $\ell(u) = \ell(v) + 1$, as we only make pushes along admissable edges.

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- For a push from v to u the edge (v, u) must become admissable. The label of v must increase by at least 2.
- Since the label of v is at most $2n - 1$, there are at most n pushes along (u, v).

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The number of non-saturating pushes performed is at most $\mathcal{O}(n^2 m)$.

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Proof.

- Define a potential function $\Phi(f) = \sum_{\text{active nodes} v} \ell(v)$
- A saturating push increases Φ by $\leq 2n$ (when the target node becomes active it may contribute at most $2n$ to the sum).
- A relabel increases Φ by at most 1.
- A non-saturating push decreases Φ by at least 1 as the node that is pushed from becomes inactive and has a label that is strictly larger than the target.

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The number of non-saturating pushes performed is at most $\mathcal{O}(n^2 m)$.

Proof.

- Define a potential function $\Phi(f) = \sum_{\text{active nodes} v} \ell(v)$
- A saturating push increases Φ by $\leq 2n$ (when the target node becomes active it may contribute at most $2n$ to the sum).
- A relabel increases Φ by at most 1.
- A non-saturating push decreases Φ by at least 1 as the node that is pushed from becomes inactive and has a label that is strictly larger than the target.
- Hence,

$$\begin{aligned}\#\text{non-saturating\_pushes} &\leq \#\text{relabels} + 2n \cdot \#\text{saturating\_pushes} \\ &\leq \mathcal{O}(n^2 m) \ .\end{aligned}$$

Analysis

Theorem 77

There is an implementation of the generic push relabel algorithm with running time $\mathcal{O}(n^2 m)$.

Analysis

Proof:

For every node maintain a list of admissable edges starting at that node. Further maintain a list of active nodes.

A push along an edge (u,v) can be performed in constant time

- check whether edge (v,u) needs to be added to E_f
- check whether (u,v) needs to be deleted (saturating push)
- check whether v becomes inactive and has to be deleted from the set of active nodes

A relabel at a node u can be performed in time $\Theta(n)$

- check for all outgoing edges if they become admissable
- check for all incoming edges if they become non-admissable

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Proof:

For every node maintain a list of admissable edges starting at that node. Further maintain a list of active nodes.

A push along an edge (u,v) can be performed in constant time

- check whether edge (v,u) needs to be added to v
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- check whether v becomes inactive and has to be deleted from the set of active nodes

A relabel at a node u can be performed in time $\Theta(n)$

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- check whether u becomes inactive and has to be deleted from the set of active nodes

A relabel at a node u can be performed in time $\mathcal{O}(n)$

- check for all outgoing edges if they become admissable
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A relabel at a node u can be performed in time $\mathcal{O}(n)$

- check for all outgoing edges if they become admissable
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Analysis

For special variants of push relabel algorithms we organize the neighbours of a node into a linked list (possible neighbours in the residual graph G_f). Then we use the discharge-operation:

Algorithm 4 discharge(u)

```
1: while u is active do
2:     v ← u.current-neighbour
3:     if v = null then
4:         relabel(u)
5:         u.current-neighbour ← u.neighbour-list-head
6:     else
7:         if (u, v) admissable then push(u, v)
8:         else u.current-neighbour ← v.next-in-list
```

Note that *u.current-neighbour* is a global variable. It is only changed within the discharge routine, but keeps its value between consecutive calls to discharge.

Lemma 78

If $v = \text{null}$ in Line 3, then there is no outgoing admissable edge from u.

Proof.

- While pushing from u the current-neighbour pointer is only advanced if the current edge is not admissable.
- The only thing that could make the edge admissable again would be a relabel at u.
- If we reach the end of the list ($v = \text{null}$) all edges are not admissable. □

This shows that discharge(u) is correct, and that we can perform a relabel in Line 4.

14.2 Relabel to Front

Algorithm 50 relabel-to-front(G, s, t)

```
 1: initialize preflow
 2: initialize node list L containing V \ {s,t} in any order
 3: foreach u ∈ V \ {s,t} do
 4:       u.current-neighbour ← u.neighbour-list-head
 5: u ← L.head
 6: while u ≠ null do
 7:       old-height ← ℓ(u)
 8:       discharge(u)
 9:       if ℓ(u) > old-height then // relabel happened
10:             move u to the front of L
11:       u ← u.next
```

14.2 Relabel to Front

Lemma 79 (Invariant)

In Line 6 of the relabel-to-front algorithm the following invariant holds.

1. *The sequence L is topologically sorted w.r.t. the set of admissable edges; this means for an admissable edge (x, y) the node x appears before y in sequence L.*
2. *No node before u in the list L is active.*

Proof:

- Initialization:
  1. In the beginning s has label $n \geq 2$, and all other nodes have label 0. Hence, no edge is admissable, which means that any ordering L is permitted.
  2. We start with u being the head of the list; hence no node before u can be active
- Maintenance:
  1. - Pushes do no create any new admissable edges. Therefore, if discharge() does not relabel u, L is still topologically sorted.
     - After relabeling, u cannot have admissable incoming edges as such an edge (x, u) would have had a difference $\ell(x) - \ell(u) \geq 2$ before the re-labeling (such edges do not exist in the residual graph).
       Hence, moving u to the front does not violate the sorting property for any edge; however it fixes this property for all admissable edges leaving u that were generated by the relabeling.

14.2 Relabel to Front

Proof:

- Maintenance:
  2. If we do a relabel there is nothing to prove because the only node before u' (u in the next iteration) will be the current u; the discharge(u) operation only terminates when u is not active anymore.

     For the case that we do not relabel, observe that the only way a predecessor could be active is that we push flow to it via an admissable arc. However, all admissable arc point to successors of u.

Note that the invariant means that for $u = \text{null}$ we have a preflow with a valid labelling that does not have active nodes. This means we have a maximum flow.

14.2 Relabel to Front

Lemma 80

There are at most $\mathcal{O}(n^3)$ calls to discharge(u).

Every discharge operation without a relabel advances u (the current node within list L). Hence, if we have n discharge operations without a relabel we have $u = \text{null}$ and the algorithm terminates.

Therefore, the number of calls to discharge is at most $n(\#relabels + 1) = \mathcal{O}(n^3)$.

14.2 Relabel to Front

Lemma 81

The cost for all relabel-operations is only $\mathcal{O}(n^2)$.

A relabel-operation at a node is constant time (increasing the label and resetting $u.\mathit{current\text{-}neighbour}$). In total we have $\mathcal{O}(n^2)$ relabel-operations.

14.2 Relabel to Front

Note that by definition a saturing push operation ($\min\{c_f(e), f(u)\} = c_f(e)$) can at the same time be a non-saturating push operation ($\min\{c_f(e), f(u)\} = f(u)$).

Lemma 82

*The cost for all saturating push-operations that are **not** also non-saturating push-operations is only $\mathcal{O}(mn)$.*

Note that such a push-operation leaves the node u active but makes the edge e disappear from the residual graph. Therefore the push-operation is immediately followed by an increase of the pointer $u.current\text{-}neighbour$.
This pointer can traverse the neighbour-list at most $\mathcal{O}(n)$ times (upper bound on number of relabels) and the neighbour-list has only $degree(u) + 1$ many entries ($+1$ for null-entry).

14.2 Relabel to Front

Lemma 83

The cost for all non-saturating push-operations is only $\mathcal{O}(n^3)$.

A non-saturating push-operation takes constant time and ends the current call to discharge(). Hence, there are only $\mathcal{O}(n^3)$ such operations.

Theorem 84

The push-relabel algorithm with the rule relabel-to-front takes time $\mathcal{O}(n^3)$.

14.3 Highest label

Algorithm 6 highest-label(G, s, t)

1: initialize preflow
2: **foreach** $u \in V \setminus \{s, t\}$ **do**
3: $\quad u.\mathit{current\text{-}neighbour} \leftarrow u.\mathit{neighbour\text{-}list\text{-}head}$
4: **while** $\exists$ active node u **do**
5: $\quad$ select active node u with highest label
6: $\quad$ discharge(u)

14.3 Highest label

Lemma 85

When using highest label the number of non-saturating pushes is only $\mathcal{O}(n^3)$.

A push from a node on level ℓ can only "activate" nodes on levels strictly less than ℓ.

This means, after a non-saturating push from u a relabel is required to make u active again.

Hence, after n non-saturating pushes without an intermediate relabel there are no active nodes left.

Therefore, the number of non-saturating pushes is at most $n(\#relabels + 1) = \mathcal{O}(n^3)$.

14.3 Highest label

Since a discharge-operation is terminated by a non-saturating push this gives an upper bound of $\mathcal{O}(n^3)$ on the number of discharge-operations.

The cost for relabels and saturating pushes can be estimated in exactly the same way as in the case of the generic push-relabel algorithm.

Question:
How do we find the next node for a discharge operation?

14.3 Highest label

Maintain lists L_i, $i \in \{0, \ldots, 2n\}$, where list L_i contains active nodes with label i (maintaining these lists induces only constant additional cost for every push-operation and for every relabel-operation).

After a discharge operation terminated for a node u with label k, traverse the lists $L_k, L_{k-1}, \ldots, L_0$, (in that order) until you find a non-empty list.

Unless the last (non-saturating) push was to s or t the list $k-1$ must be non-empty (i.e., the search takes constant time).

14.3 Highest label

Hence, the total time required for searching for active nodes is at most

$$\mathcal{O}(n^3) + n(\#\textit{non-saturating-pushes-to-s-or-t})$$

Lemma 86

The number of non-saturating pushes to s or t is at most $\mathcal{O}(n^2)$.

With this lemma we get

Theorem 87

The push-relabel algorithm with the rule highest-label takes time $\mathcal{O}(n^3)$.

14.3 Highest label

Proof of the Lemma.

- We only show that the number of pushes to the source is at most $\mathcal{O}(n^2)$. A similar argument holds for the target.
- After a node v (which must have $\ell(v) = n+1$) made a non-saturating push to the source there needs to be another node whose label is increased from $\leq n+1$ to $n+2$ before v can become active again.
- This happens for every push that v makes to the source. Since, every node can pass the threshold $n+2$ at most once, v can make at most n pushes to the source.
- As this holds for every node the total number of pushes to the source is at most $\mathcal{O}(n^2)$.

Mincost Flow

Problem Definition:

$$\begin{array}{ll} \min & \sum_e c(e) f(e) \\ \text{s.t.} & \forall e \in E: \;\; 0 \le f(e) \le u(e) \\ & \forall v \in V: \;\; f(v) = b(v) \end{array}$$

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- $G = (V, E)$ is a directed graph.
- $u : E \to \mathbb{R}_0^+ \cup \{\infty\}$ is the capacity function.
- $c : E \to \mathbb{R}$ is the cost function
  (note that $c(e)$ may be negative).
- $b : V \to \mathbb{R}$, $\sum_{v \in V} b(v) = 0$ is a demand function.

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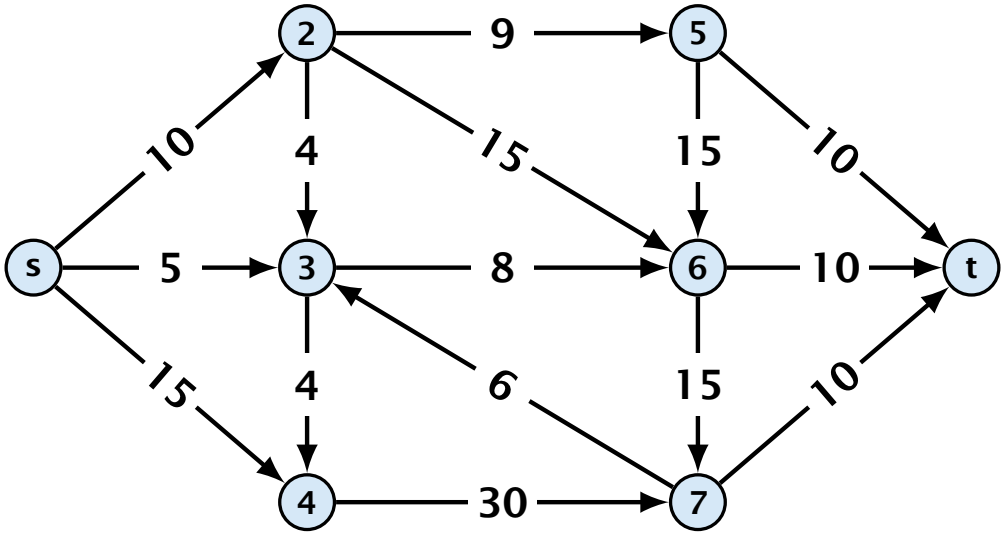
Solve Maxflow Using Mincost Flow



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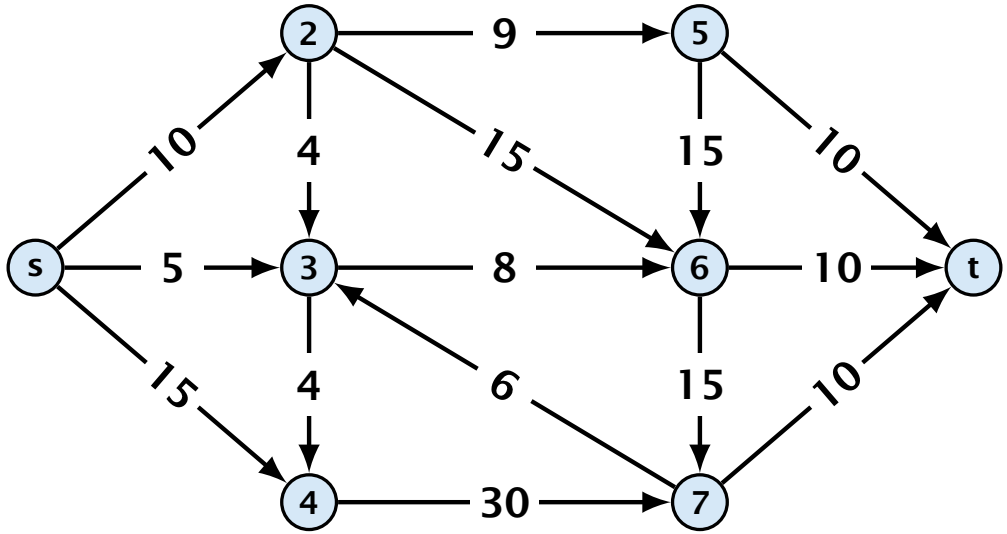
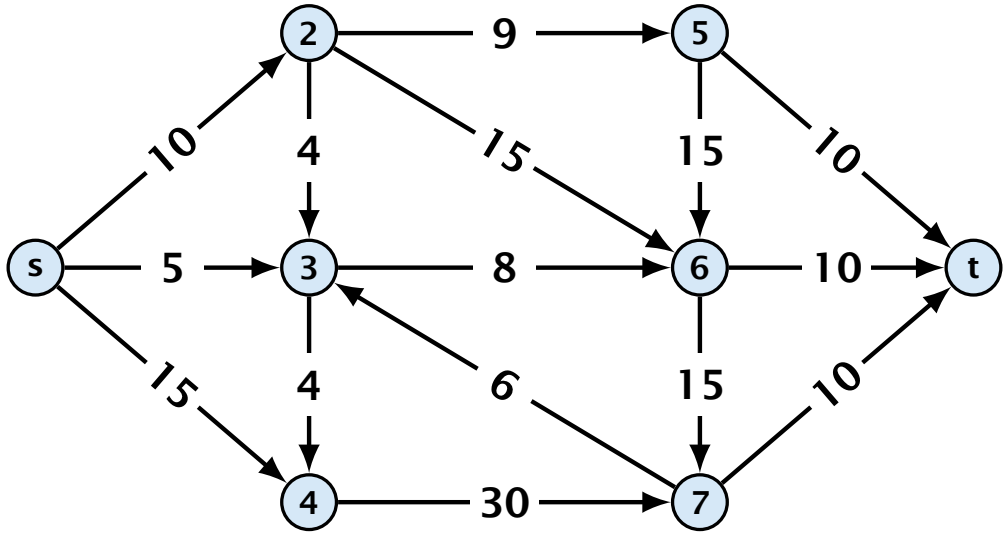


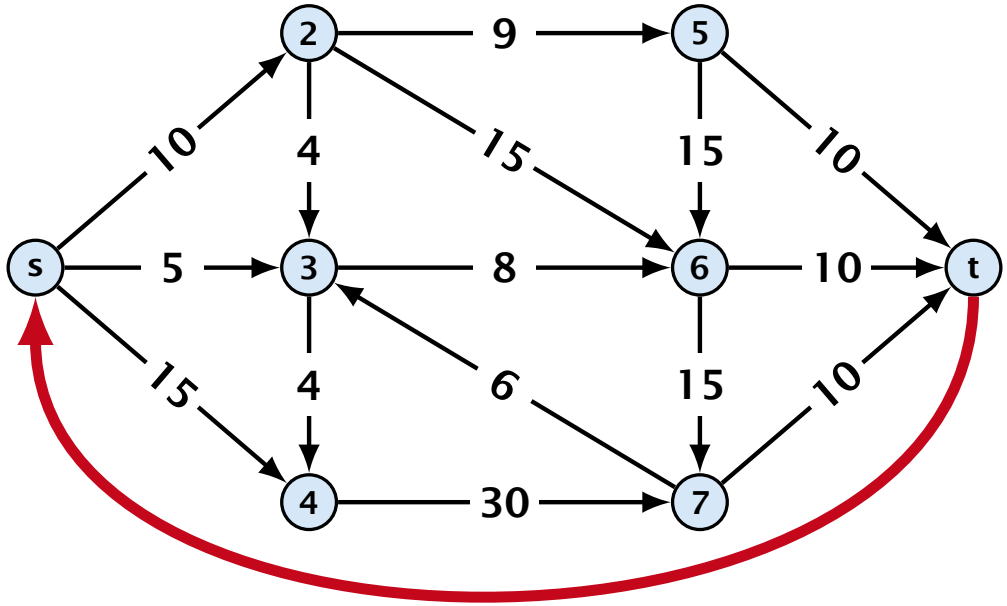
- Given a flow network for a standard maxflow problem.

Solve Maxflow Using Mincost Flow



- Given a flow network for a standard maxflow problem.
- Set $b(v) = 0$ for every node. Keep the capacity function u for all edges. Set the cost $c(e)$ for every edge to 0.

Solve Maxflow Using Mincost Flow



- Given a flow network for a standard maxflow problem.
- Set $b(v) = 0$ for every node. Keep the capacity function u for all edges. Set the cost $c(e)$ for every edge to 0.
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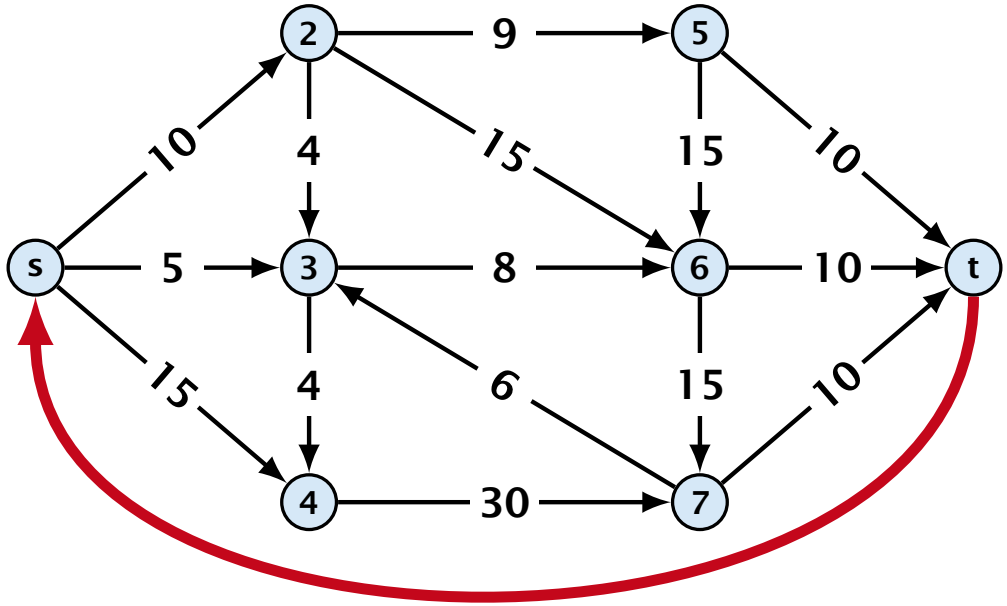

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- Set $b(v) = 0$ for every node. Keep the capacity function u for all edges. Set the cost $c(e)$ for every edge to 0.
- Add an edge from t to s with infinite capacity and cost -1.
- Then, $\mathrm{val}(f^*) = -\mathrm{cost}(f_{\min})$, where f^* is a maxflow, and $f_{\min}$ is a mincost-flow.

Solve Maxflow Using Mincost Flow

Solve decision version of maxflow:

- Given a flow network for a standard maxflow problem, and a value k.
- Set $b(v) = 0$ for every node apart from s or t. Set $b(s) = -k$ and $b(t) = k$.
- Set edge-costs to zero, and keep the capacities.
- There exists a maxflow of value k if and only if the mincost-flow problem is feasible.

Solve Maxflow Using Mincost Flow

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Generalization

Our model:

$$\begin{array}{ll} \min & \sum_e c(e) f(e) \\ \text{s.t.} & \forall e \in E: \ \ 0 \le f(e) \le u(e) \\ & \forall v \in V: \ \ f(v) = b(v) \end{array}$$

where $b : V \to \mathbb{R}$, $\sum_v b(v) = 0$; $u : E \to \mathbb{R}_0^+ \cup \{\infty\}$; $c : E \to \mathbb{R}$;

A more general model?

$$\begin{array}{ll} \min & \sum_e c(e) f(e) \\ \text{s.t.} & \forall e \in E: \ \ \ell(e) \le f(e) \le u(e) \\ & \forall v \in V: \ \ a(v) \le f(v) \le b(v) \end{array}$$

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Generalization

Differences

- Flow along an edge e may have non-zero lower bound $\ell(e)$.
- Flow along e may have negative upper bound $u(e)$.
- The demand at a node v may have lower bound $a(v)$ and upper bound $b(v)$ instead of just lower bound = upper bound = $b(v)$.

Reduction I

$$\begin{aligned} \min \quad & \textstyle\sum_e c(e) f(e) \\ \text{s.t.} \quad & \forall e \in E: \ \ \ell(e) \le f(e) \le u(e) \\ & \forall v \in V: \ \ a(v) \le f(v) \le b(v) \end{aligned}$$

We can assume that $a(v) = b(v)$:

Add new node r.

Add edge (v, r) for all $v \in V$.

Set $\ell(v, r) = u(v, r) = b(v)$ for these edges.

Set $\ell(r, v) = u(r, v) = a(v)$ for edge (r, v).

Set $c(e) = 0$ for all new edges e.

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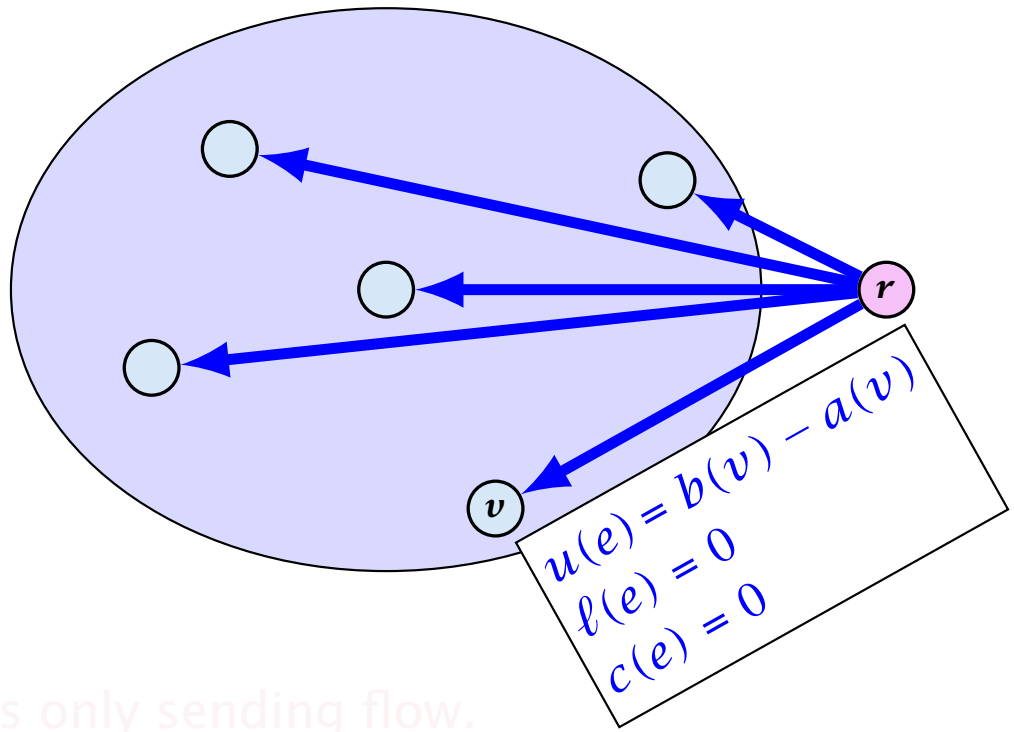
Set $u(e) = b(v) - a(v)$ for edge (r, v).

Set $a(v) = b(v)$ for all $v \in V$.

Set $b(r) = -\sum_{v \in V} b(v)$.

$-\sum_v b(v)$ is negative; hence r is only sending flow.



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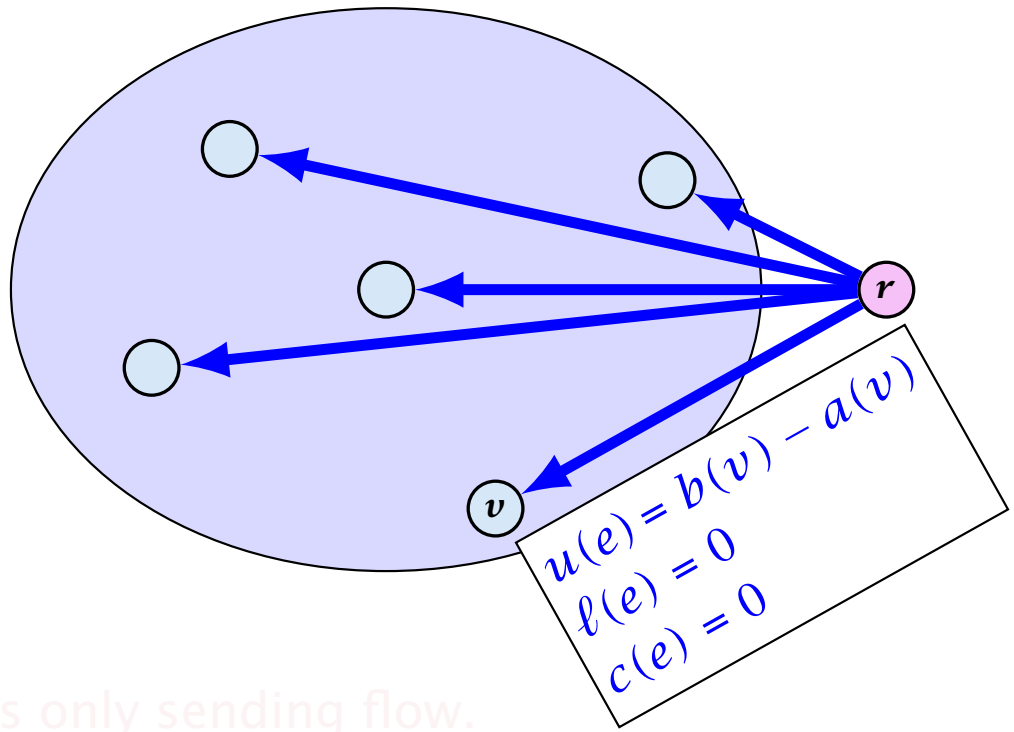

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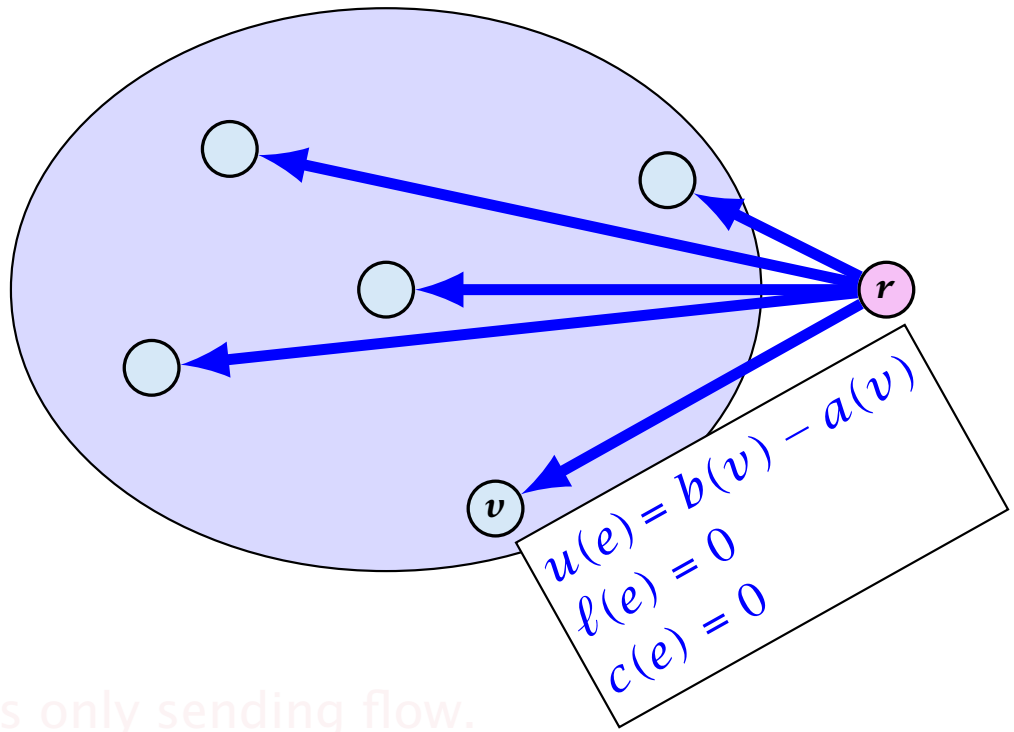
$-\sum_v b(v)$ is negative; hence r is only sending flow.



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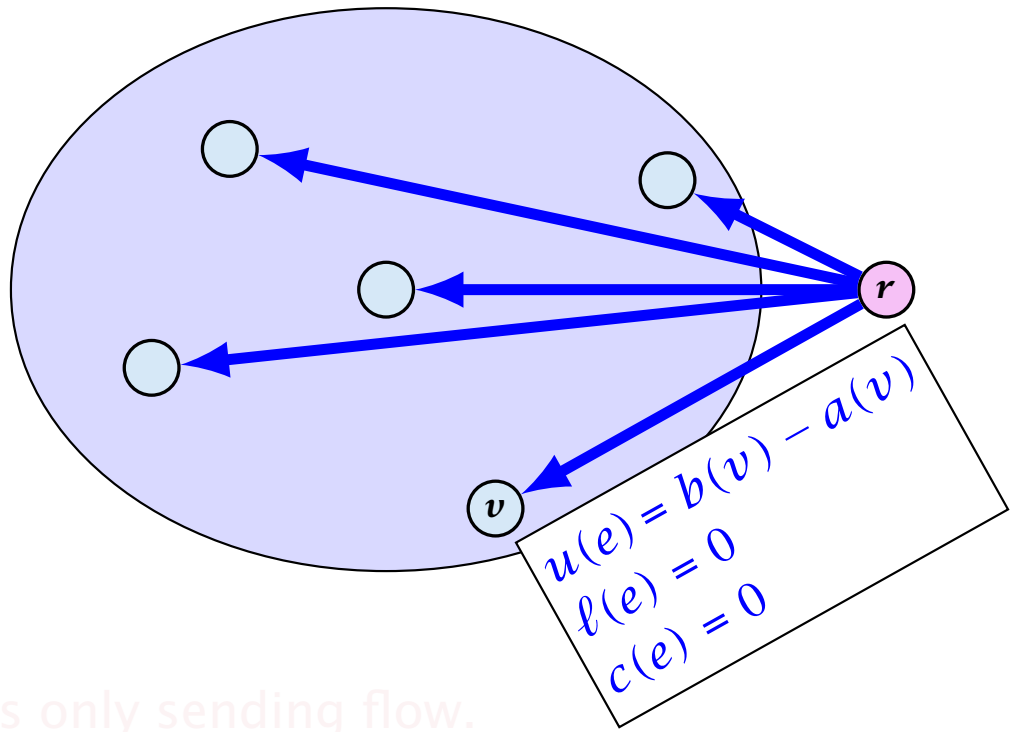

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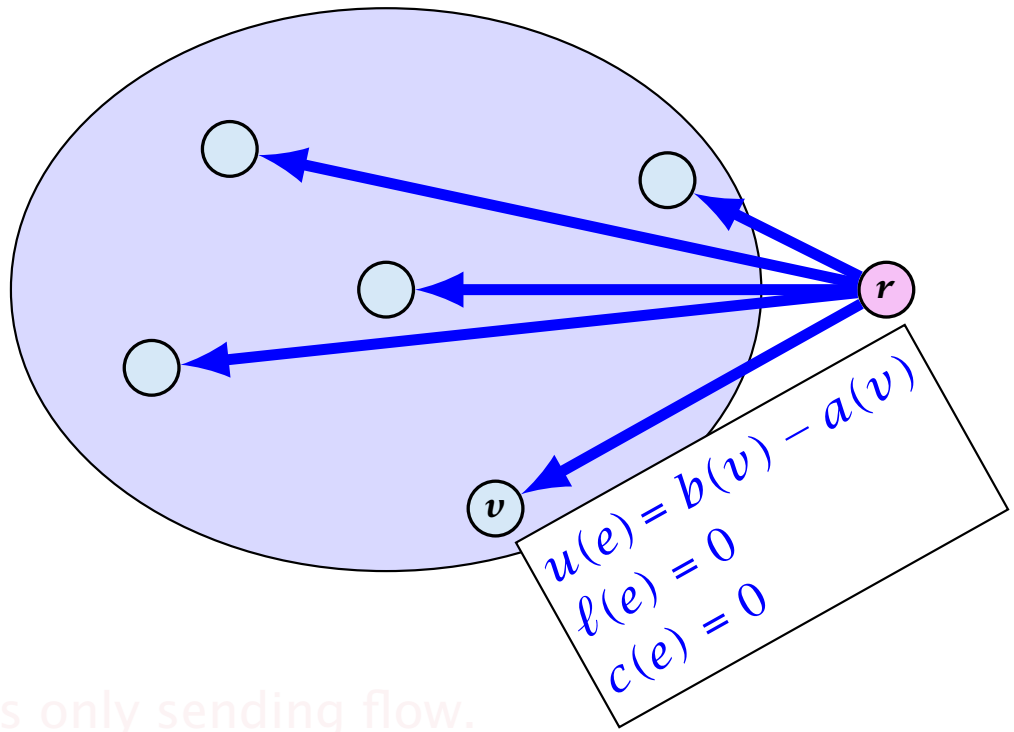

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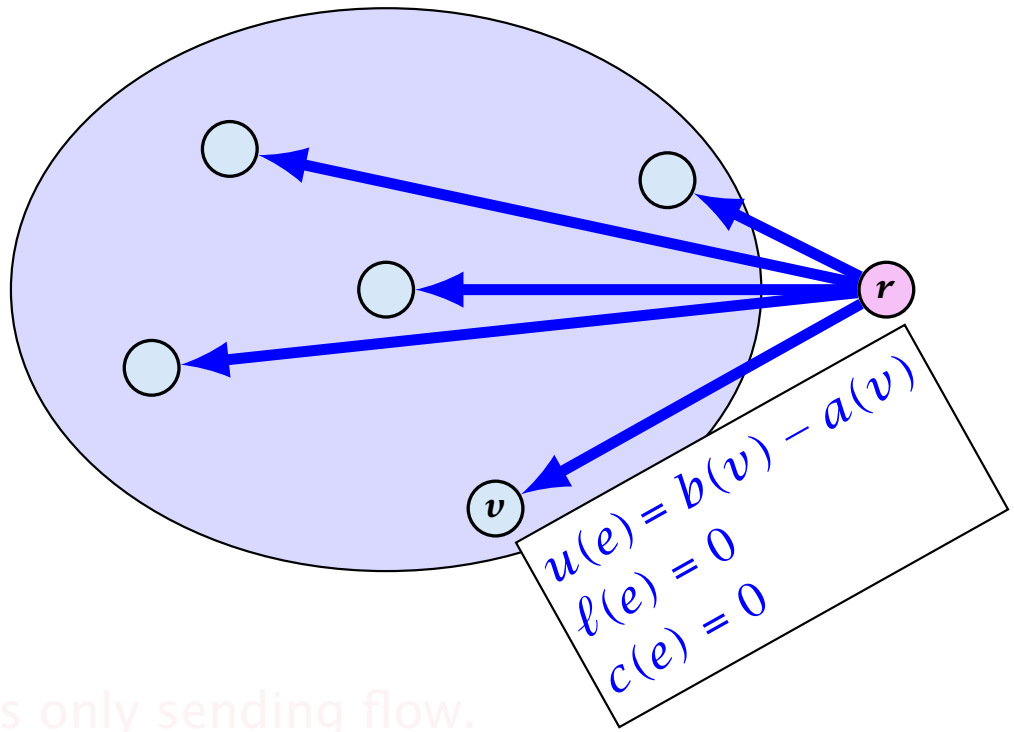

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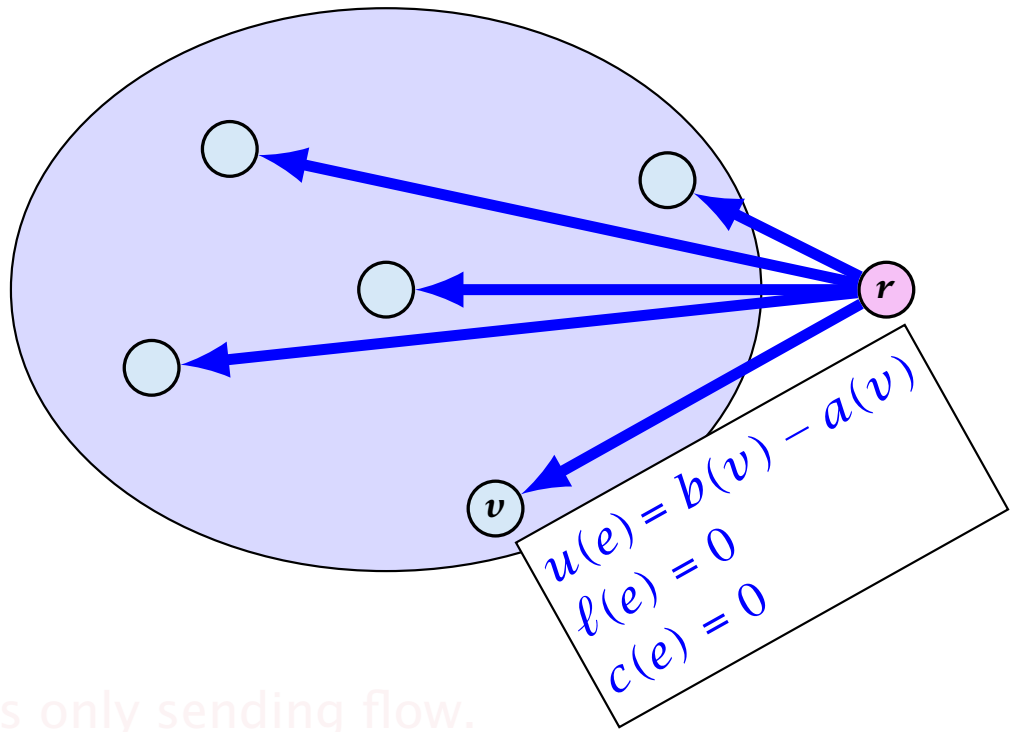

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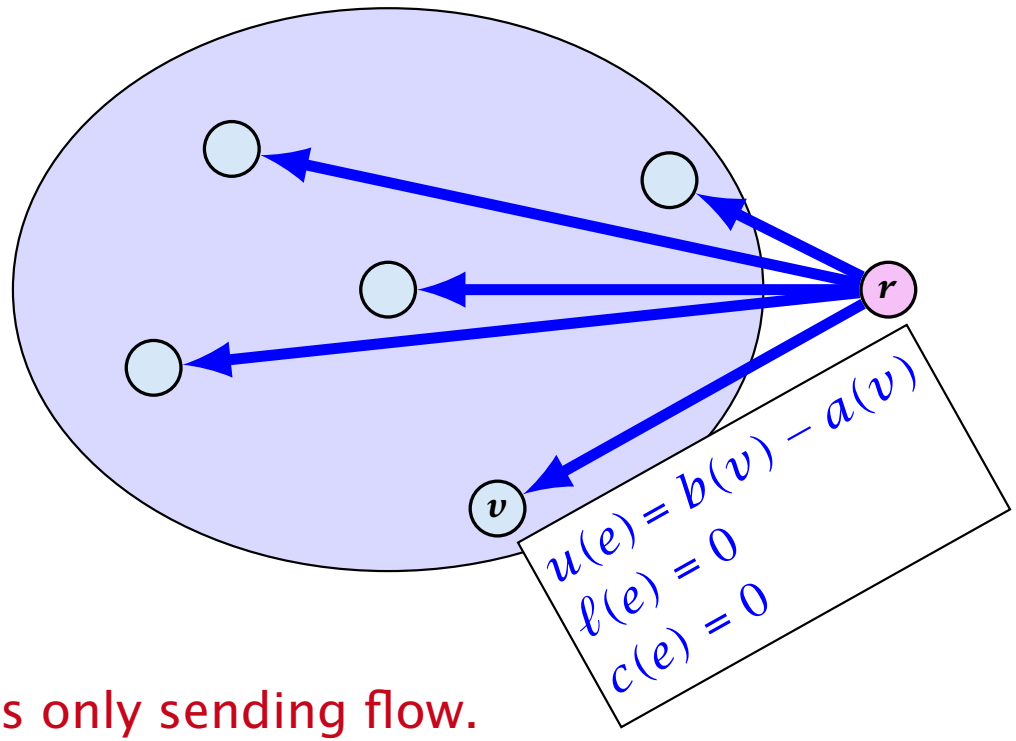

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Reduction II

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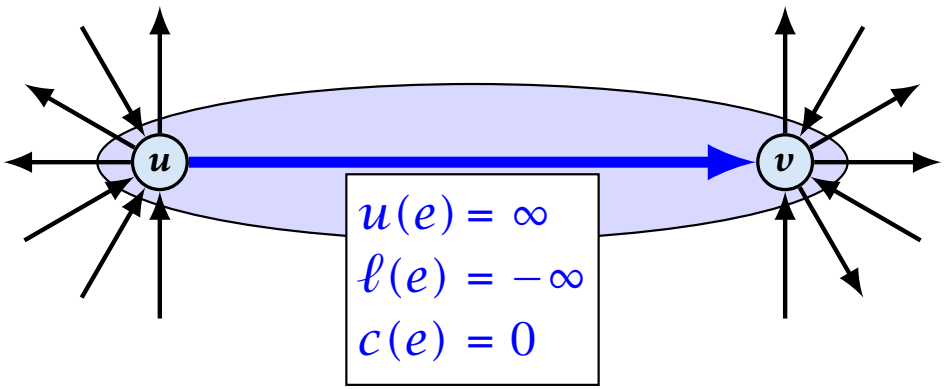

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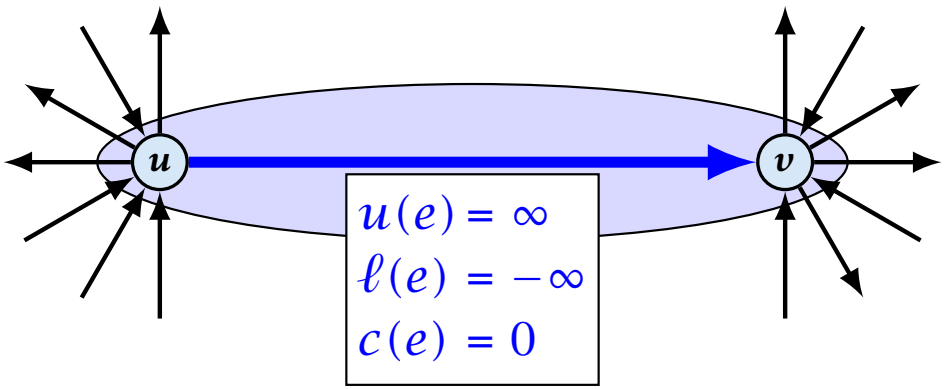

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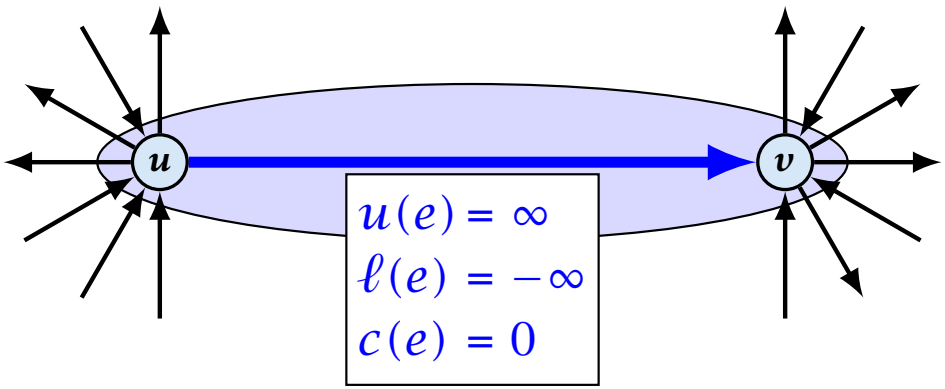

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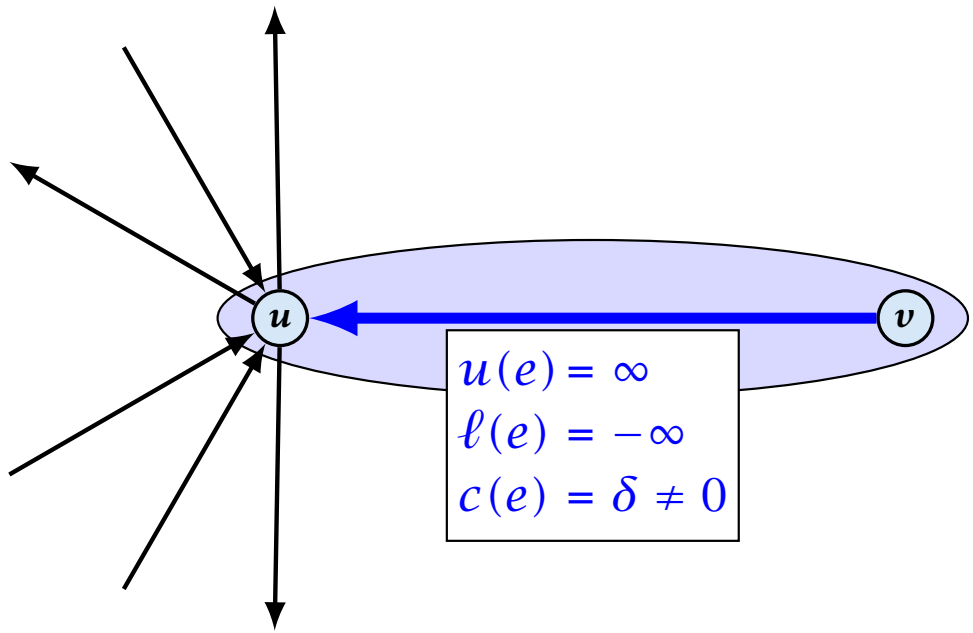

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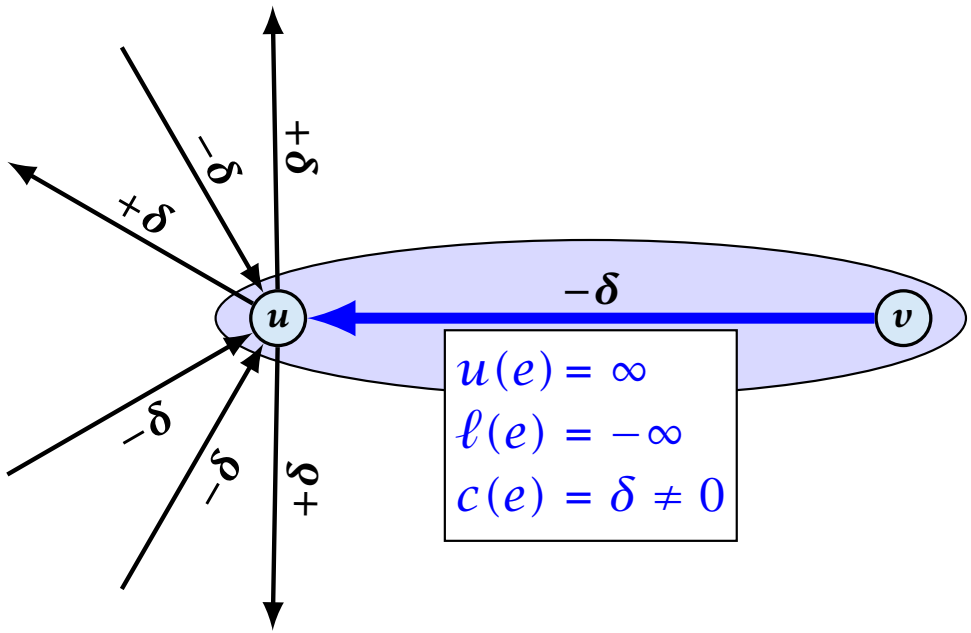

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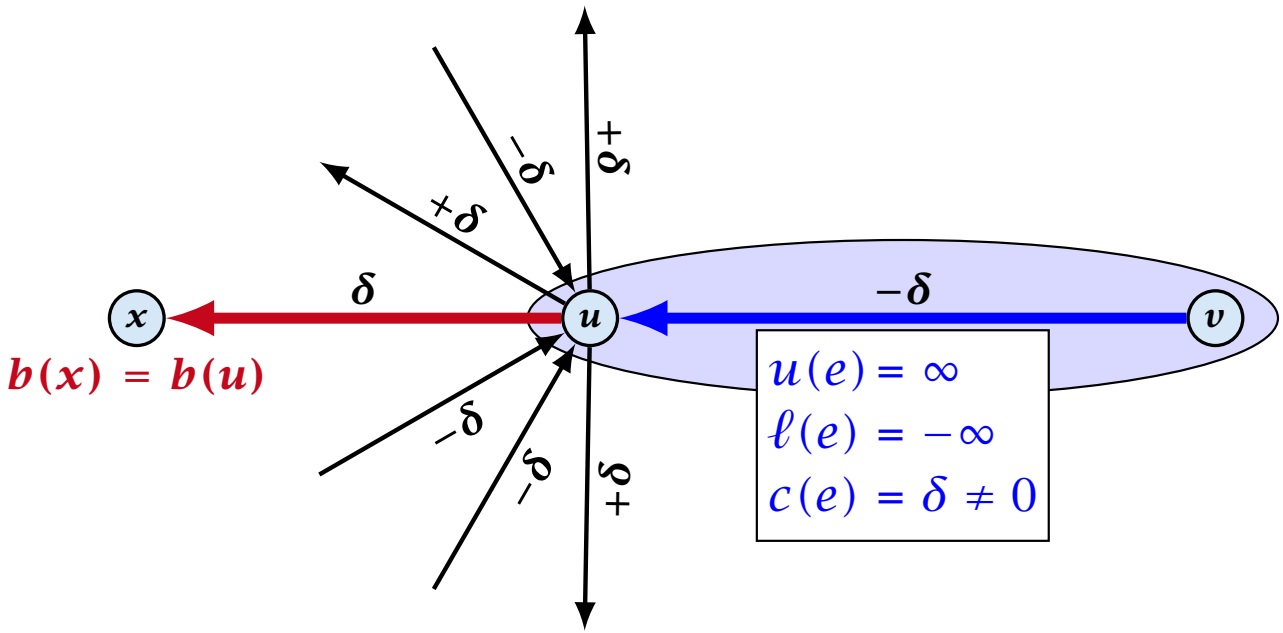

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Reduction III

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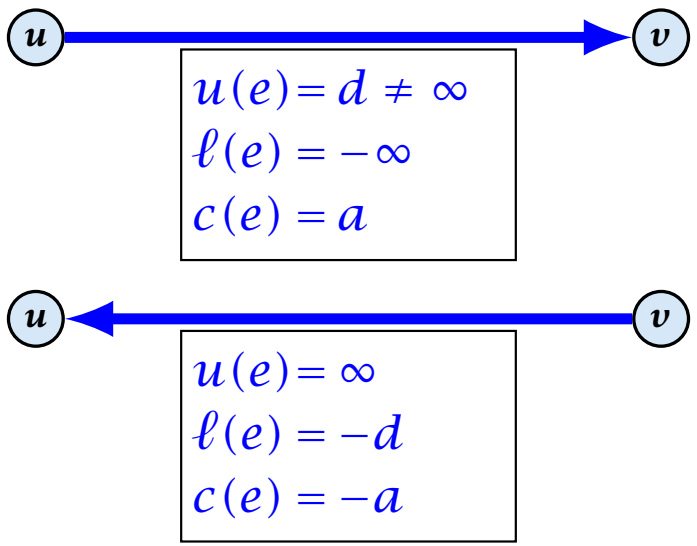

Replace the edge by an edge in opposite direction.

Reduction IV

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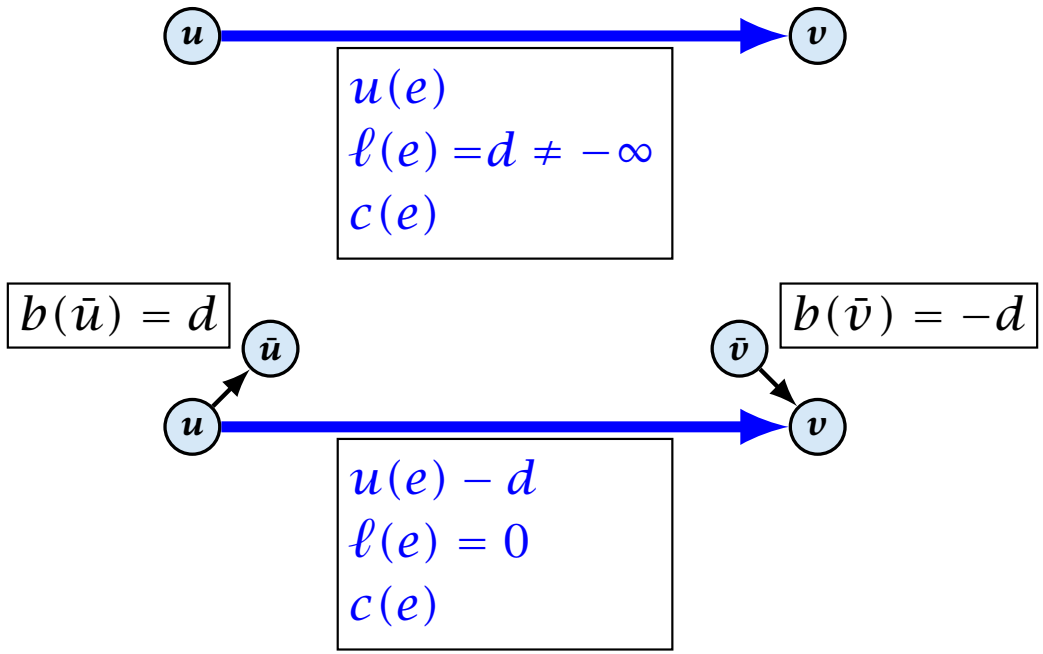

The added edges have infinite capacity and cost $c(e)/2$.

Applications

Caterer Problem

- She needs to supply r_i napkins on N successive days.
- She can buy new napkins at p cents each.
- She can launder them at a fast laundry that takes m days and cost f cents a napkin.
- She can use a slow laundry that takes $k > m$ days and costs s cents each.
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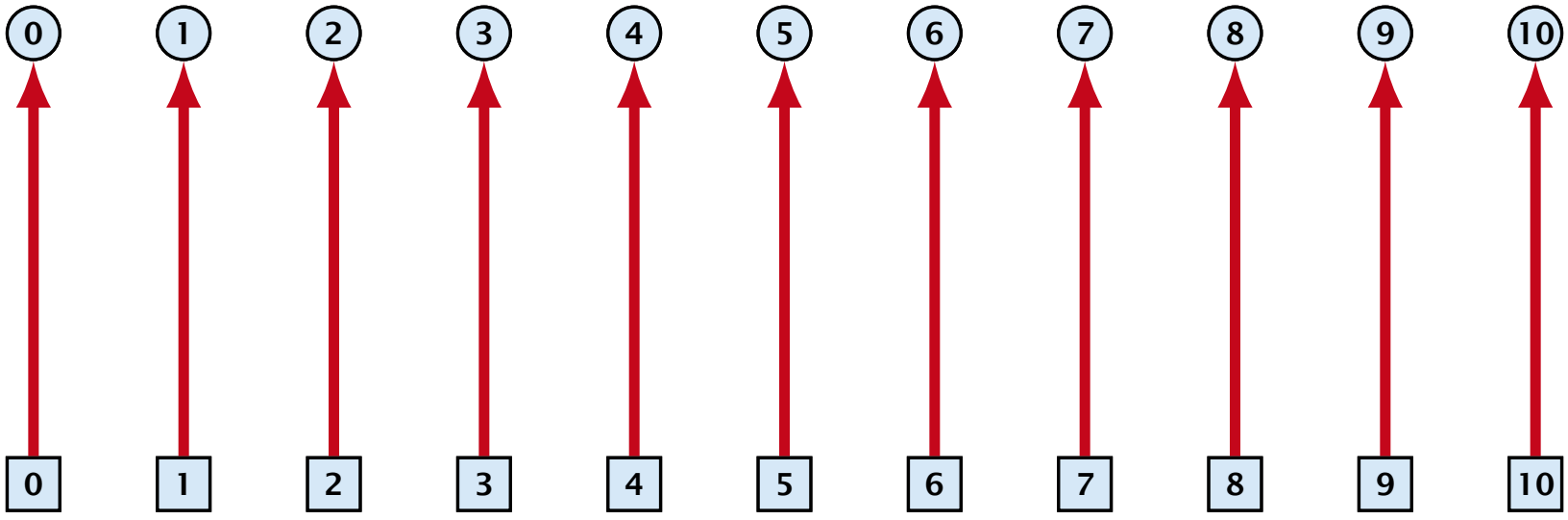
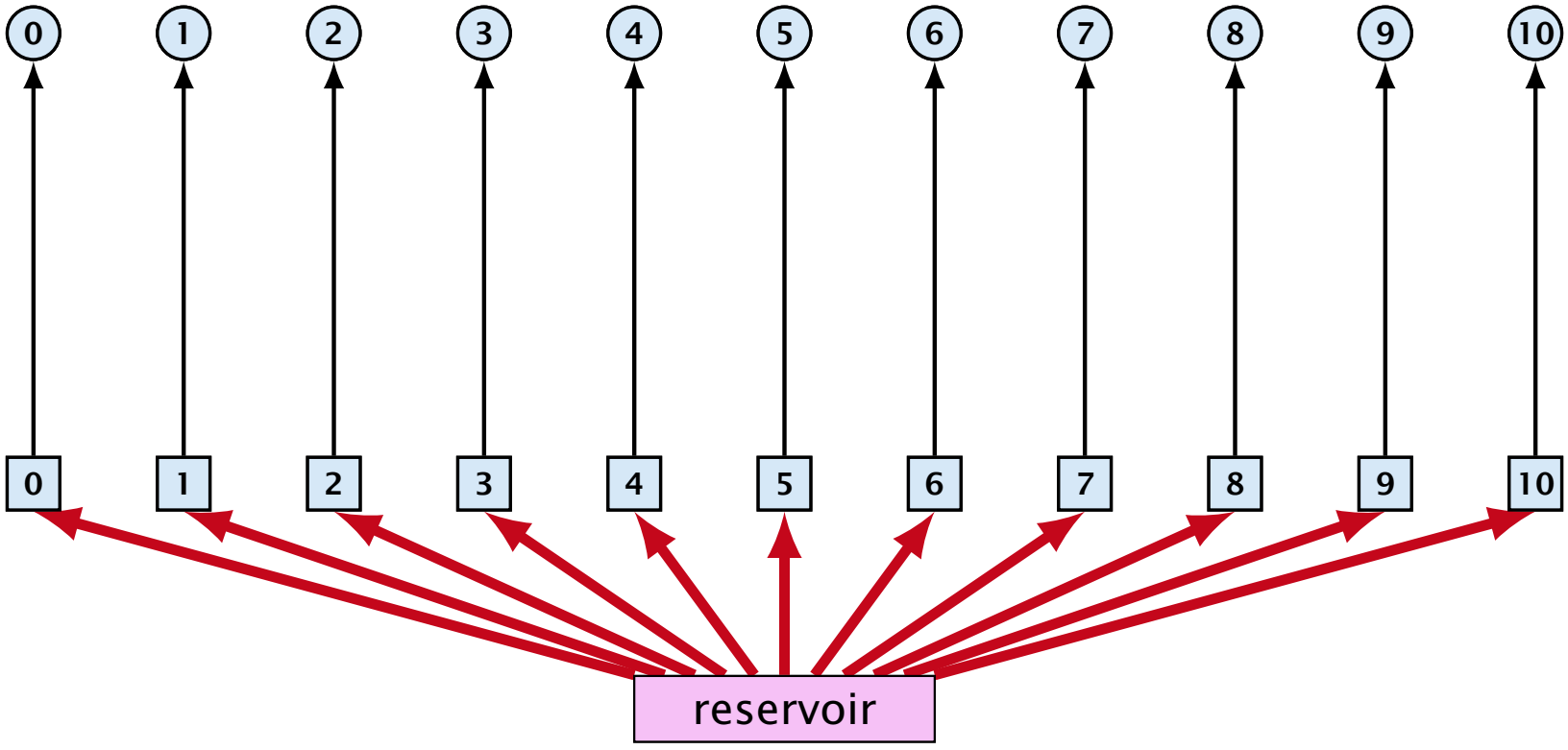


day edges:

upper bound: $u(e_i) = \infty$;
lower bound: $\ell(e_i) = r_i$;
cost: $c(e) = 0$



buy edges:

upper bound: $u(e_i) = \infty$;
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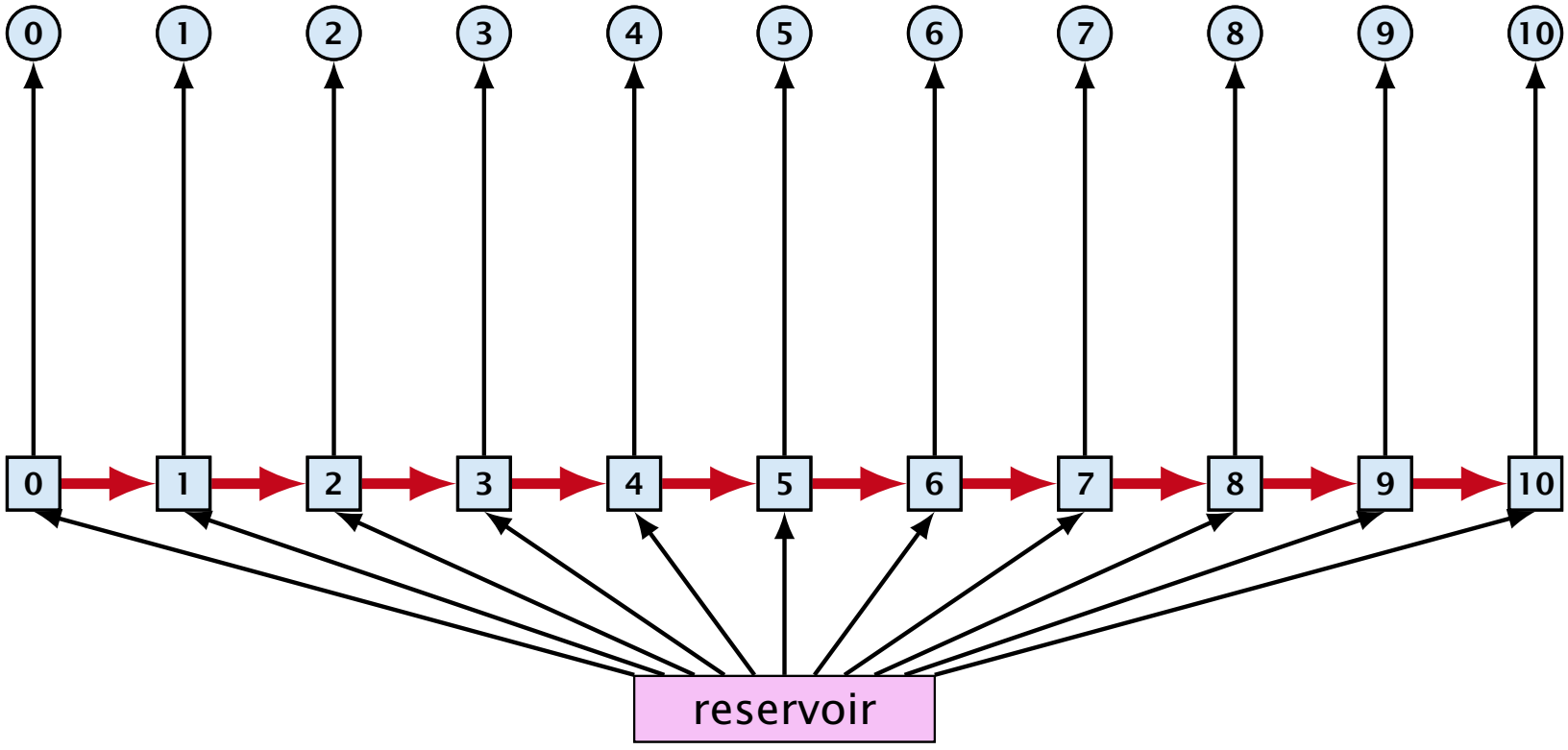
cost: $c(e) = p$



forward edges:

upper bound: $u(e_i) = \infty$;
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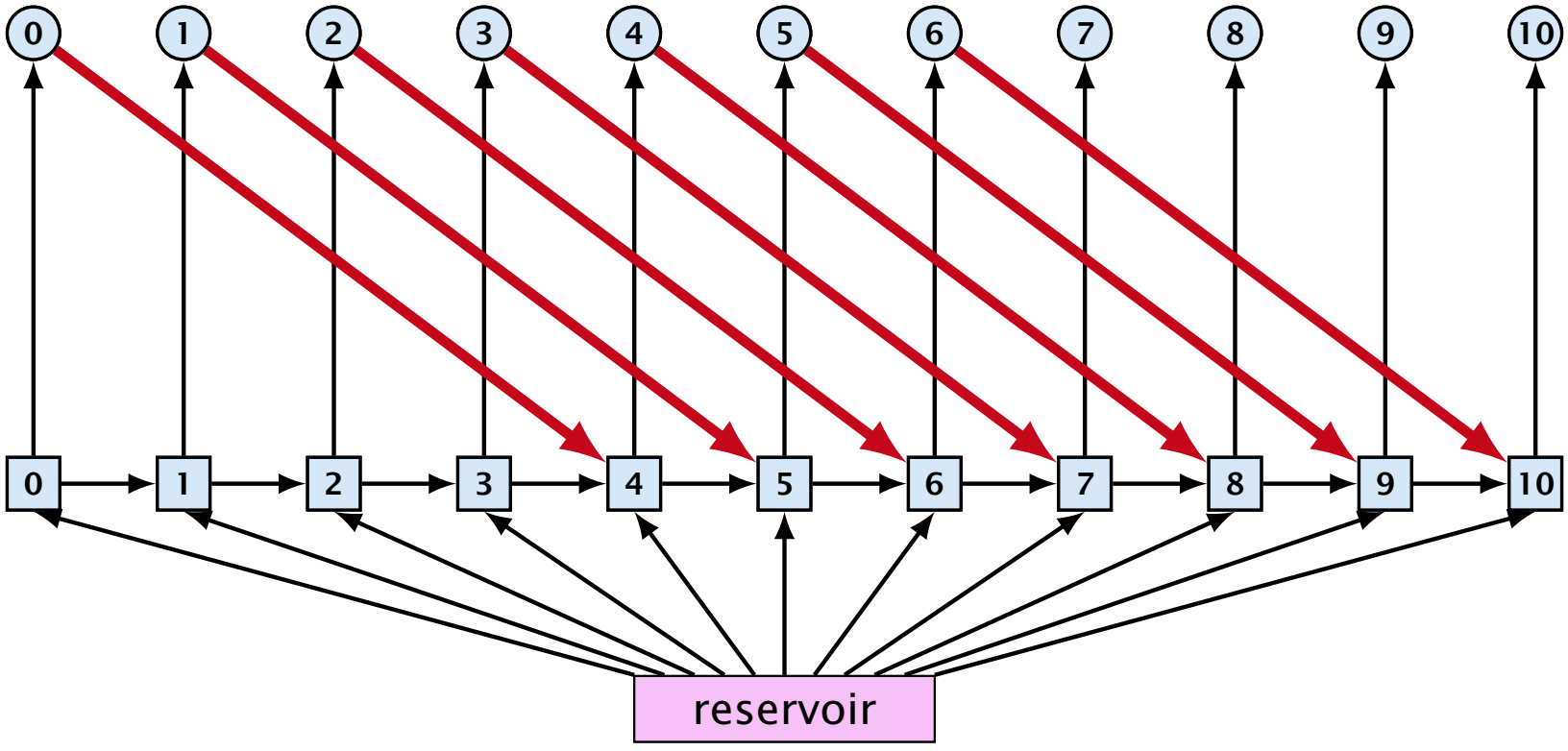

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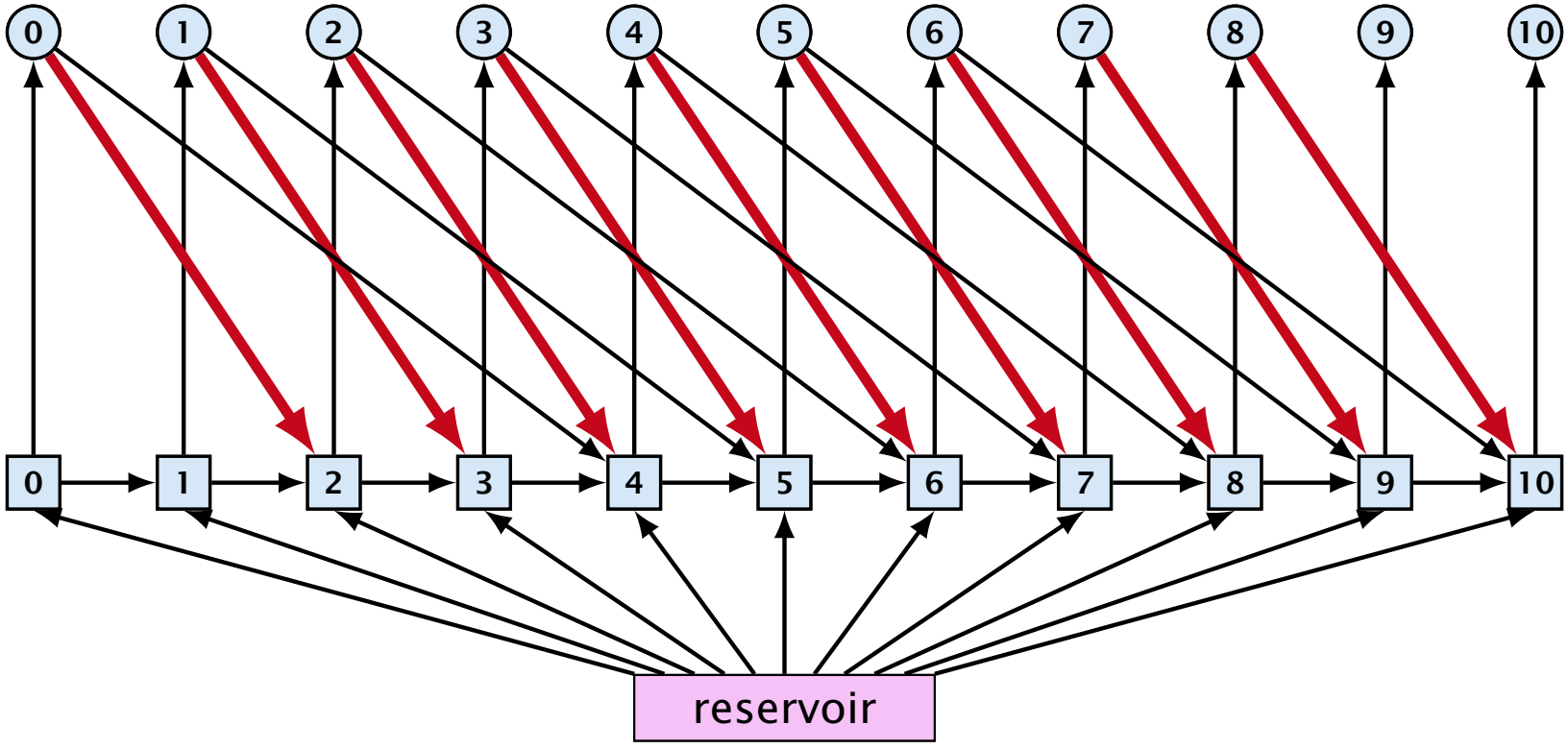

fast edges:

upper bound: $u(e_i) = \infty$;
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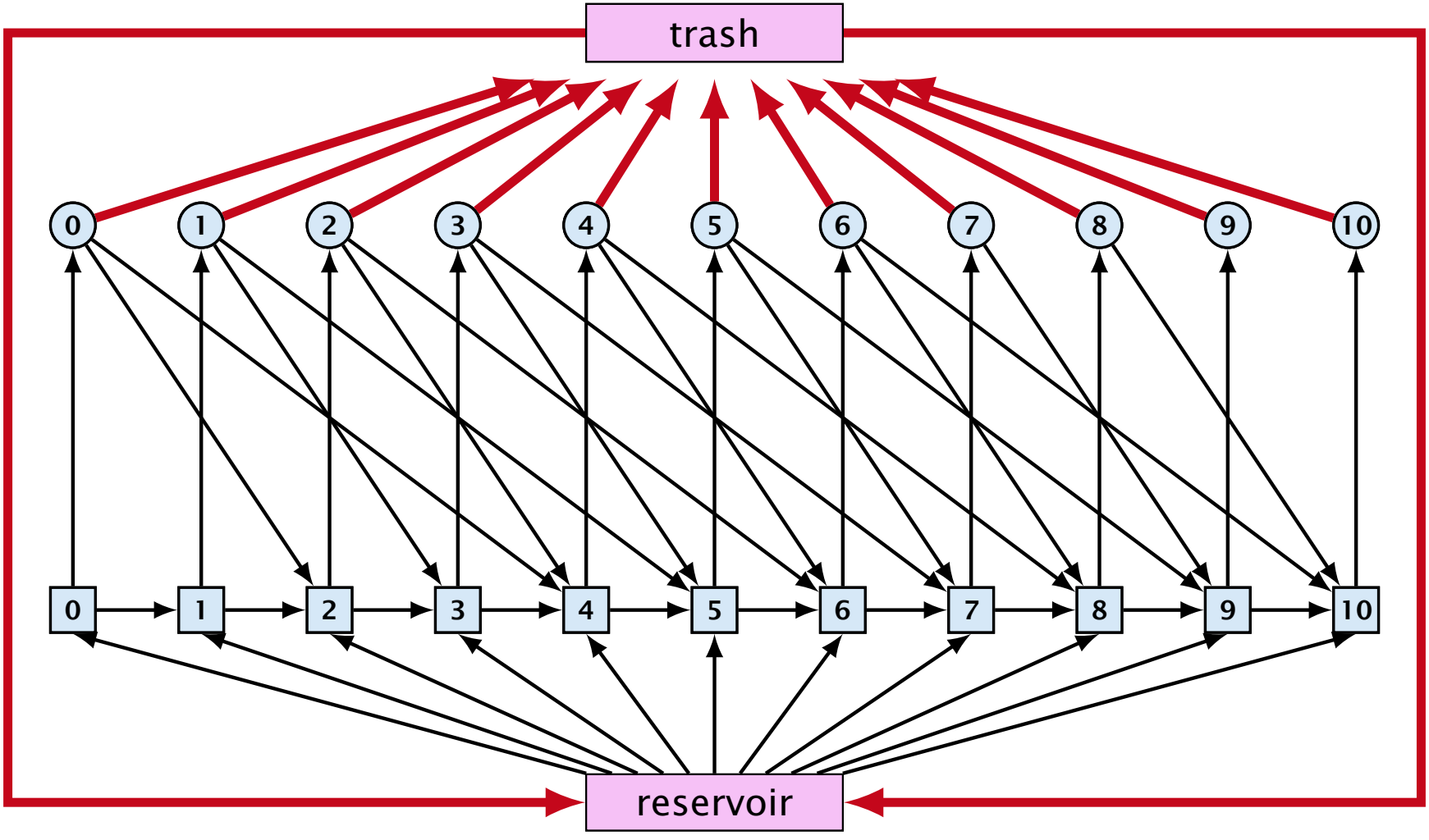

trash edges:

upper bound: $u(e_i) = \infty$;
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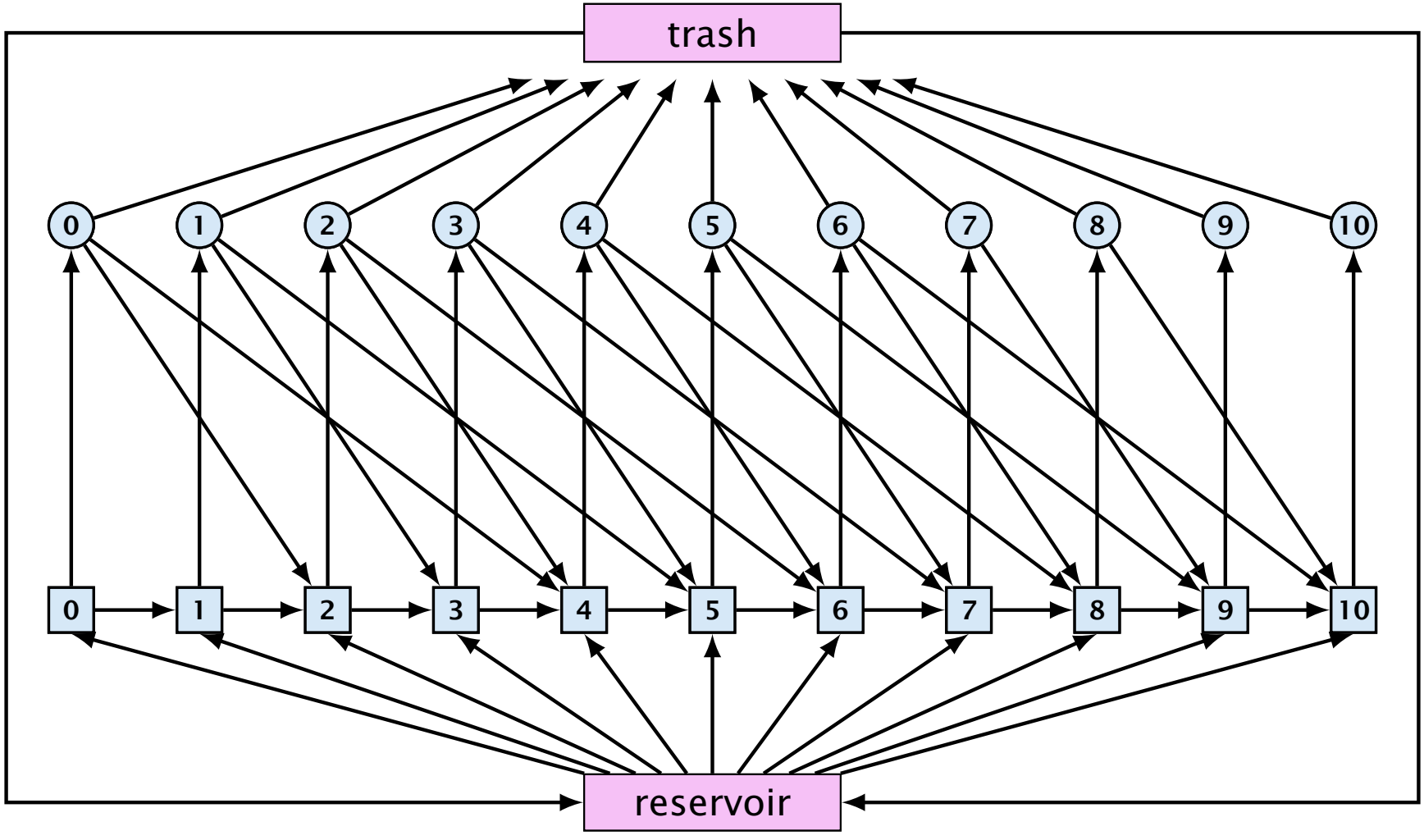
cost: $c(e) = 0$


trash
0
1
2
3
4
5
6
7
8
9
10
0
1
2
3
4
5
6
7
8
9
10
reservoir

Residual Graph

The residual graph for a mincost flow is exactly defined as the residual graph for standard flows, with the only exception that one needs to define a cost for the residual edge.

For a flow of z from u to v the residual edge (v,u) has capacity z and a cost of $-c((u,v))$.

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15 Mincost Flow

A circulation in a graph $G = (V, E)$ is a function $f : E \to \mathbb{R}^+$ that has an excess flow $f(v) = 0$ for every node $v \in V$.

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Then $f + g$ is a feasible flow with cost $\text{cost}(f) + \text{cost}(g) < \text{cost}(f)$. Hence, f is not minimum cost.

$\Leftarrow$ Let f be a non-mincost flow, and let f^* be a min-cost flow. We need to show that the residual graph has a feasible circulation with negative cost.

Clearly $f^* - f$ is a circulation of negative cost. One can also easily see that it is feasible for the residual graph. (after sending f in the residual graph (pushing all flow that f sends) we arrive at the original graph, for this $f^* - f$ is clearly feasible)

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Proof.

- Suppose that we have a negative cost circulation.
- Find directed path only using edges that have more zero flow.
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- Otherwise send flow in opposite direction around the cycle until the bottleneck edge(s) does not carry any flow.
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- Find directed path only using edges that have non-zero flow.
- If this path has negative cost you are done.
- Otherwise send flow in opposite direction along the cycle until the bottleneck edge(s) does not carry any flow.
- You still have a circulation with negative cost.
- Repeat.

Lemma 89

A graph (without zero-capacity edges) has a feasible circulation of negative cost if and only if it has a negative cycle w.r.t. edge-weights $c : E \to \mathbb{R}$.

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15 Mincost Flow

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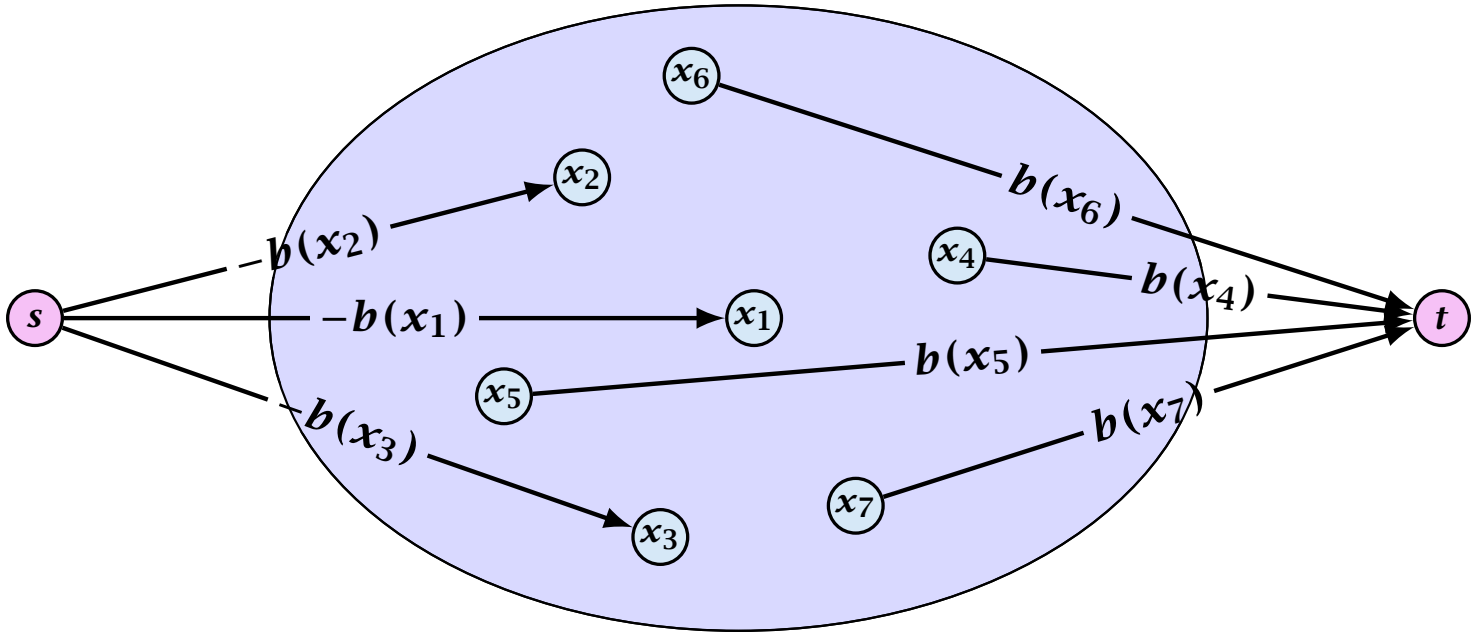
Algorithm 51 CycleCanceling$(G = (V, E), c, u, b)$

establish a feasible flow f in G
while G_f contains negative cycle **do**
    use Bellman-Ford to find a negative circuit Z
    $\delta \leftarrow \min\{u_f(e) \mid e \in Z\}$
    augment δ units along Z and update G_f

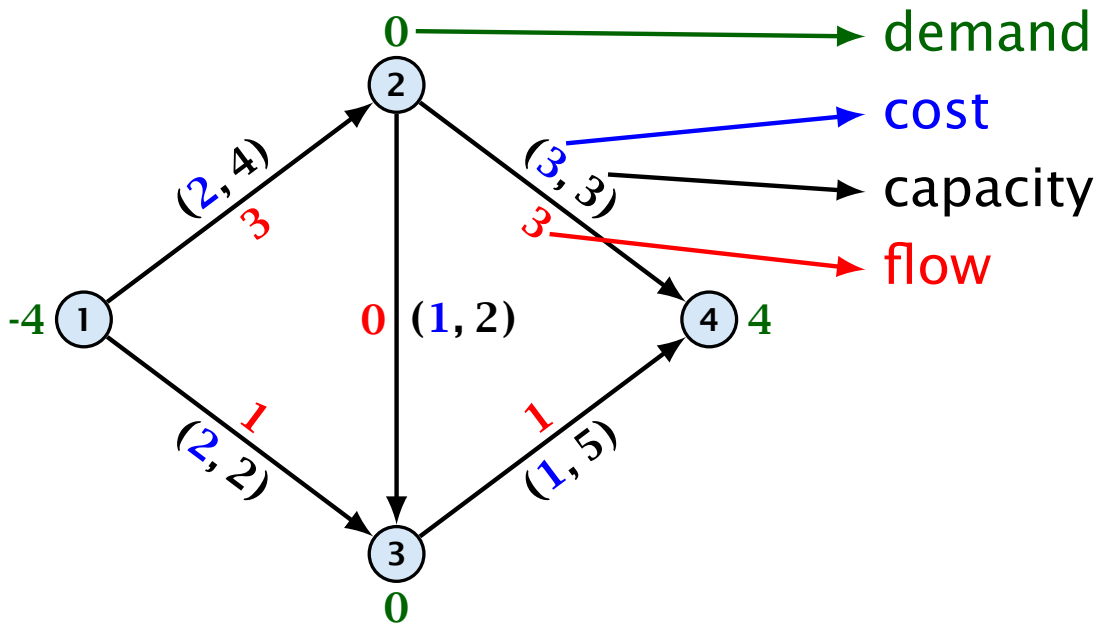
How do we find the initial feasible flow?



- Connect new node s to all nodes with negative $b(v)$-value.
- Connect nodes with positive $b(v)$-value to a new node t.
- There exist a feasible flow in the original graph iff in the resulting graph there exists an s-t flow of value

$$\sum_{v:b(v)<0} (-b(v)) = \sum_{v:b(v)>0} b(v) \ .$$

15 Mincost Flow



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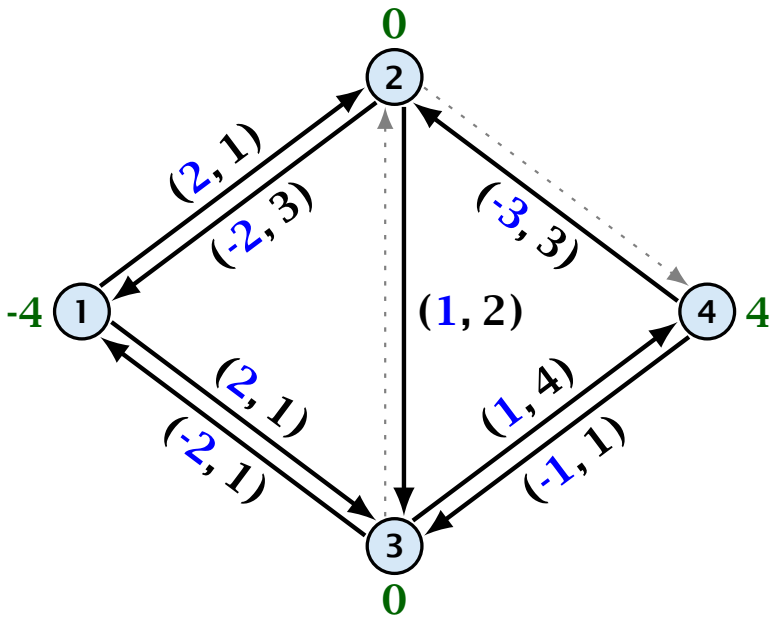

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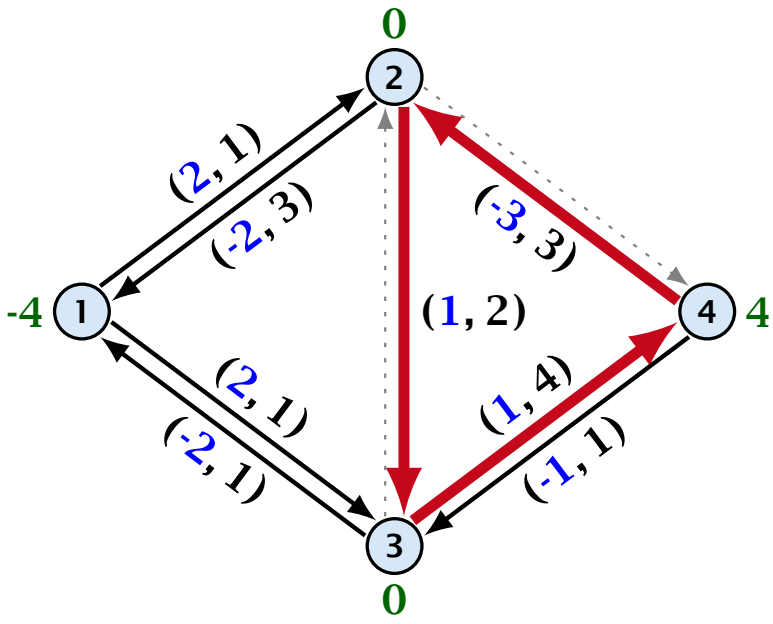

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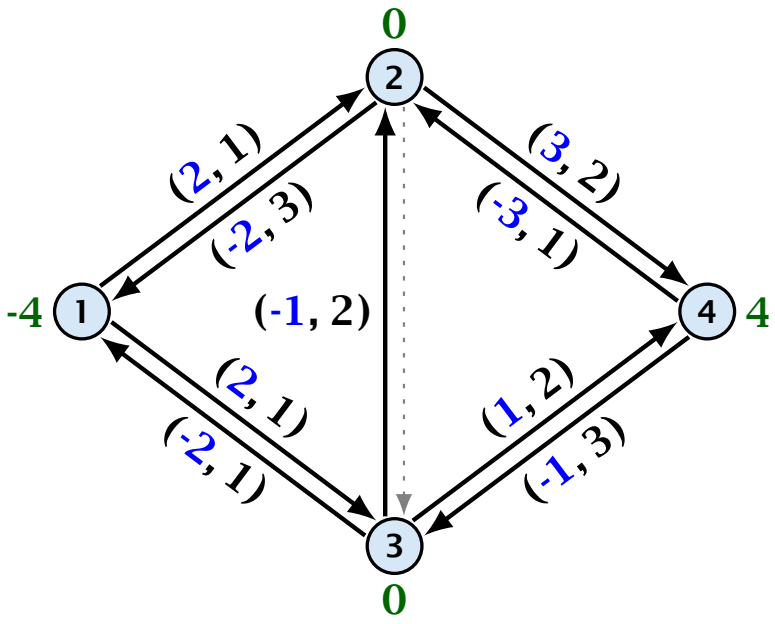

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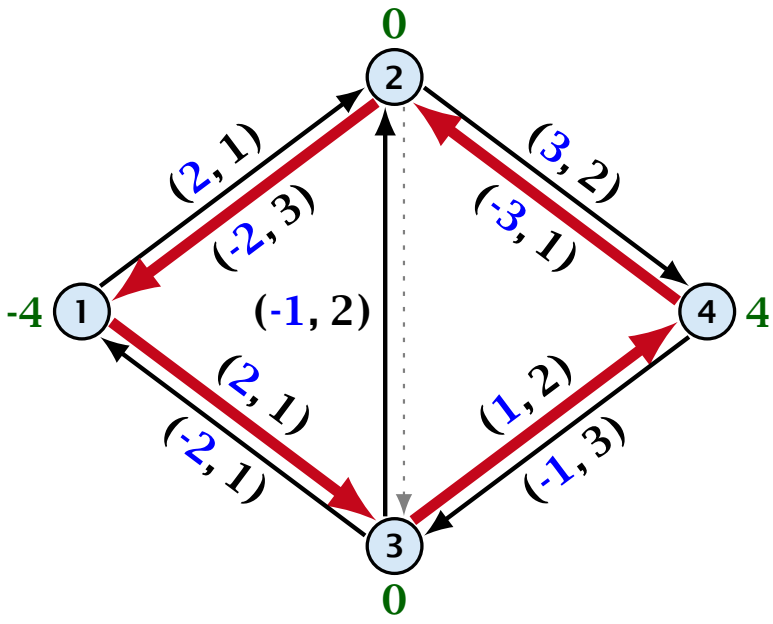

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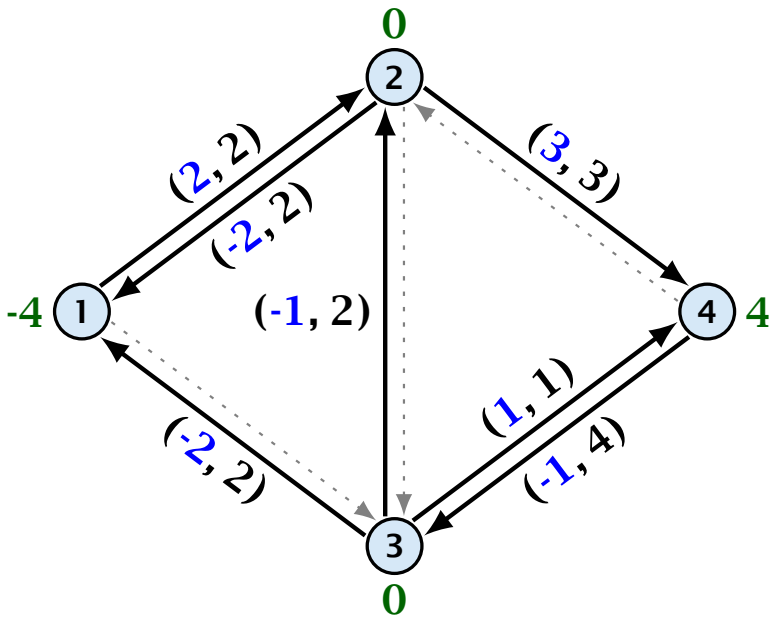


15 Mincost Flow

Lemma 90

The improving cycle algorithm runs in time $\mathcal{O}(nm^2CU)$, for integer capacities and costs, when for all edges e, $|c(e)| \leq C$ and $|u(e)| \leq U$.

- Running time of Bellman-Ford is $\mathcal{O}(mn)$.
- Pushing flow along the cycle can be done in time $\mathcal{O}(n)$.
- Each iteration decreases the total cost by at least 1.
- The true optimum cost must lie in the interval $[-mCU, \ldots, +mCU]$.

Note that this lemma is weak since it does not allow for edges with infinite capacity.

15 Mincost Flow

A general mincost flow problem is of the following form:

$$\begin{array}{ll} \min & \sum_e c(e) f(e) \\ \text{s.t.} & \forall e \in E: \ \ \ell(e) \leq f(e) \leq u(e) \\ & \forall v \in V: \ \ a(v) \leq f(v) \leq b(v) \end{array}$$

where $a : V \to \mathbb{R}$, $b : V \to \mathbb{R}$; $\ell : E \to \mathbb{R} \cup \{-\infty\}$, $u : E \to \mathbb{R} \cup \{\infty\}$ $c : E \to \mathbb{R}$;

Lemma 91 (without proof)

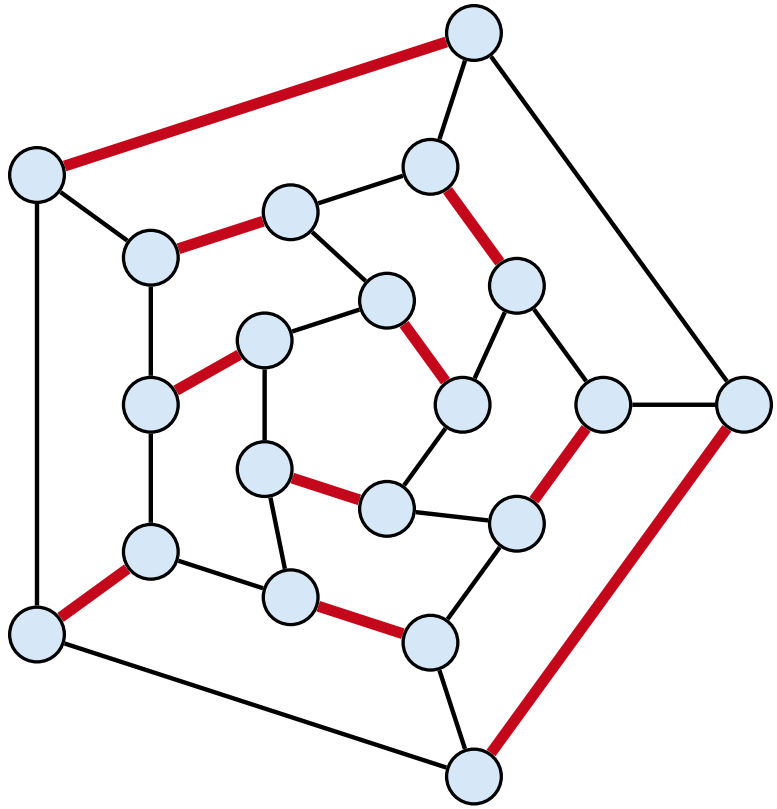
A general mincost flow problem can be solved in polynomial time.

Part V

Matchings

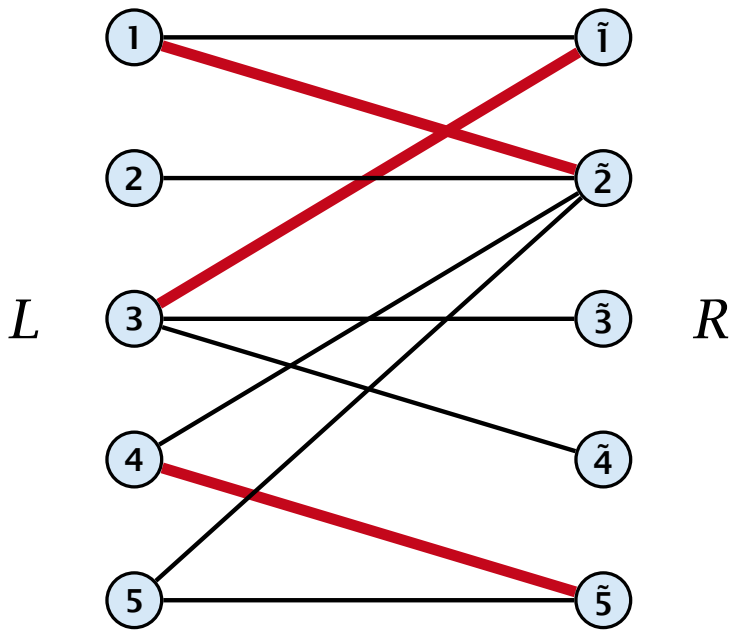
Matching

- Input: undirected graph $G = (V, E)$.
- $M \subseteq E$ is a matching if each node appears in at most one edge in M.
- Maximum Matching: find a matching of maximum cardinality



Bipartite Matching

- Input: undirected, bipartite graph $G = (L \uplus R, E)$.
- $M \subseteq E$ is a matching if each node appears in at most one edge in M.
- Maximum Matching: find a matching of maximum cardinality



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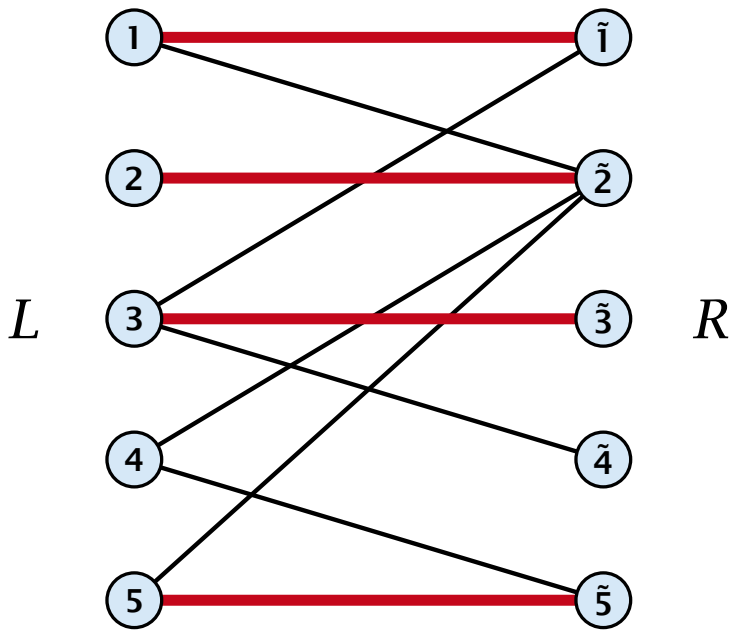
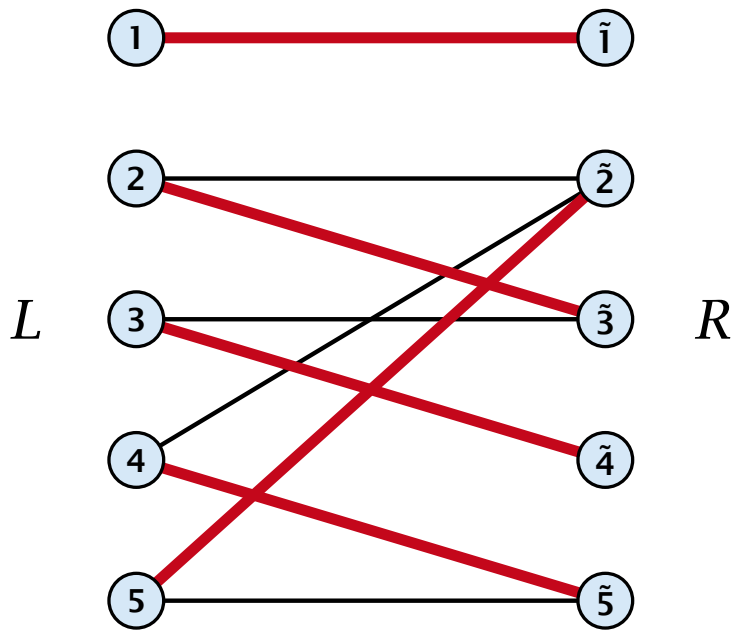


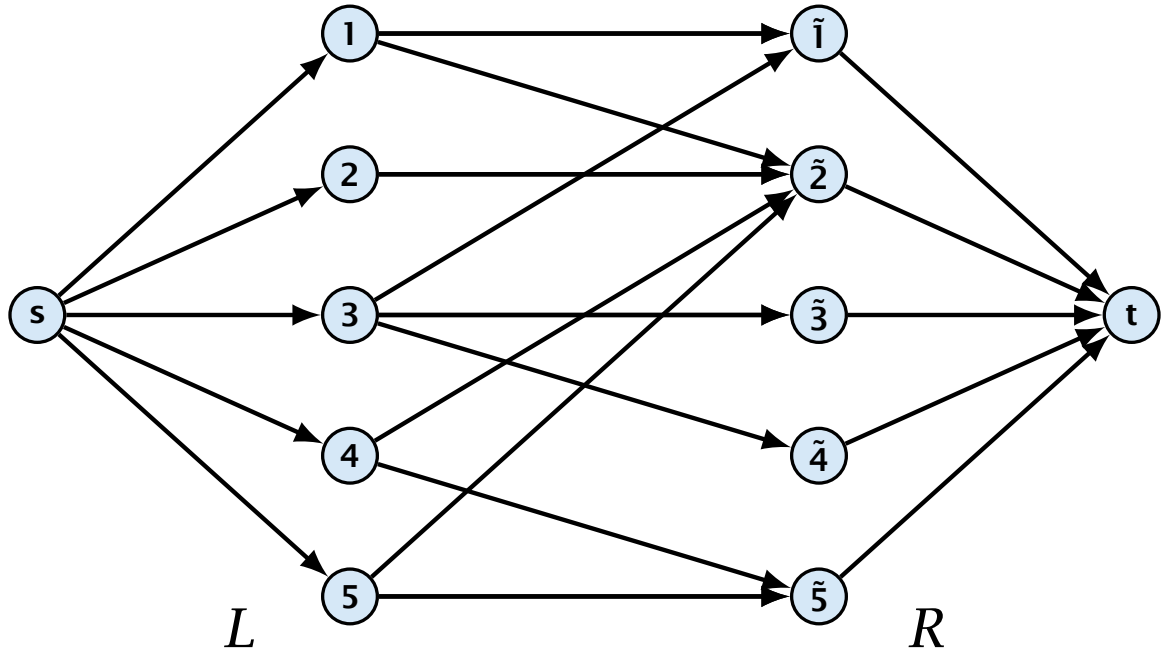
Bipartite Matching

- A matching M is perfect if it is of cardinality $|M| = |V|/2$.
- For a bipartite graph $G = (L \uplus R, E)$ this means $|M| = |L| = |R| = n$.



17 Bipartite Matching via Flows

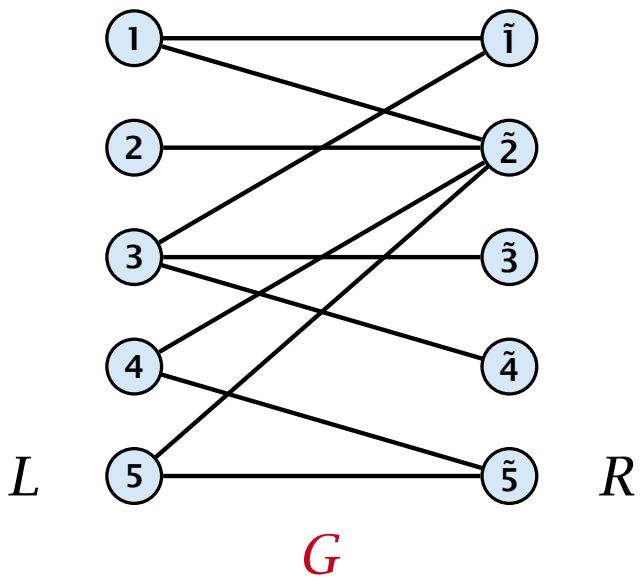
- Input: undirected, bipartite graph $G = (L \uplus R \uplus \{s, t\}, E')$.
- Direct all edges from L to R.
- Add source s and connect it to all nodes on the left.
- Add t and connect all nodes on the right to t.
- All edges have unit capacity.

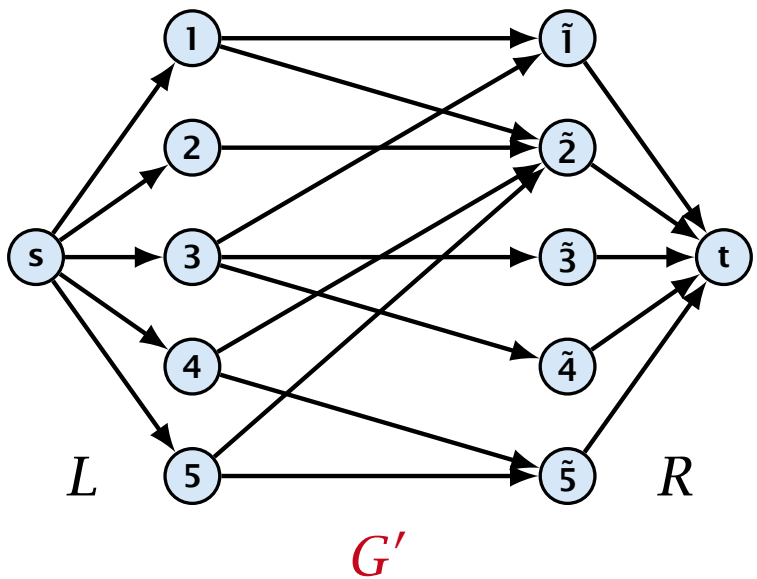


Proof

Max cardinality matching in $G \leq$ value of maxflow in G'

- Given a maximum matching M of cardinality k.
- Consider flow f that sends one unit along each of k paths.
- f is a flow and has cardinality k.





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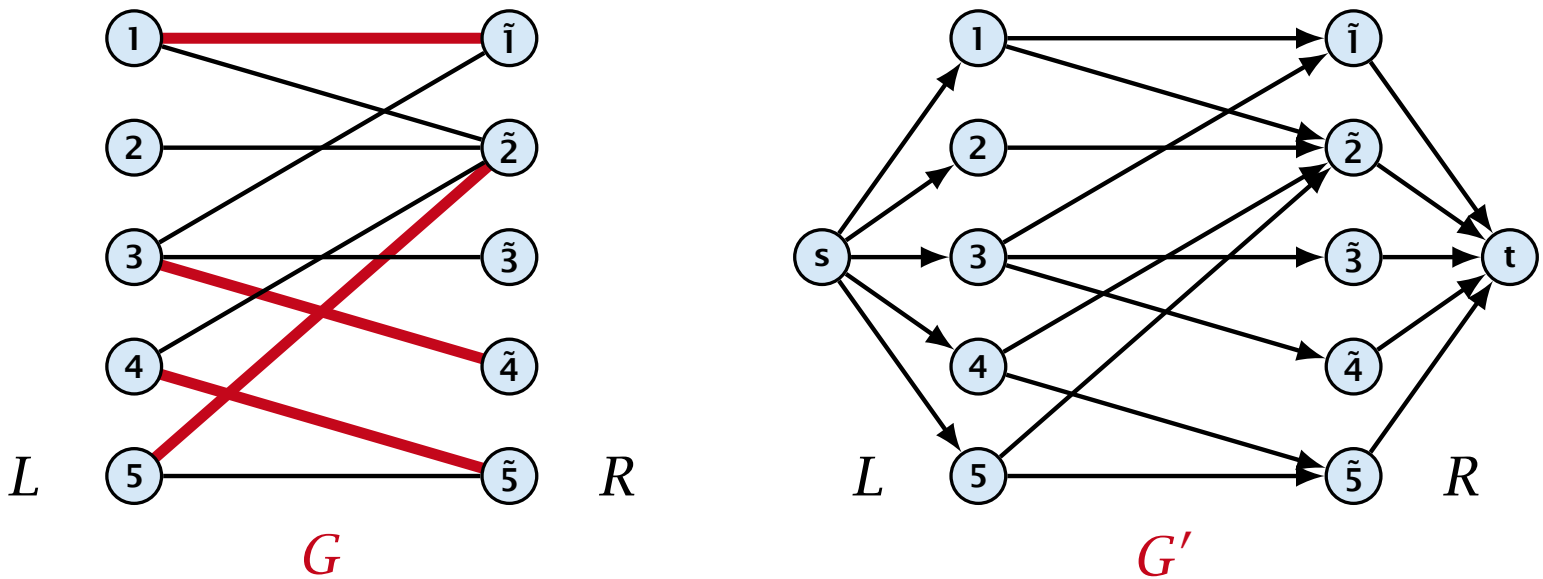

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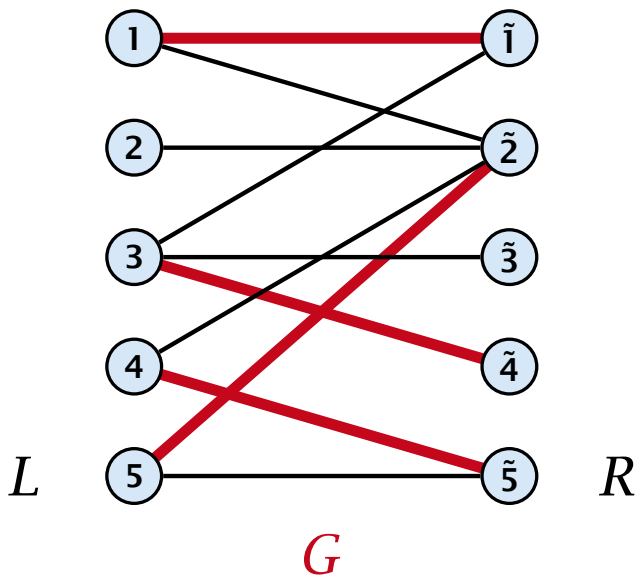

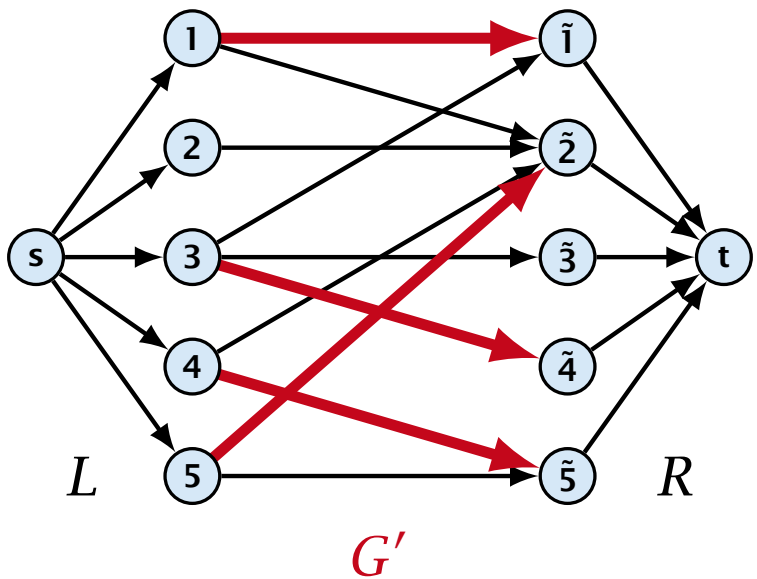

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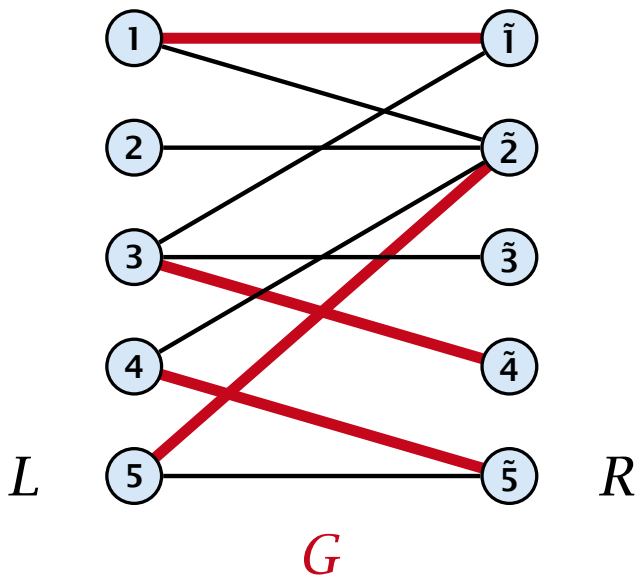
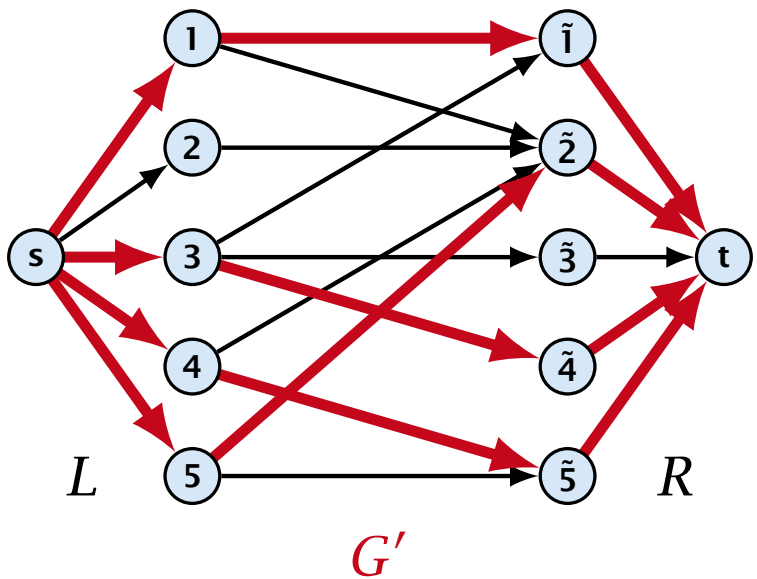
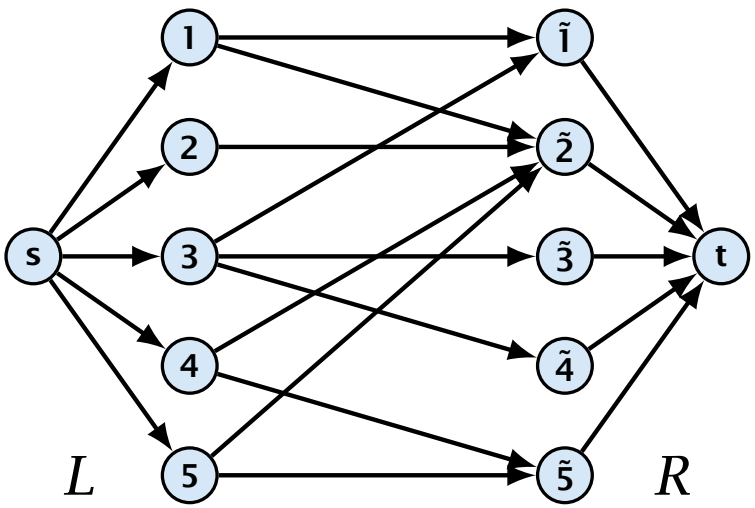




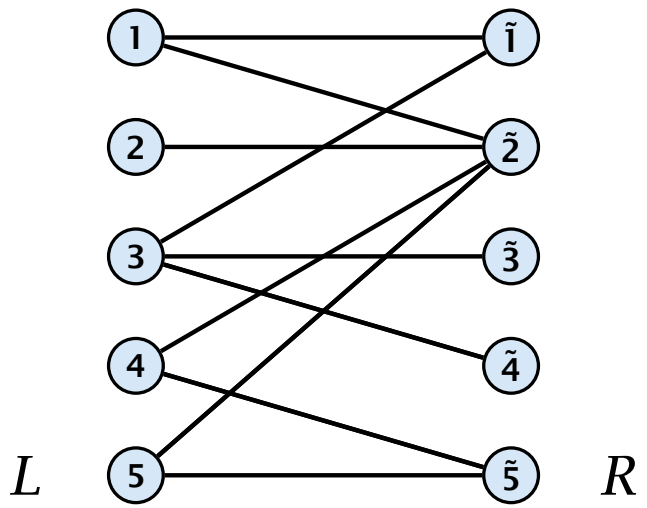
Proof

Max cardinality matching in $G \geq$ value of maxflow in G'

- Let f be a maxflow in G' of value k
- Integrality theorem $\Rightarrow$ k integral; we can assume f is 0/1.
- Consider M= set of edges from L to R with $f(e) = 1$.
- Each node in L and R participates in at most one edge in M.
- $|M| = k$, as the flow must use at least k middle edges.



G'



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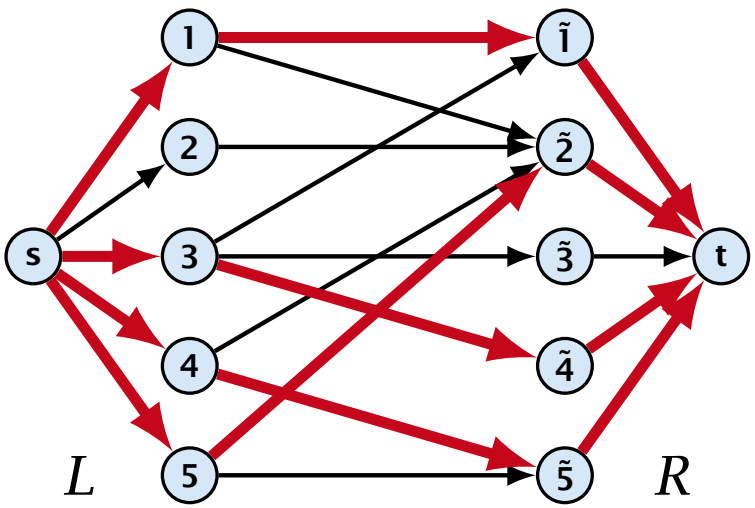

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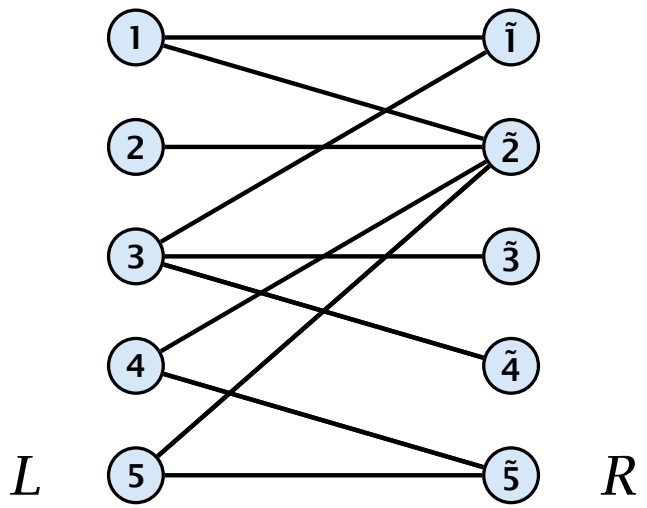

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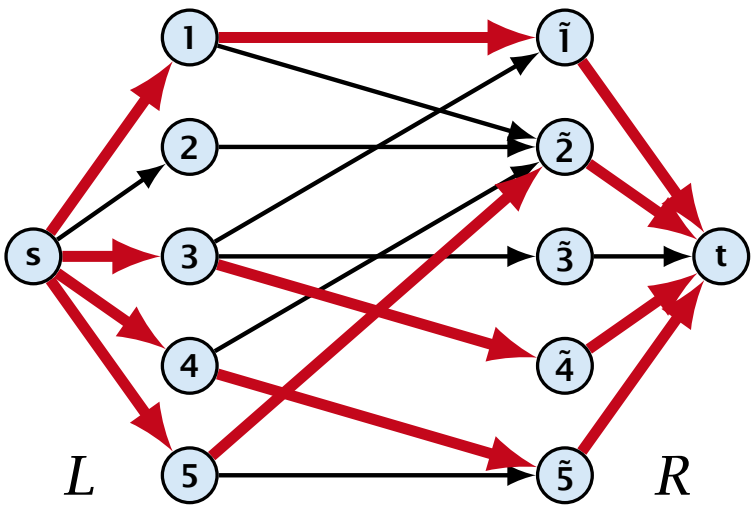

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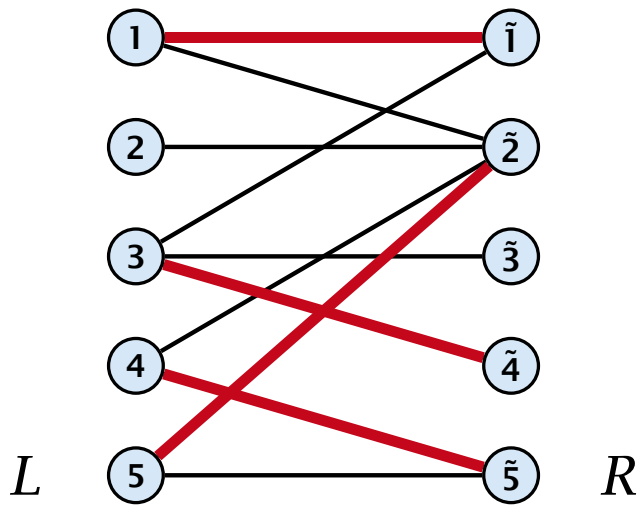

G

17 Bipartite Matching via Flows

Which flow algorithm to use?

- Generic augmenting path: $\mathcal{O}(m \operatorname{val}(f^*)) = \mathcal{O}(mn)$.
- Capacity scaling: $\mathcal{O}(m^2 \log C) = \mathcal{O}(m^2)$.

18 Augmenting Paths for Matchings

Definitions.

- Given a matching M in a graph G, a vertex that is not incident to any edge of M is called a free vertex w. r. .t. M.
- For a matching M a path P in G is called an alternating path if edges in M alternate with edges not in M.
- An alternating path is called an augmenting path for matching M if it ends at distinct free vertices.

Theorem 92

A matching M is a maximum matching if and only if there is no augmenting path w. r. t. M.

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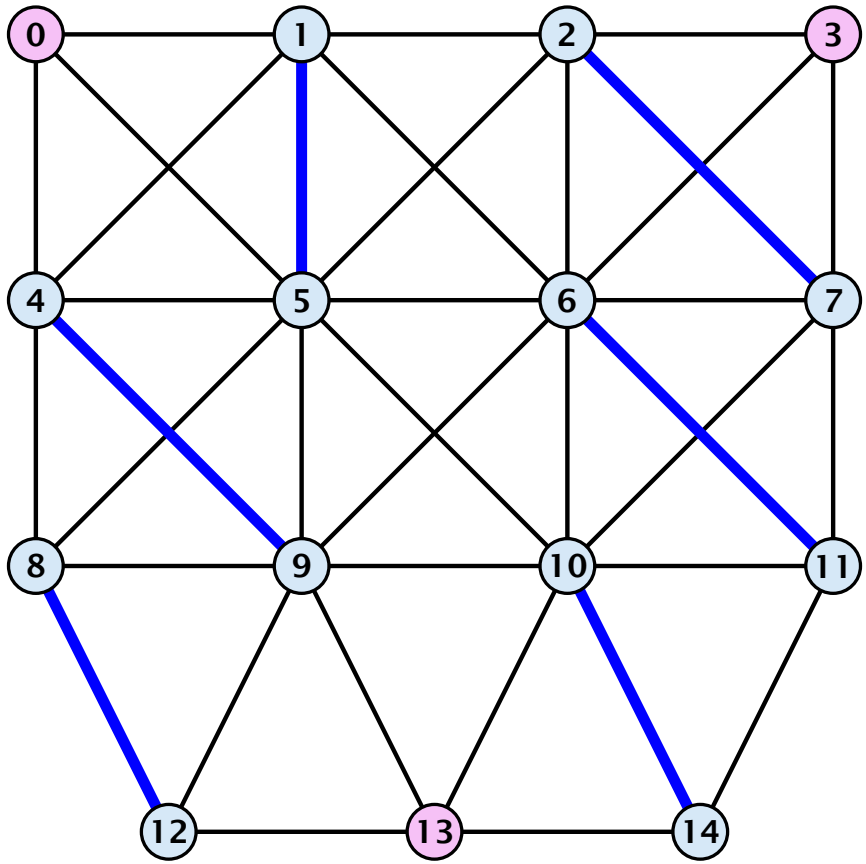

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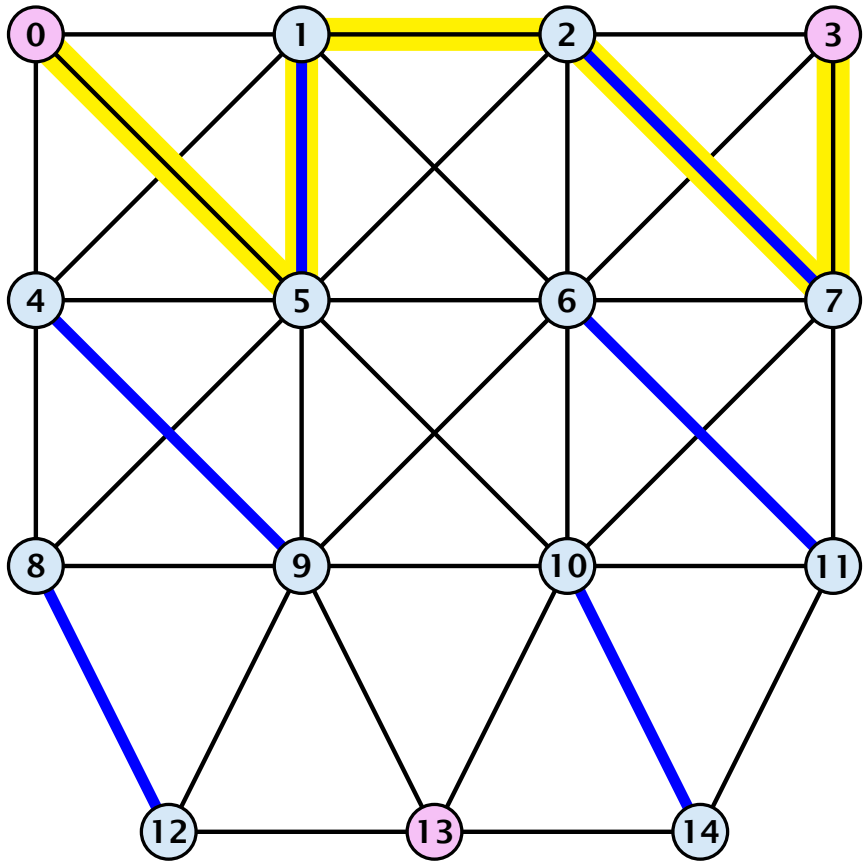

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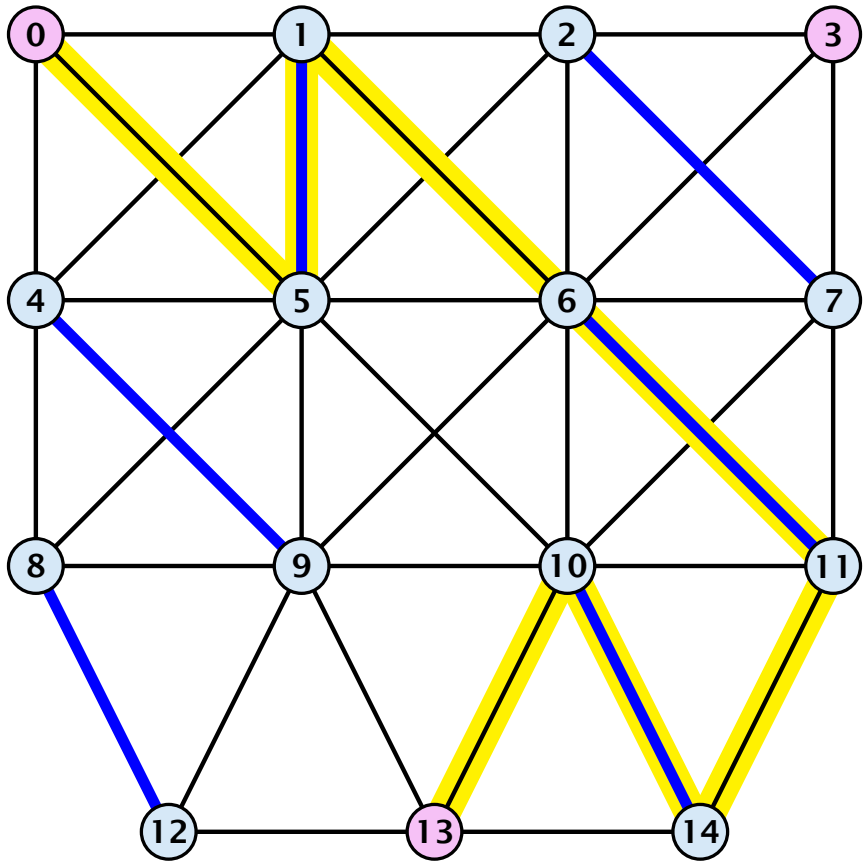

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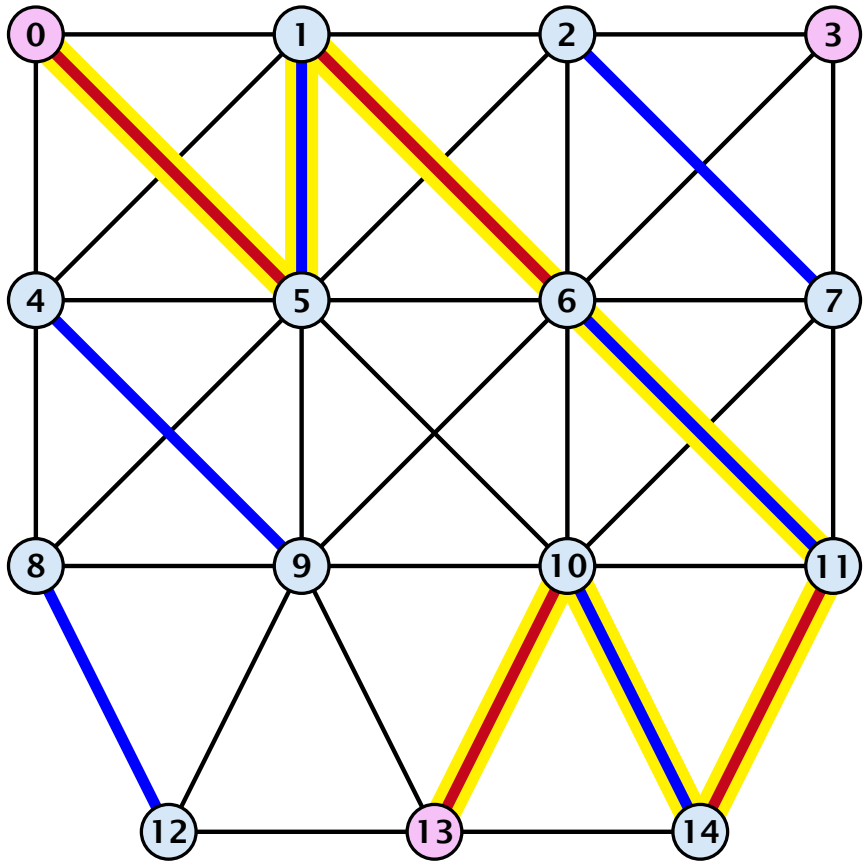

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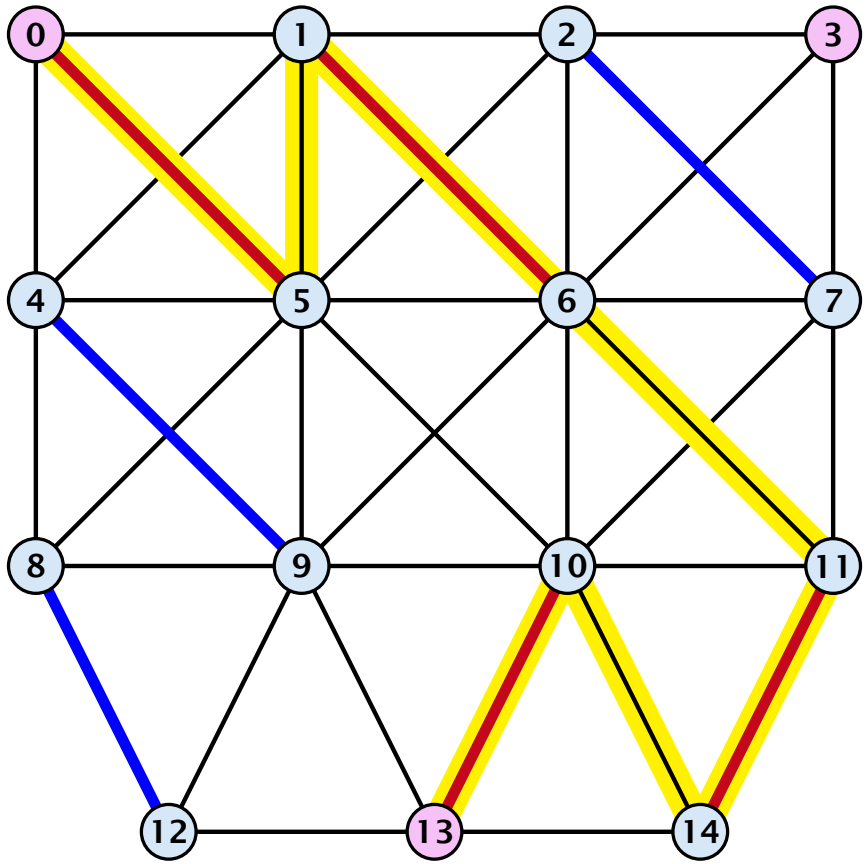

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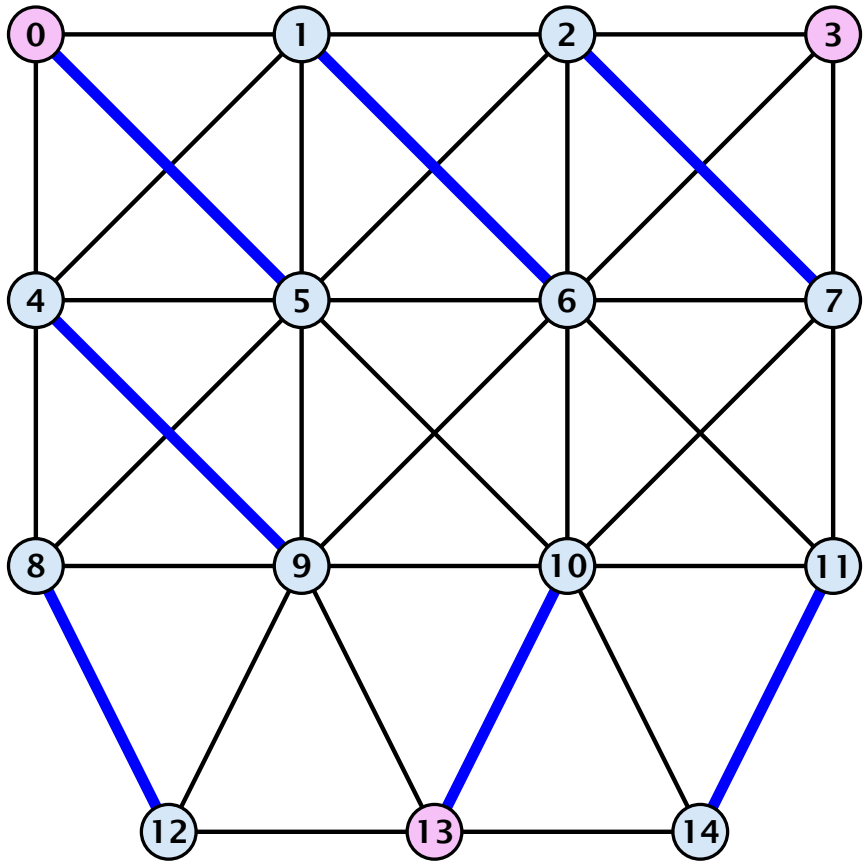

18 Augmenting Paths for Matchings

Proof.

⇒ If M is maximum there is no augmenting path P, because we could switch matching and non-matching edges along P. This gives matching $M' = M \oplus P$ with larger cardinality.

⇐ Suppose there is a matching M' with larger cardinality. Consider the graph H with edge-set $M' \oplus M$ (i.e., only edges that are in either M or M' but not in both).

Each vertex can be incident to at most two edges (one from M and one from M'). Hence, the connected components are alternating cycles or alternating path.

As $|M'| > |M|$ there is one connected component that is a path P for which both endpoints are incident to edges from M'. P is an alternating path.

18 Augmenting Paths for Matchings

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- ⇐ Suppose there is a matching M' with larger cardinality. Consider the graph H with edge-set $M' \oplus M$ (i.e., only edges that are in either M or M' but not in both).

  Each vertex can be incident to at most two edges (one from M and one from M'). Hence, the connected components are alternating cycles or alternating path.

  As $|M'| > |M|$ there is one connected component that is a path P for which both endpoints are incident to edges from M'. P is an alternating path.

18 Augmenting Paths for Matchings

Proof.

- $\Rightarrow$ If M is maximum there is no augmenting path P, because we could switch matching and non-matching edges along P. This gives matching $M' = M \oplus P$ with larger cardinality.
- $\Leftarrow$ Suppose there is a matching M' with larger cardinality. Consider the graph H with edge-set $M' \oplus M$ (i.e., only edges that are in either M or M' but not in both).

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  As $|M'| > |M|$ there is one connected component that is a path P for which both endpoints are incident to edges from M'. P is an alternating path.

18 Augmenting Paths for Matchings

Algorithmic idea:
As long as you find an augmenting path augment your matching using this path. When you arrive at a matching for which no augmenting path exists you have a maximum matching.

Theorem 93
Let G be a graph, M a matching in G, and let u be a free vertex w.r.t. M. Further let P denote an augmenting path w.r.t. M and let $M' = M \oplus P$ denote the matching resulting from augmenting M with P. If there was no augmenting path starting at u in M then there is no augmenting path starting at u in M'.

18 Augmenting Paths for Matchings

Algorithmic idea:
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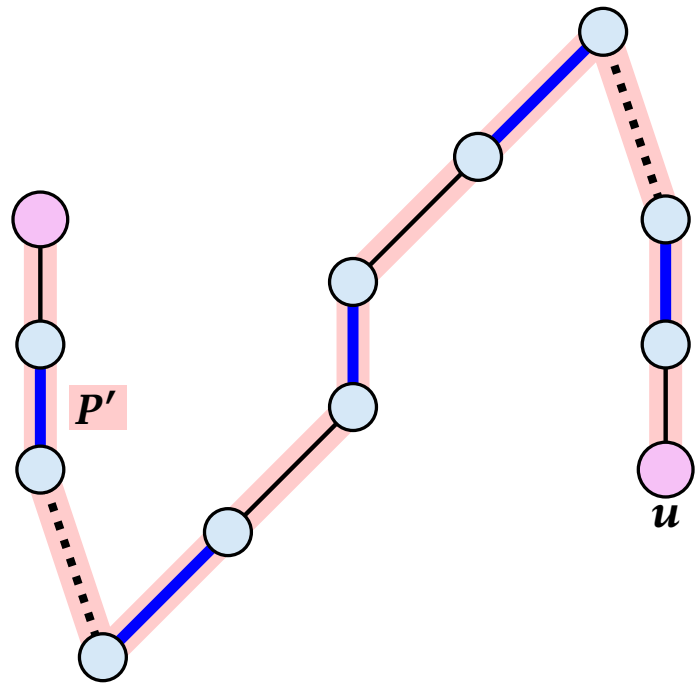
Proof

18 Augmenting Paths for Matchings

Proof

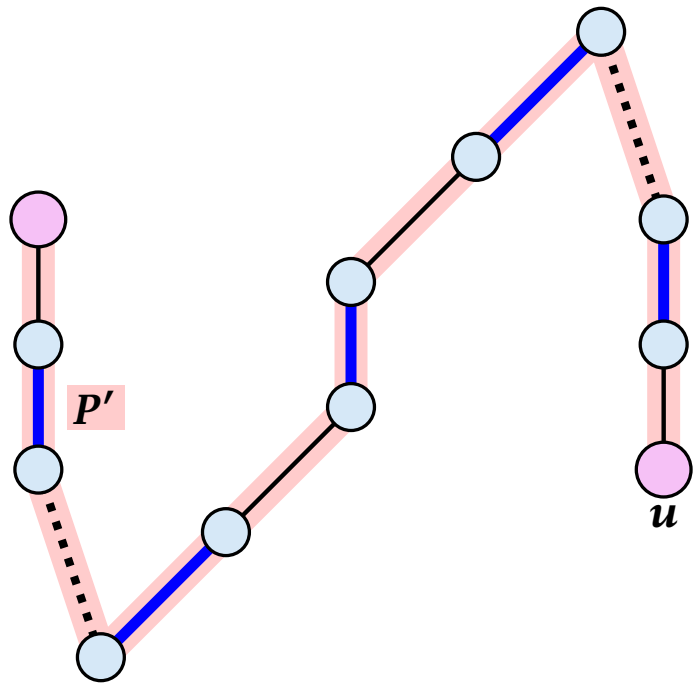
- Assume there is an augmenting path P' w.r.t. M' starting at u.



18 Augmenting Paths for Matchings

Proof

- Assume there is an augmenting path P' w.r.t. M' starting at u.
- If P' and P are node-disjoint, P' is also augmenting path w.r.t. M (↯).



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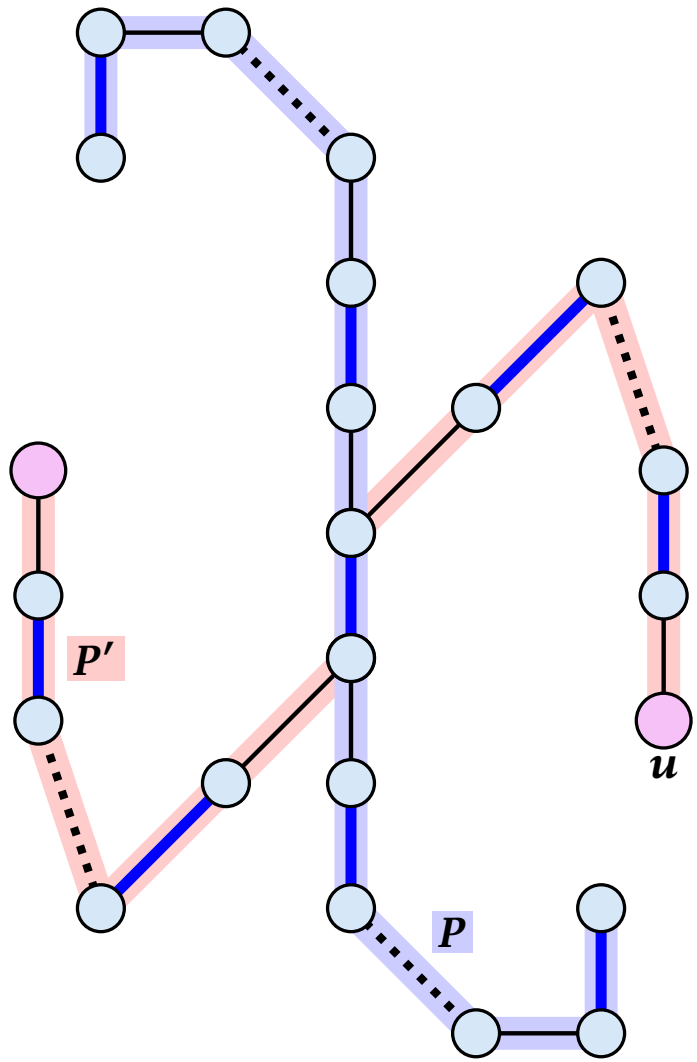

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- Assume there is an augmenting path P' w.r.t. M' starting at u.
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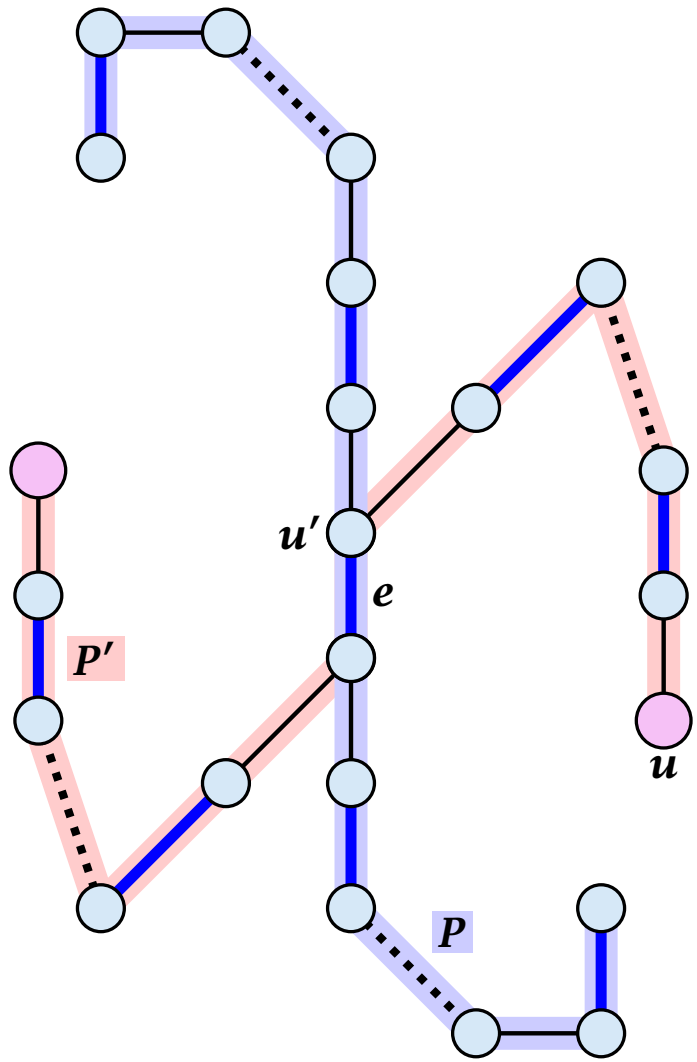
- Let u' be the first node on P' that is in P, and let e be the matching edge from M' incident to u'.



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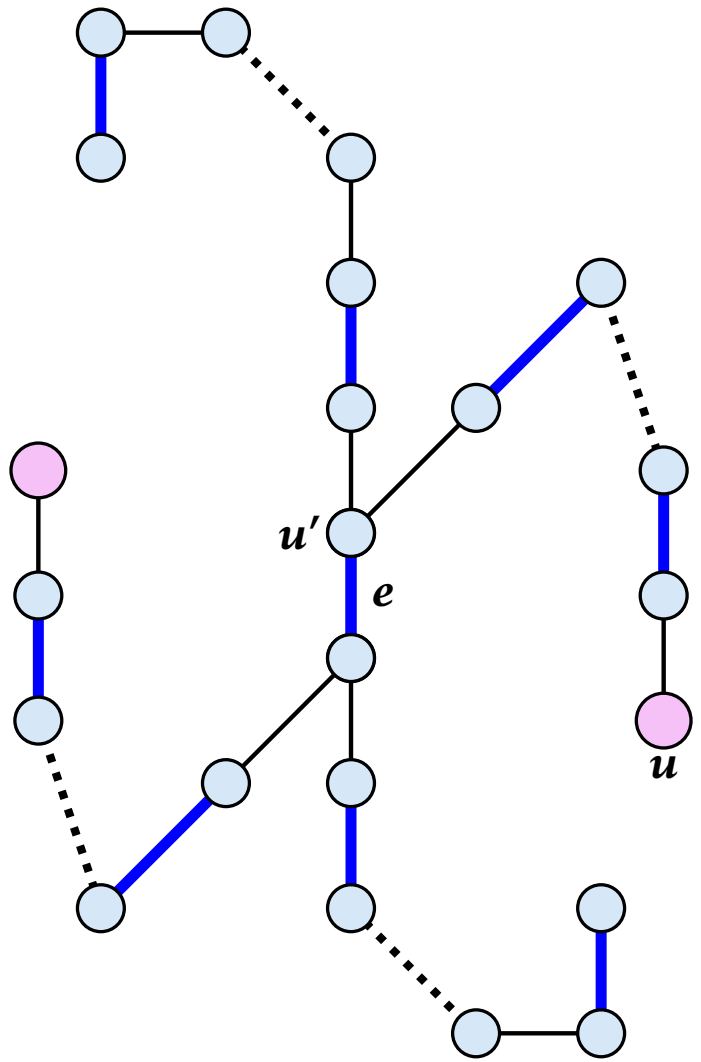

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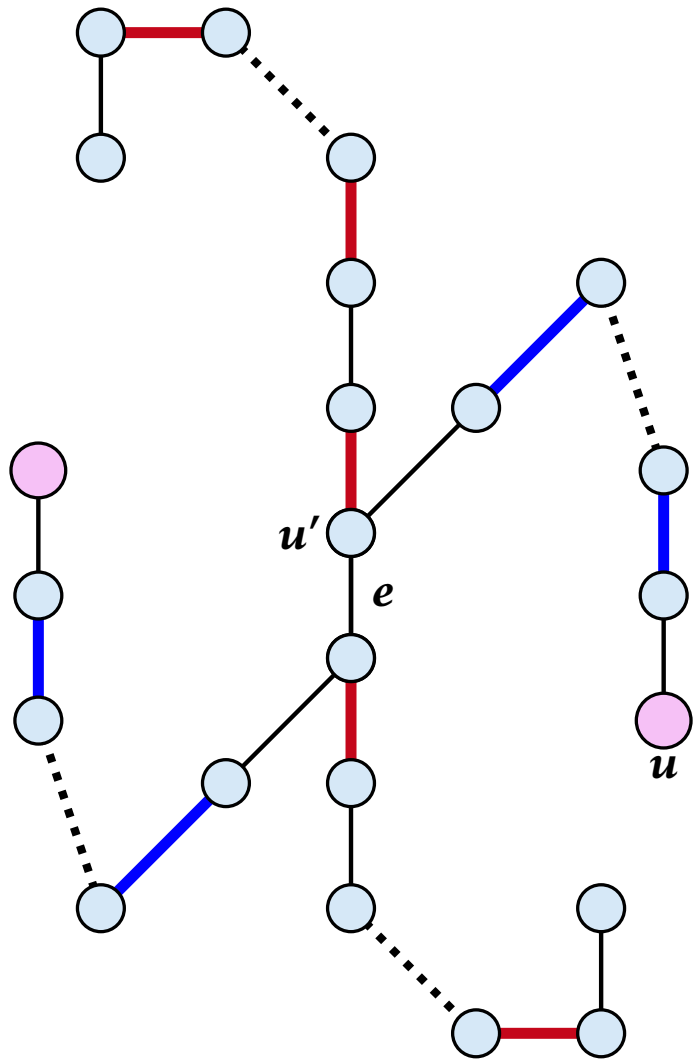

18 Augmenting Paths for Matchings

Proof

- Assume there is an augmenting path P' w.r.t. M' starting at u.
- If P' and P are node-disjoint, P' is also augmenting path w.r.t. M (↯).
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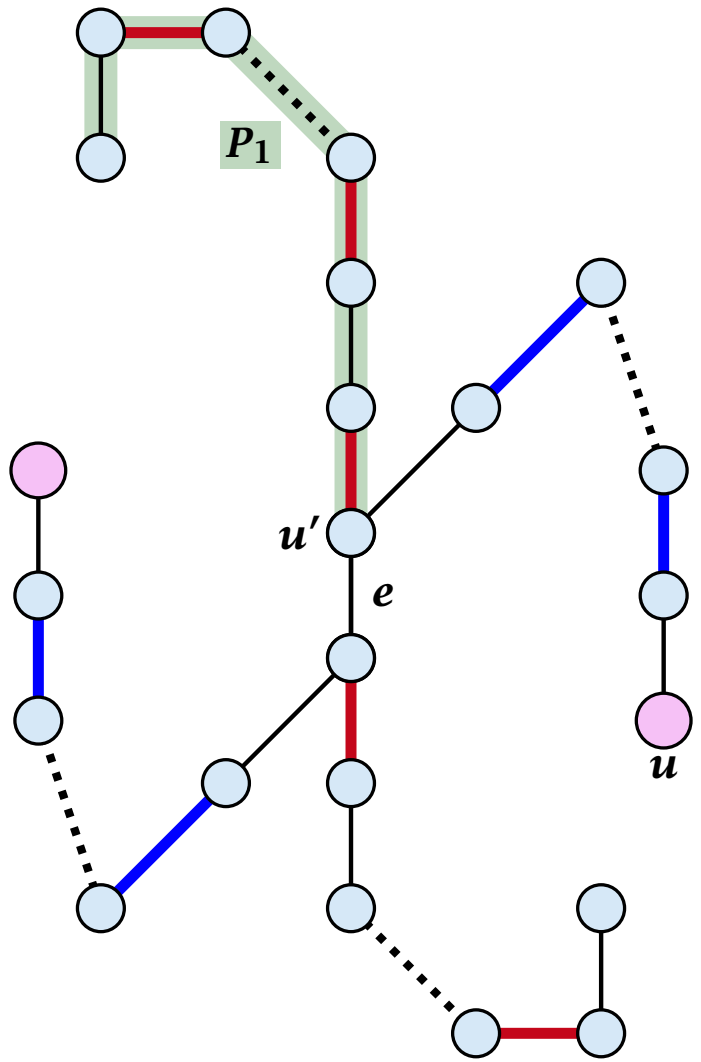

18 Augmenting Paths for Matchings

Proof

- Assume there is an augmenting path P' w.r.t. M' starting at u.
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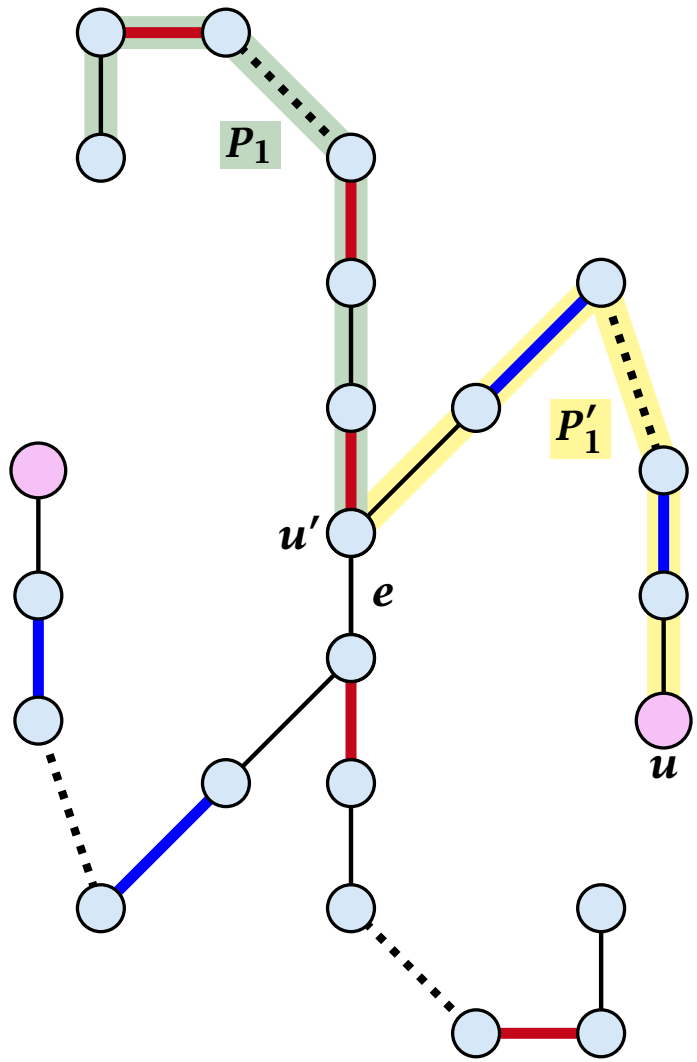
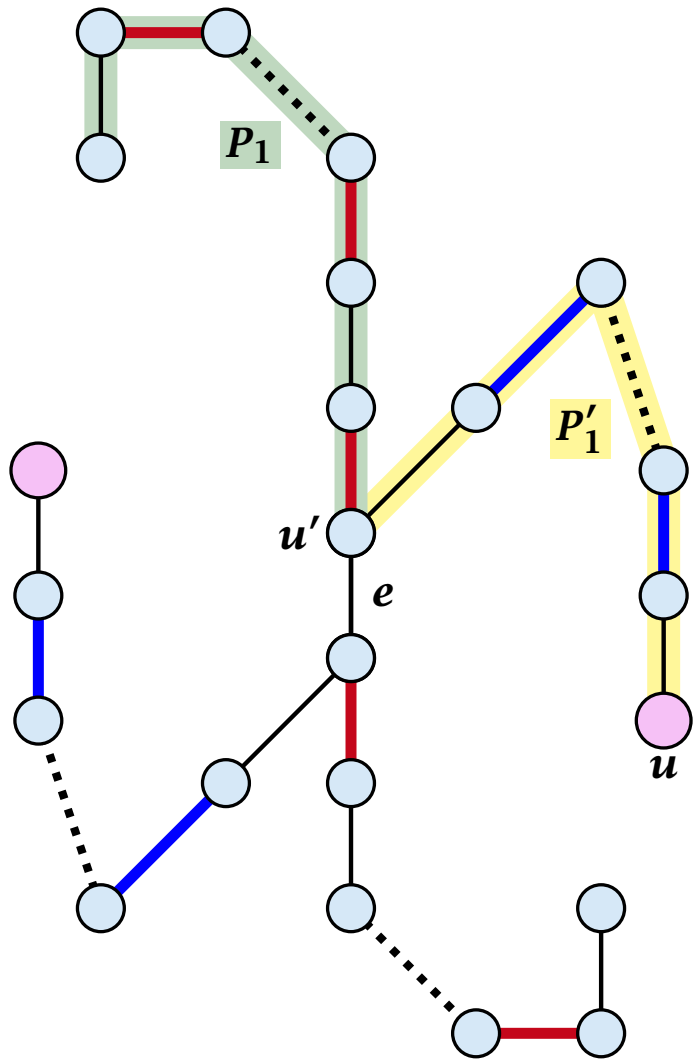


18 Augmenting Paths for Matchings

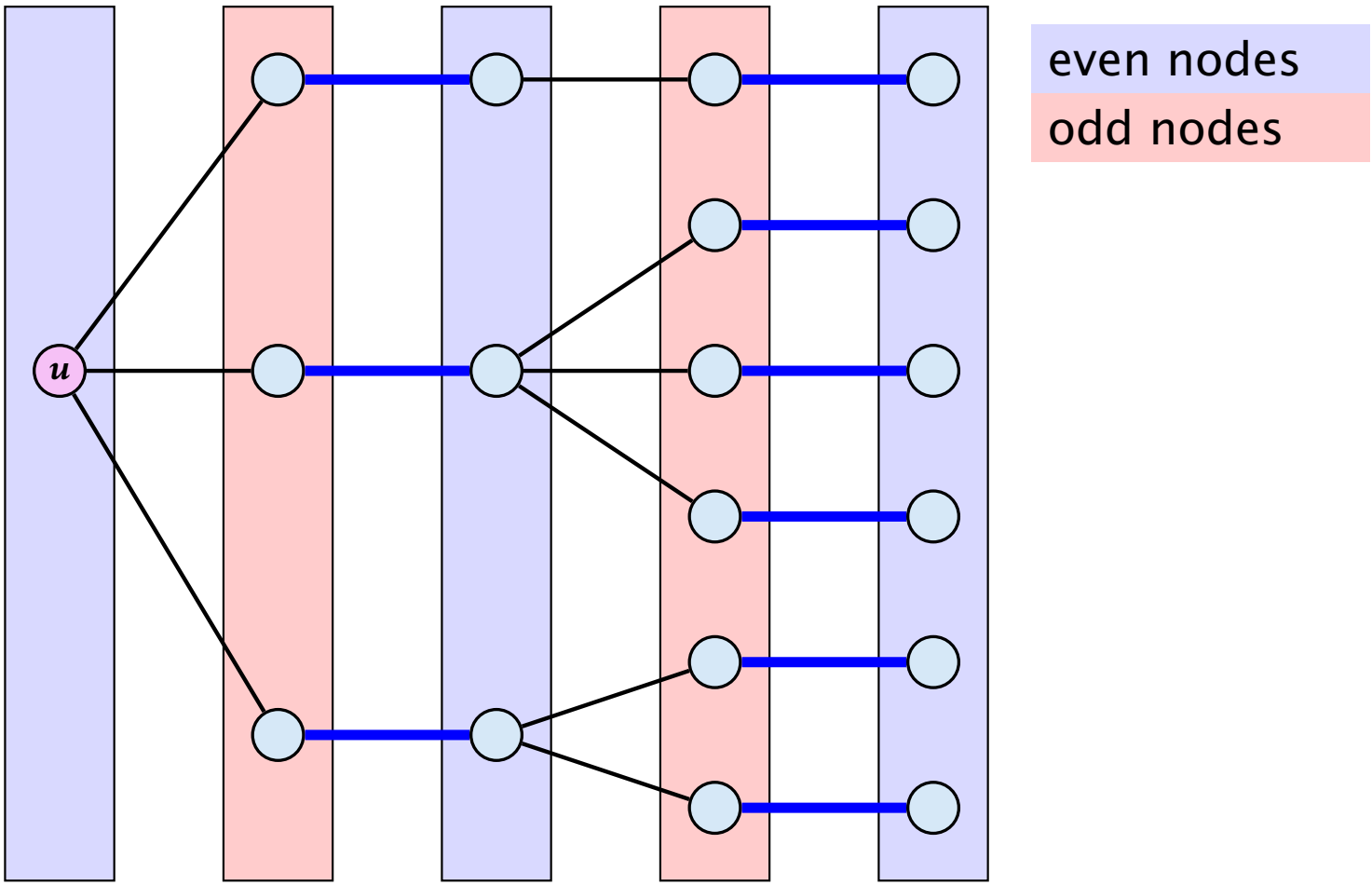
Proof

- Assume there is an augmenting path P' w.r.t. M' starting at u.
- If P' and P are node-disjoint, P' is also augmenting path w.r.t. M (↯).
- Let u' be the first node on P' that is in P, and let e be the matching edge from M' incident to u'.
- u' splits P into two parts one of which does not contain e. Call this part P_1. Denote the sub-path of P' from u to u' with P'_1.
- $P_1 \circ P'_1$ is augmenting path in M (↯).



How to find an augmenting path?

Construct an alternating tree.



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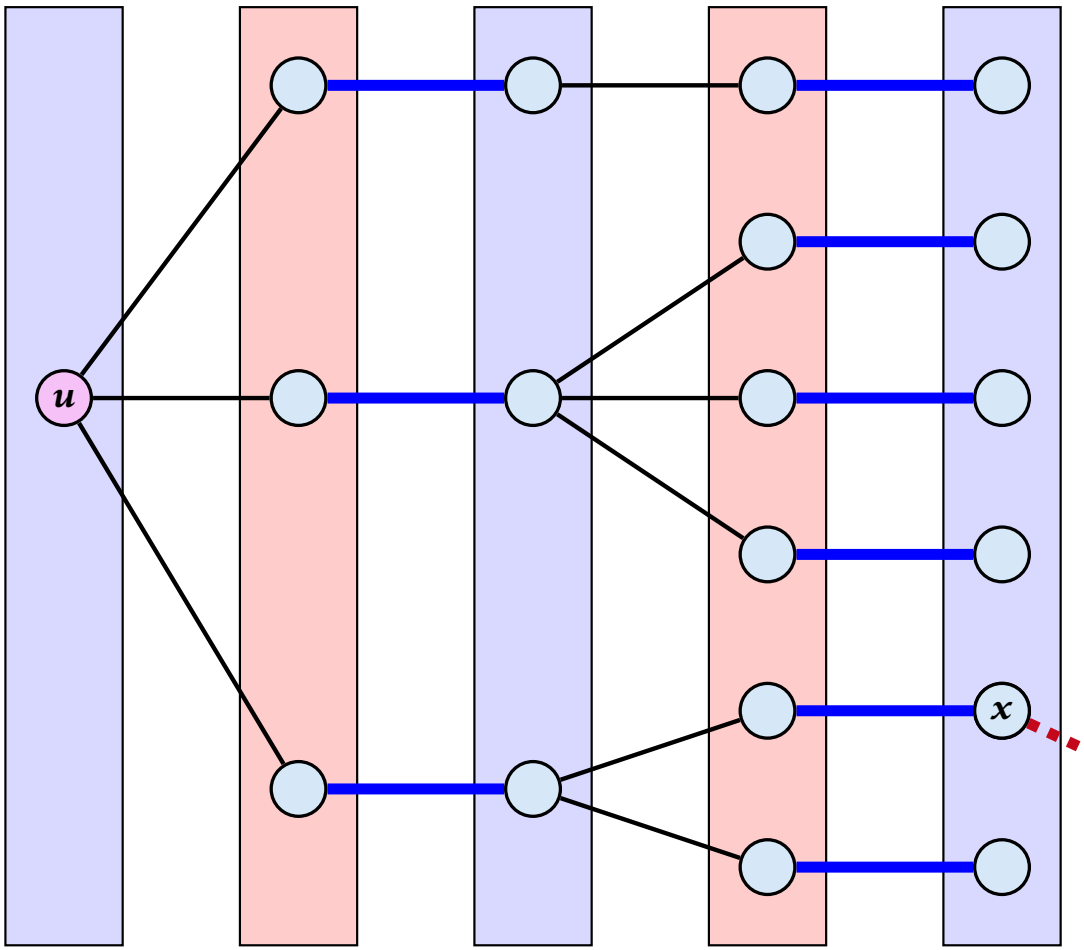


even nodes

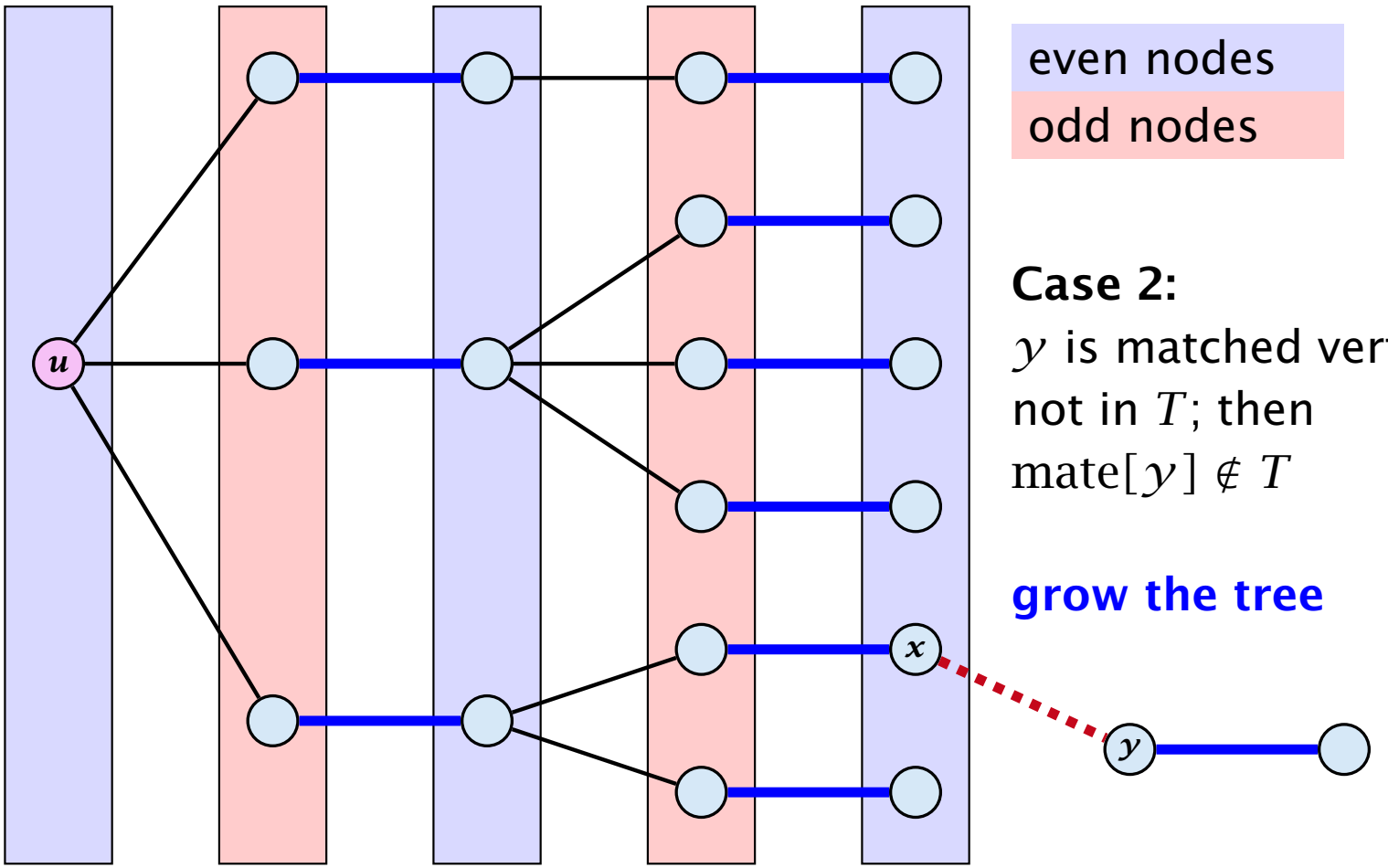
odd nodes

Case 1:
y is free vertex not contained in T

you found alternating path

How to find an augmenting path?

Construct an alternating tree.



even nodes

odd nodes

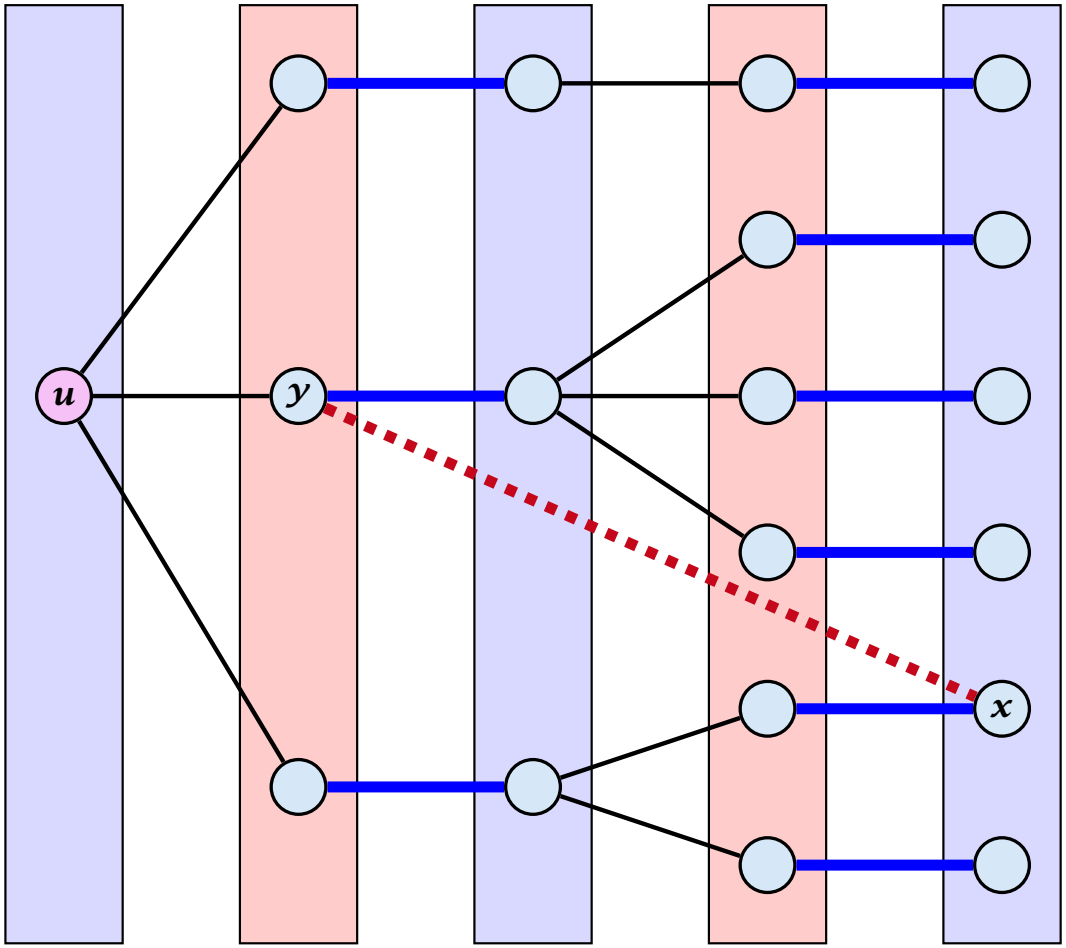
Case 2:

y is matched vertex not in T; then $\text{mate}[y] \notin T$

grow the tree

How to find an augmenting path?

Construct an alternating tree.



even nodes

odd nodes

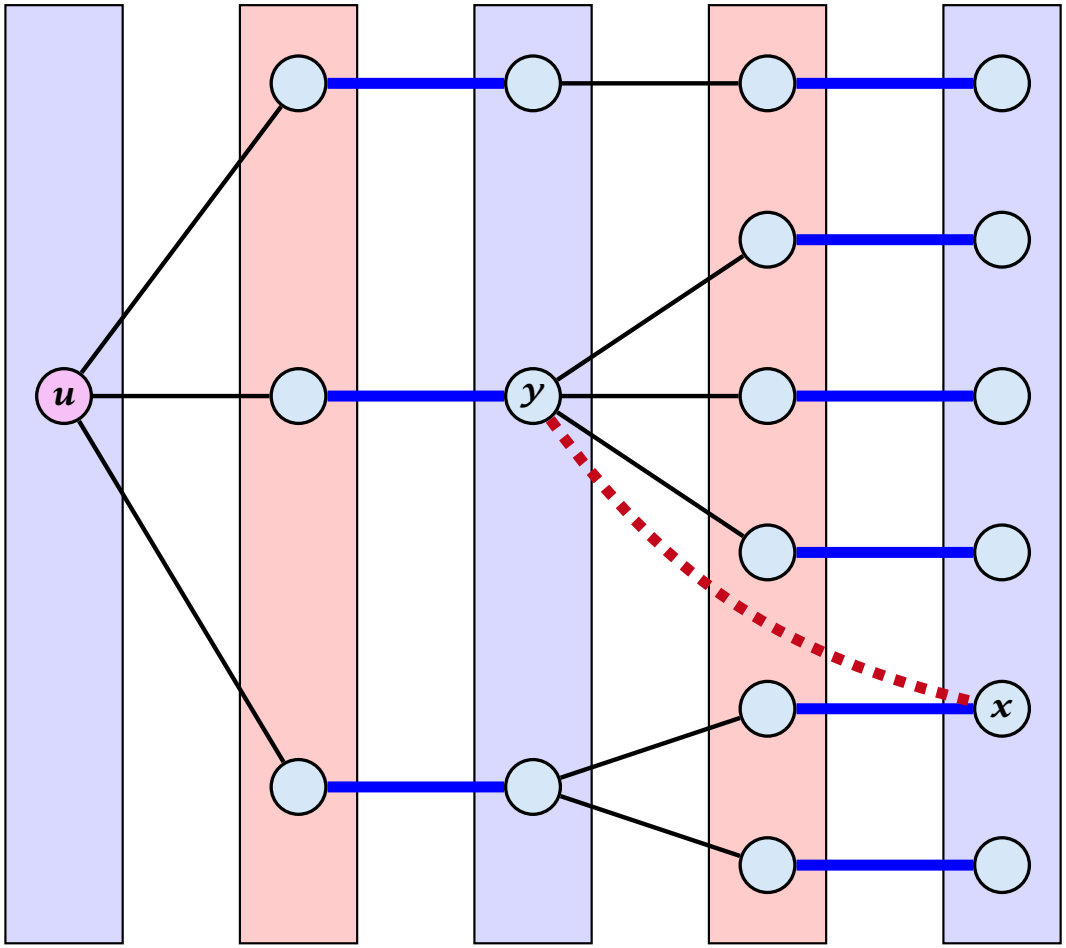
Case 3:

y is already contained in T as an odd vertex

ignore successor y

How to find an augmenting path?

Construct an alternating tree.



even nodes

odd nodes

Case 4:

y is already contained in T as an even vertex

can't ignore y

does not happen in bipartite graphs

Algorithm 52 BiMatch($G, match$)

```
 1: for x ∈ V do mate[x] ← 0;
 2: r ← 0; free ← n;
 3: while free ≥ 1 and r < n do
 4:     r ← r + 1
 5:     if mate[r] = 0 then
 6:         for i = 1 to m do parent[i'] ← 0
 7:         Q ← ∅; Q.append(r); aug ← false;
 8:         while aug = false and Q ≠ ∅ do
 9:             x ← Q.dequeue();
10:             for y ∈ A_x do
11:                 if mate[y] = 0 then
12:                     augm(mate, parent, y);
13:                     aug ← true;
14:                     free ← free − 1;
15:                 else
16:                     if parent[y] = 0 then
17:                         parent[y] ← x;
18:                         Q.enqueue(mate[y]);
```

graph $G = (S \cup S', E)$

$S = \{1, \ldots, n\}$

$S' = \{1', \ldots, n'\}$

Algorithm 52 BiMatch$(G, match)$

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 1: for x ∈ V do mate[x] ← 0;
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```

start with an
empty matching

Algorithm 52 BiMatch$(G, match)$

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16:                     if parent[y] = 0 then
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18:                         Q.enqueue(mate[y]);
```

free: number of unmatched nodes in S

r: root of current tree

Algorithm 52 BiMatch$(G, match)$

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 1: for x ∈ V do mate[x] ← 0;
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14:                     free ← free − 1;
15:                 else
16:                     if parent[y] = 0 then
17:                         parent[y] ← x;
18:                         Q.enqueue(mate[y]);
```

as long as there are unmatched nodes and we did not yet try to grow from all nodes we continue

Algorithm 52 BiMatch(G, *match*)

```
 1: for x ∈ V do mate[x] ← 0;
 2: r ← 0; free ← n;
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18:                         Q.enqueue(mate[y]);
```

r is the new node that we grow from.

Algorithm 52 BiMatch$(G, match)$

```
 1: for x ∈ V do mate[x] ← 0;
 2: r ← 0; free ← n;
 3: while free ≥ 1 and r < n  do
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```

If r is free start tree construction

Algorithm 52 BiMatch$(G, match)$

```
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 3: while free ≥ 1 and r < n do
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```

Initialize an empty tree.
Note that only nodes i' have parent pointers.

Algorithm 52 BiMatch(G, *match*)

```
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14:                     free ← free − 1;
15:                 else
16:                     if parent[y] = 0 then
17:                         parent[y] ← x;
18:                         Q.enqueue(mate[y]);
```

Q is a queue (BFS!!!).

aug is a Boolean that stores whether we already found an augmenting path.

Algorithm 52 BiMatch$(G, match)$

```
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14:                     free ← free − 1;
15:                 else
16:                     if parent[y] = 0  then
17:                         parent[y] ← x;
18:                         Q.enqueue(mate[y]);
```

as long as we did not augment and there are still unexamined leaves continue...

Algorithm 52 BiMatch(G, *match*)

```
 1: for x ∈ V do mate[x] ← 0;
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take next unexamined leaf

Algorithm 52 BiMatch(G, *match*)

```
 1: for x ∈ V do mate[x] ← 0;
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```

if x has unmatched neighbour we found an augmenting path (note that $y \neq r$ because we are in a bipartite graph)

Algorithm 52 BiMatch(G, *match*)

```
 1: for x ∈ V do mate[x] ← 0;
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14:                     free ← free − 1;
15:                 else
16:                     if parent[y] = 0  then
17:                         parent[y] ← x;
18:                         Q.enqueue(mate[y]);
```

do an augmentation...

Algorithm 52 BiMatch(G, *match*)

```
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16:                     if parent[y] = 0 then
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setting $aug = \text{true}$ ensures that the tree construction will not continue

Algorithm 52 BiMatch(G, *match*)

```
 1: for x ∈ V do mate[x] ← 0;
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14:                     free ← free − 1;
15:                 else
16:                     if parent[y] = 0 then
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18:                         Q.enqueue(mate[y]);
```

reduce number of free nodes

Algorithm 52 BiMatch$(G, match)$

```
 1: for x ∈ V do mate[x] ← 0;
 2: r ← 0; free ← n;
 3: while free ≥ 1 and r < n  do
 4:     r ← r + 1
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11:                 if mate[y] = 0  then
12:                     augm(mate, parent, y);
13:                     aug ← true;
14:                     free ← free − 1;
15:                 else
16:                     if parent[y] = 0  then
17:                         parent[y] ← x;
18:                         Q.enqueue(mate[y]);
```

if y is not in the tree yet

Algorithm 52 BiMatch$(G, match)$

```
 1: for x ∈ V do mate[x] ← 0;
 2: r ← 0; free ← n;
 3: while free ≥ 1 and r < n  do
 4:     r ← r + 1
 5:     if mate[r] = 0  then
 6:         for i = 1 to m do parent[i'] ← 0
 7:         Q ← ∅; Q.append(r); aug ← false;
 8:         while aug = false and Q ≠ ∅  do
 9:             x ← Q.dequeue();
10:             for y ∈ A_x do
11:                 if mate[y] = 0  then
12:                     augm(mate, parent, y);
13:                     aug ← true;
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15:                 else
16:                     if parent[y] = 0  then
17:                         parent[y] ← x;
18:                         Q.enqueue(mate[y]);
```

...put it into the tree

Algorithm 52 BiMatch(G, *match*)

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 1: for x ∈ V do mate[x] ← 0;
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```

add its buddy to the set of unexamined leaves

19 Weighted Bipartite Matching

Weighted Bipartite Matching/Assignment

- Input: undirected, bipartite graph $G = L \cup R, E$.
- an edge $e = (\ell, r)$ has weight $w_e \geq 0$
- find a matching of maximum weight, where the weight of a matching is the sum of the weights of its edges

Simplifying Assumptions (wlog [why?]):

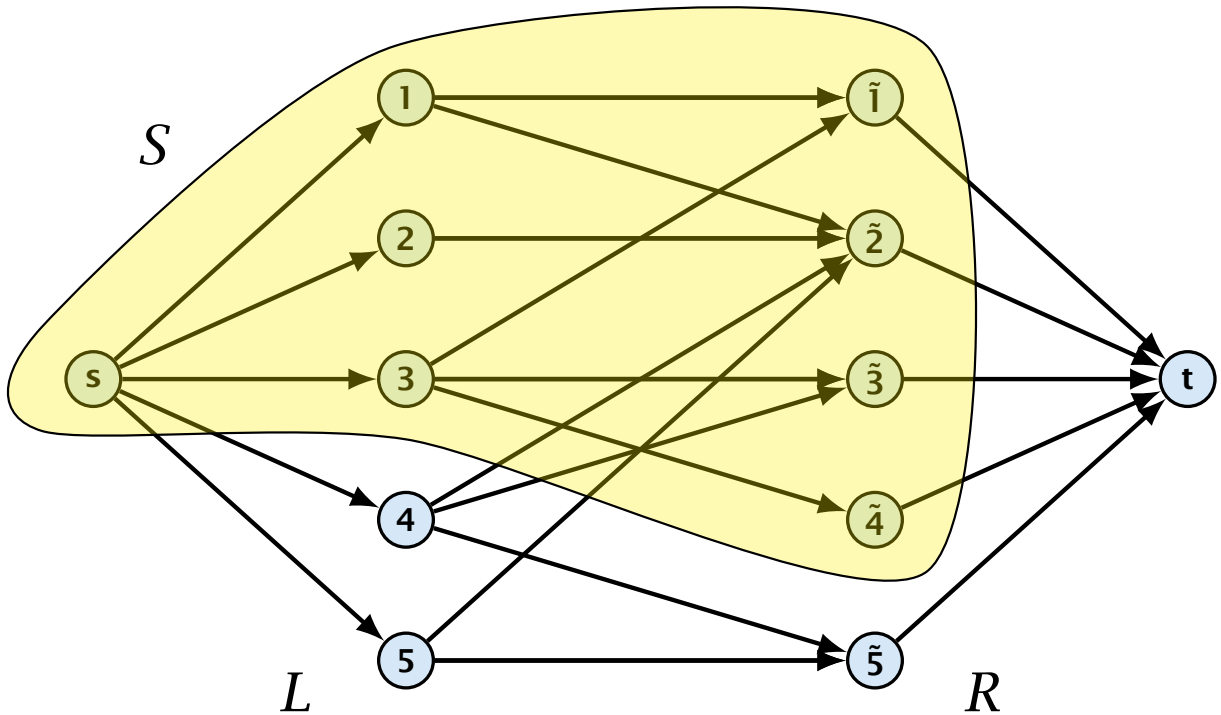
- assume that $|L| = |R| = n$
- assume that there is an edge between every pair of nodes $(\ell, r) \in V \times V$

Weighted Bipartite Matching

Theorem 94 (Halls Theorem)

A bipartite graph $G = (L \cup R, E)$ has a perfect matching if and only if for all sets $S \subseteq L$, $|\Gamma(S)| \geq |S|$, where $\Gamma(S)$ denotes the set of nodes in R that have a neighbour in S.

19 Weighted Bipartite Matching



Halls Theorem

Proof:

- $\Leftarrow$ Of course, the condition is necessary as otherwise not all nodes in S could be matched to different neigbhours.
- $\Rightarrow$ For the other direction we need to argue that the minimum cut in the graph G' is at least $|L|$.

Let S denote a minimum cut and let L_S and $L_{\bar{S}}$ denote the portion of L inside S and $\bar{S}$ respectively.

Clearly, all neighbours of nodes in L_S have to be in R_S, as otherwise we would cut an edge of infinite capacity.

This gives $|N(L_S)| \leq |R_S|$.

The size of the cut is $|L_{\bar{S}}| + |R_S|$.

Using the fact that $|N(L_S)| \geq |L_S|$ gives that this is at least $|L|$.

Halls Theorem

Proof:

- $\Leftarrow$ Of course, the condition is necessary as otherwise not all nodes in S could be matched to different neigbhours.
- $\Rightarrow$ For the other direction we need to argue that the minimum cut in the graph G' is at least $|L|$.
  - Let S denote a minimum cut and let $L_S \stackrel{\text{def}}{=} L \cap S$ and $R_S \stackrel{\text{def}}{=} R \cap S$ denote the portion of S inside L and R, respectively.
  - Clearly, all neighbours of nodes in L_S have to be in S, as otherwise we would cut an edge of infinite capacity.
  - This gives $R_S \geq |\Gamma(L_S)|$.
  - The size of the cut is $|L| - |L_S| + |R_S|$.
  - Using the fact that $|\Gamma(L_S)| \geq L_S$ gives that this is at least $|L|$.

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Algorithm Outline

Idea:

We introduce a node weighting $\vec{x}$. Let for a node $v \in V$, $x_v \geq 0$ denote the weight of node v.

- Suppose that the node weights dominate the edge weights in the following sense:

  $\forall (u,v) \in E : x_u + x_v \geq w(u,v)$

- Let $G_{\vec{x}}$ denote the subgraph of G that only contains edges that are tight w.r.t. the node weighting $\vec{x}$, i.e., edges for which $x_u + x_v = w(u,v)$.

- Try to compute a perfect matching in the subgraph $G_{\vec{x}}$. If you are successful you found an optimal matching.

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- Suppose that the node weights dominate the edge-weights in the following sense:

$$x_u + x_v \geq w_e \text{ for every edge } e = (u, v).$$

- Let $H(\vec{x})$ denote the subgraph of G that only contains edges that are tight w.r.t. the node weighting $\vec{x}$, i.e. edges $e = (u, v)$ for which $w_e = x_u + x_v$.
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Algorithm Outline

Reason:

- The weight of your matching M^* is

$$\sum_{(u,v)\in M^*} w_{(u,v)} = \sum_{(u,v)\in M^*} (x_u + x_v) = \sum_v x_v \ .$$

- Any other matching M has

$$\sum_{(u,v)\in M} w_{(u,v)} \leq \sum_{(u,v)\in M} (x_u + x_v) \leq \sum_v x_v \ .$$

Algorithm Outline

What if you don't find a perfect matching?

Then, Halls theorem guarantees you that there is a set $S \subseteq L$, with $|\Gamma(S)| < |S|$, where Γ denotes the neighbourhood w.r.t. the subgraph $H(\vec{x})$.

Idea: reweight such that:

- the total weight assigned to nodes decreases
- the weight function still dominates the edge-weights

If we can do this we have an algorithm that terminates with an optimal solution (we analyze the running time later).

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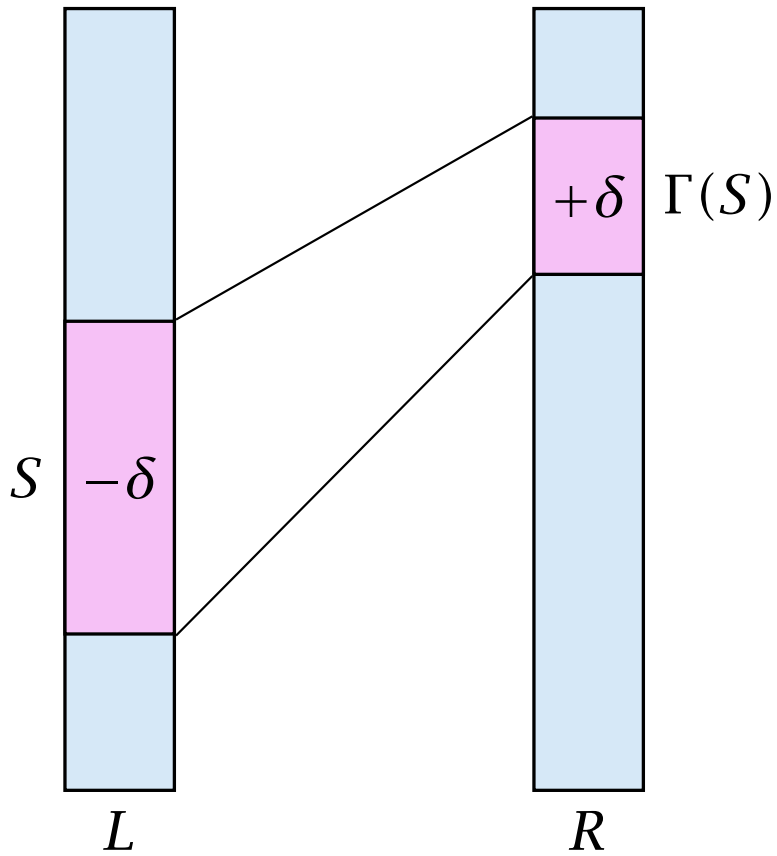
Changing Node Weights

Increase node-weights in $\Gamma(S)$ by $+\delta$, and decrease the node-weights in S by $-\delta$.

- Total node-weight decreases.
- Only edges from S to $R - \Gamma(S)$ decrease in their weight.
- Since, none of these edges is tight (otw. the edge would be contained in $H(\vec{x})$, and hence would go between S and $\Gamma(S)$) we can do this decrement for small enough $\delta > 0$ until a new edge gets tight.



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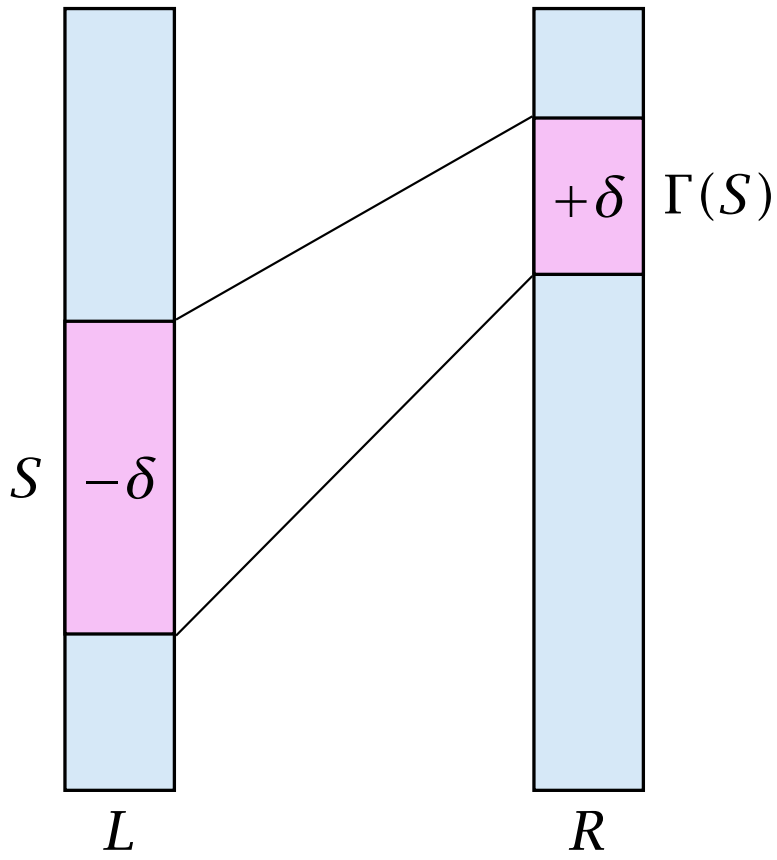

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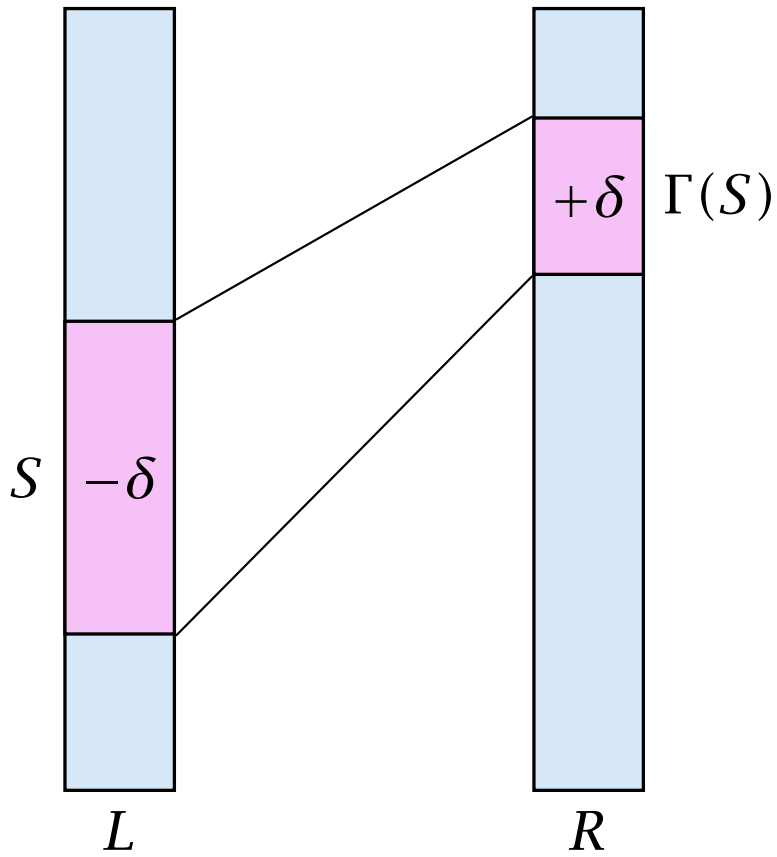

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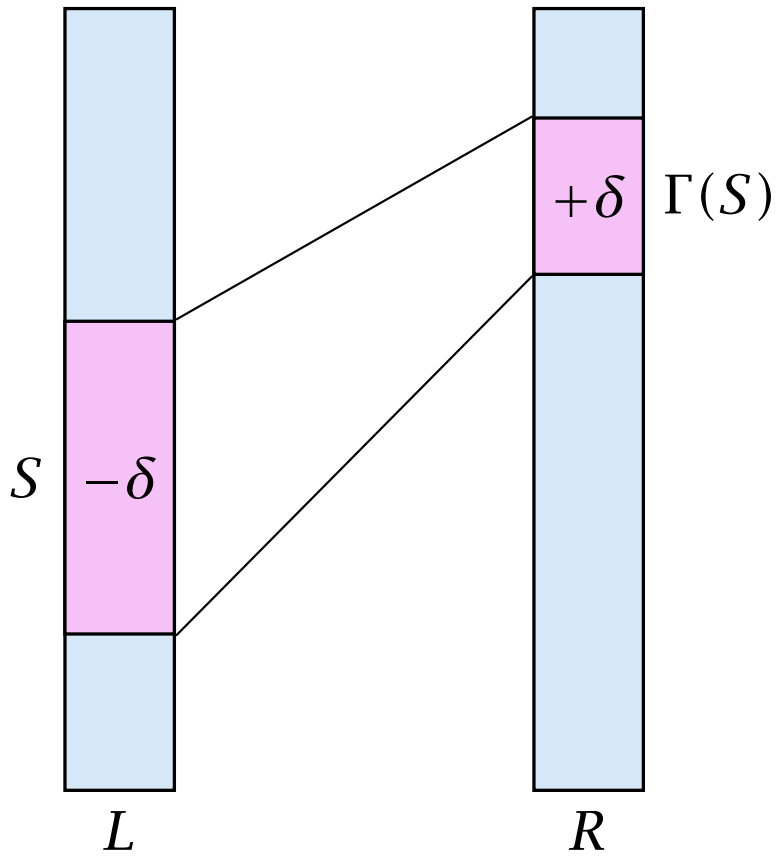
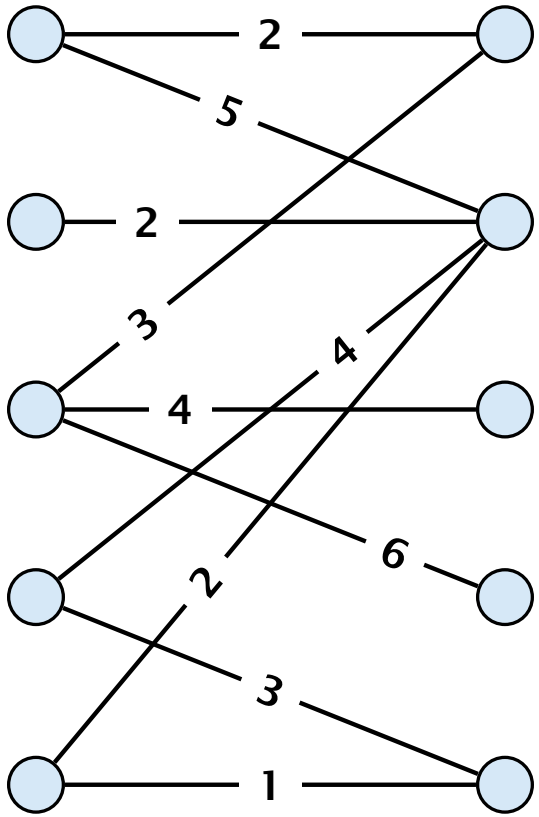


Weighted Bipartite Matching

Edges not drawn have weight 0.



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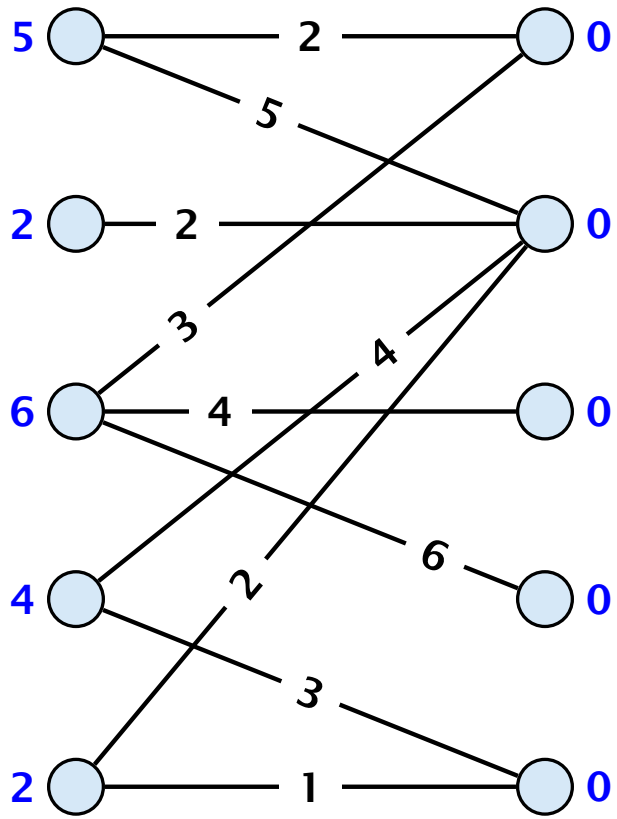

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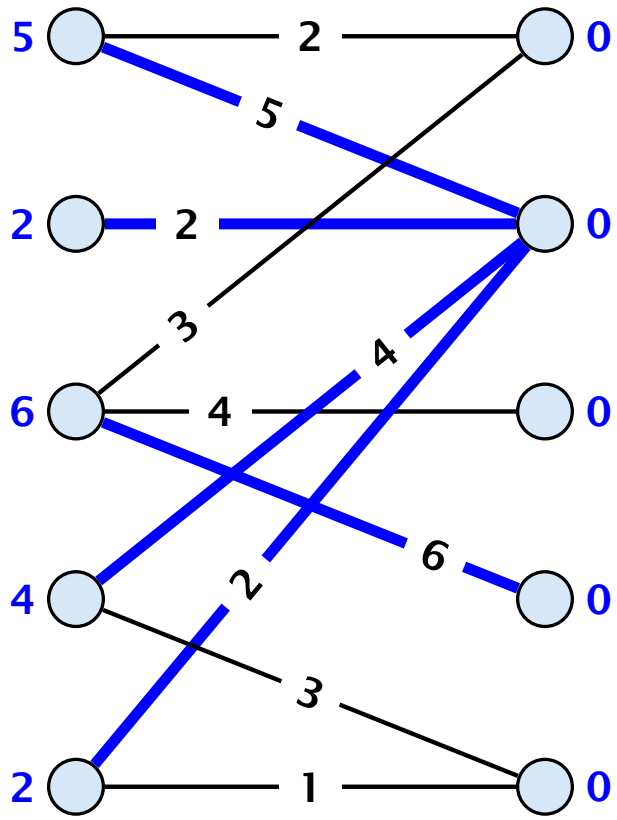

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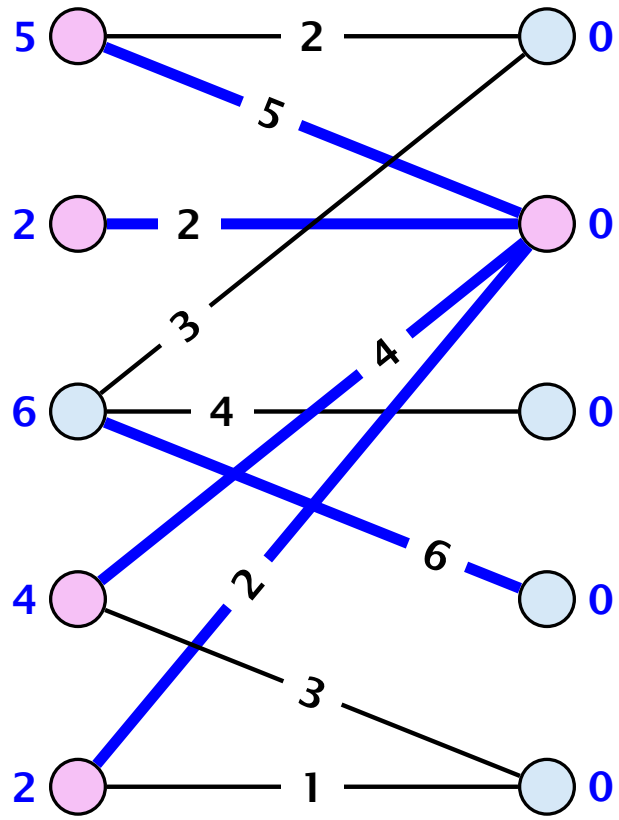

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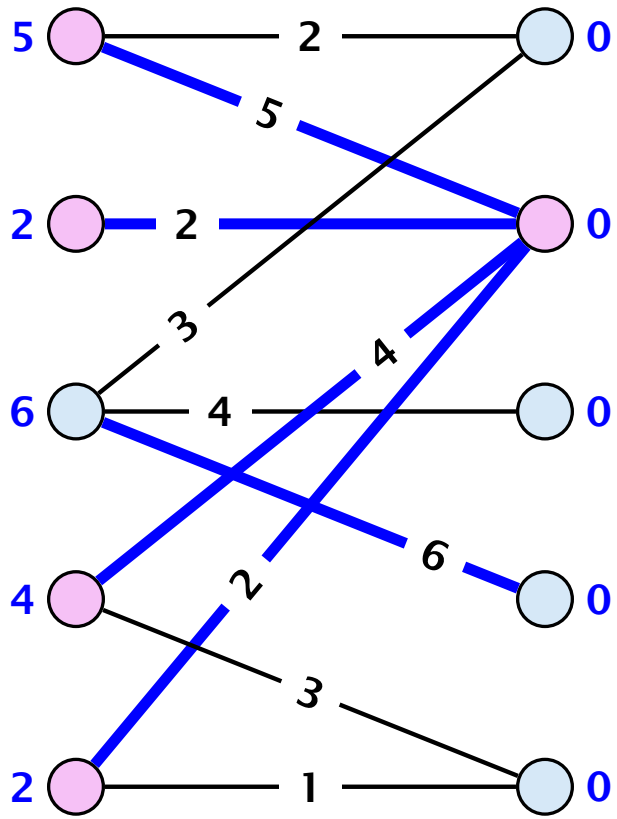

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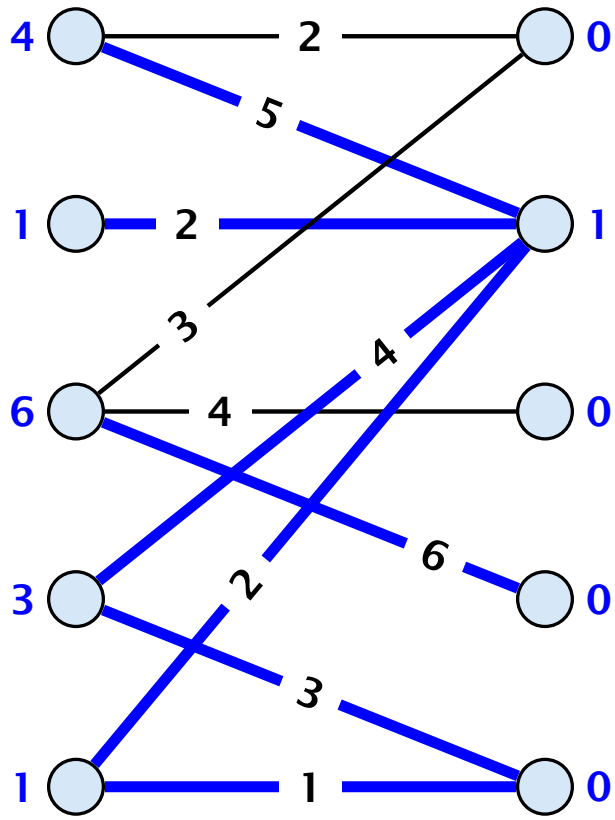

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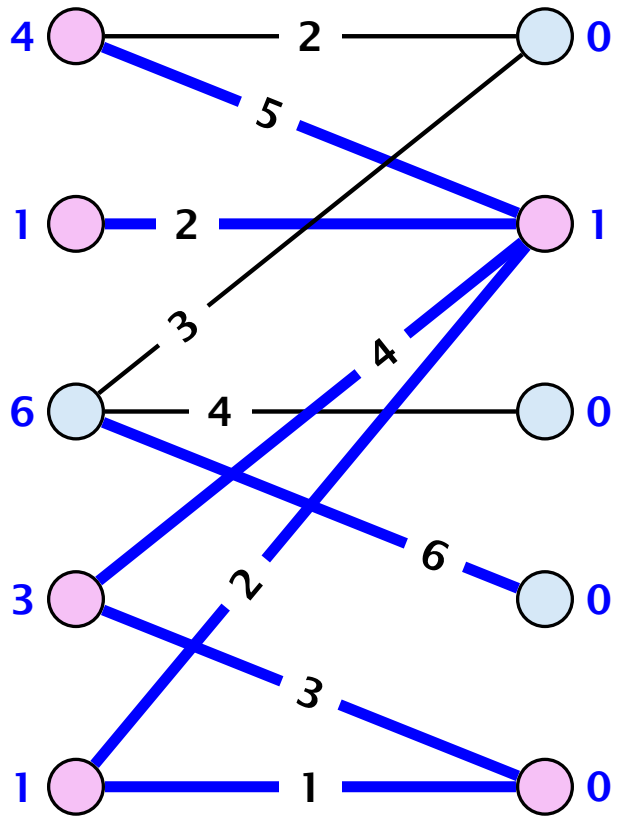

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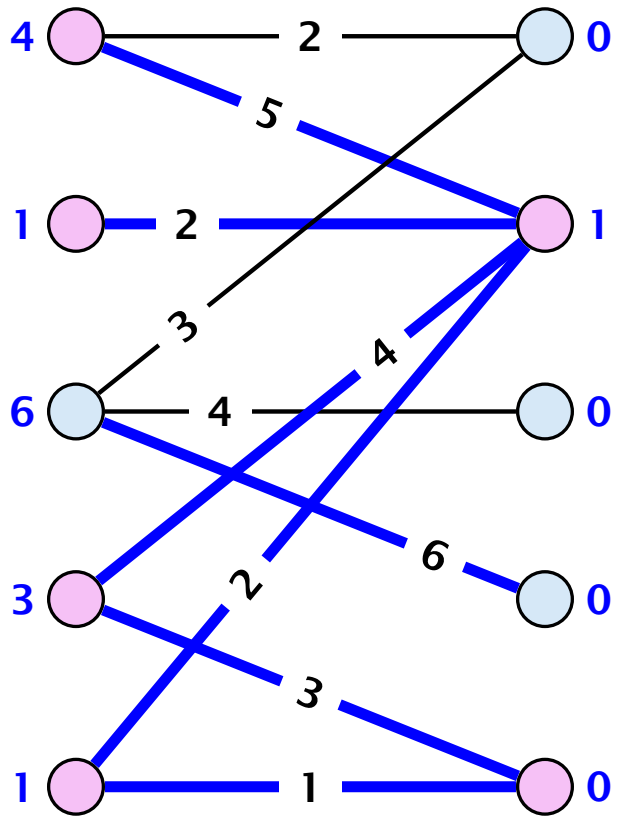

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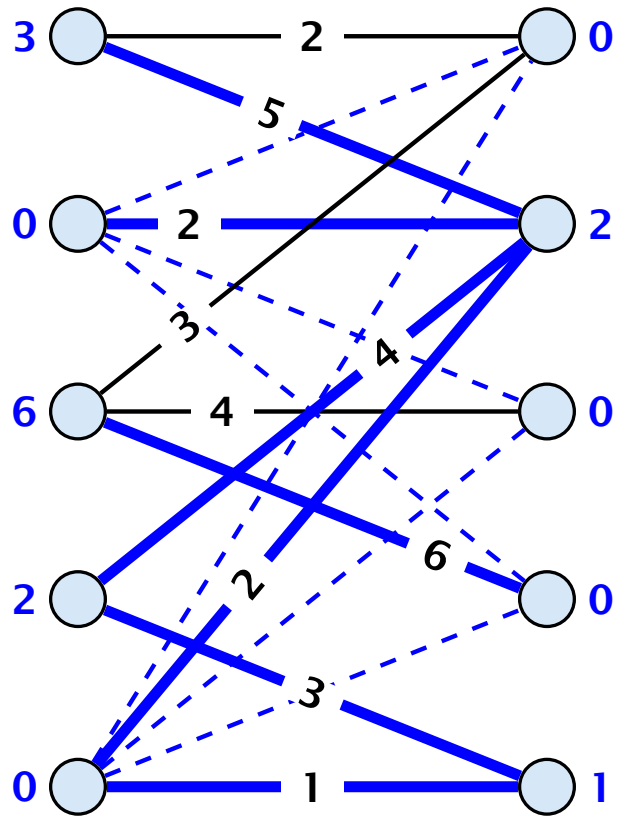

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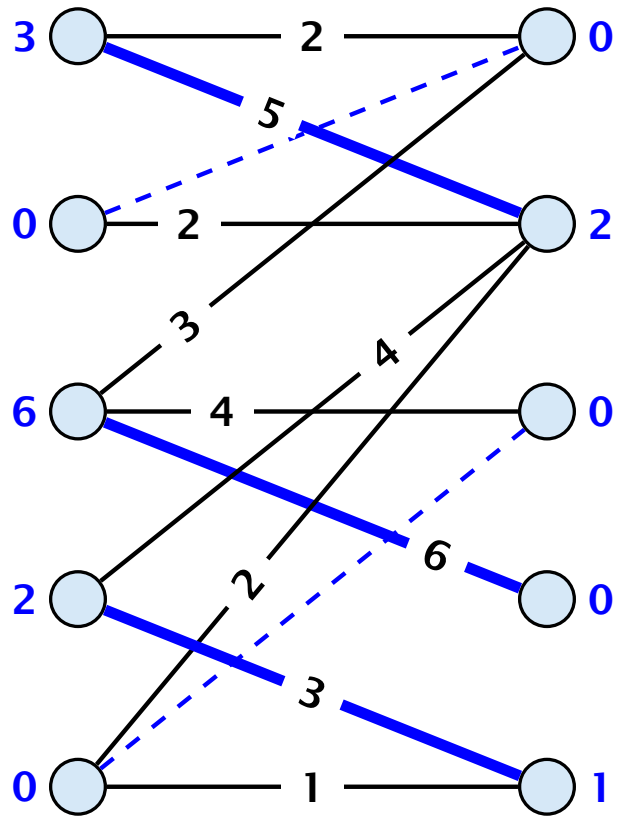

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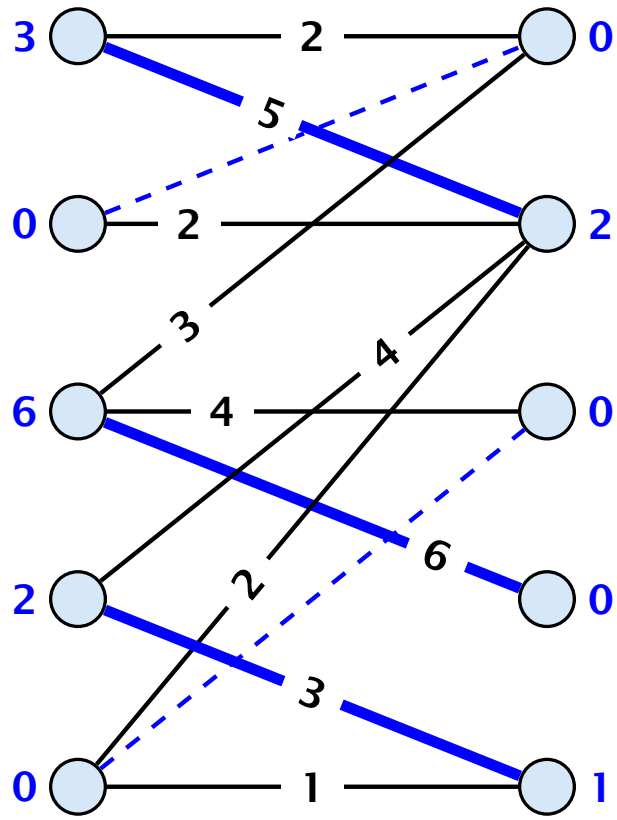

Analysis

How many iterations do we need?

- One reweighting step increases the number of edges out of S by at least one.
- Assume that we have a maximum matching that saturates the set $\Gamma(S)$, in the sense that every node in $\Gamma(S)$ is matched to a node in S (we will show that we can always find S and a matching such that this holds).
- This matching is still contained in the new graph, because all its edges either go between $\Gamma(S)$ and S or between $L - S$ and $R - \Gamma(S)$.
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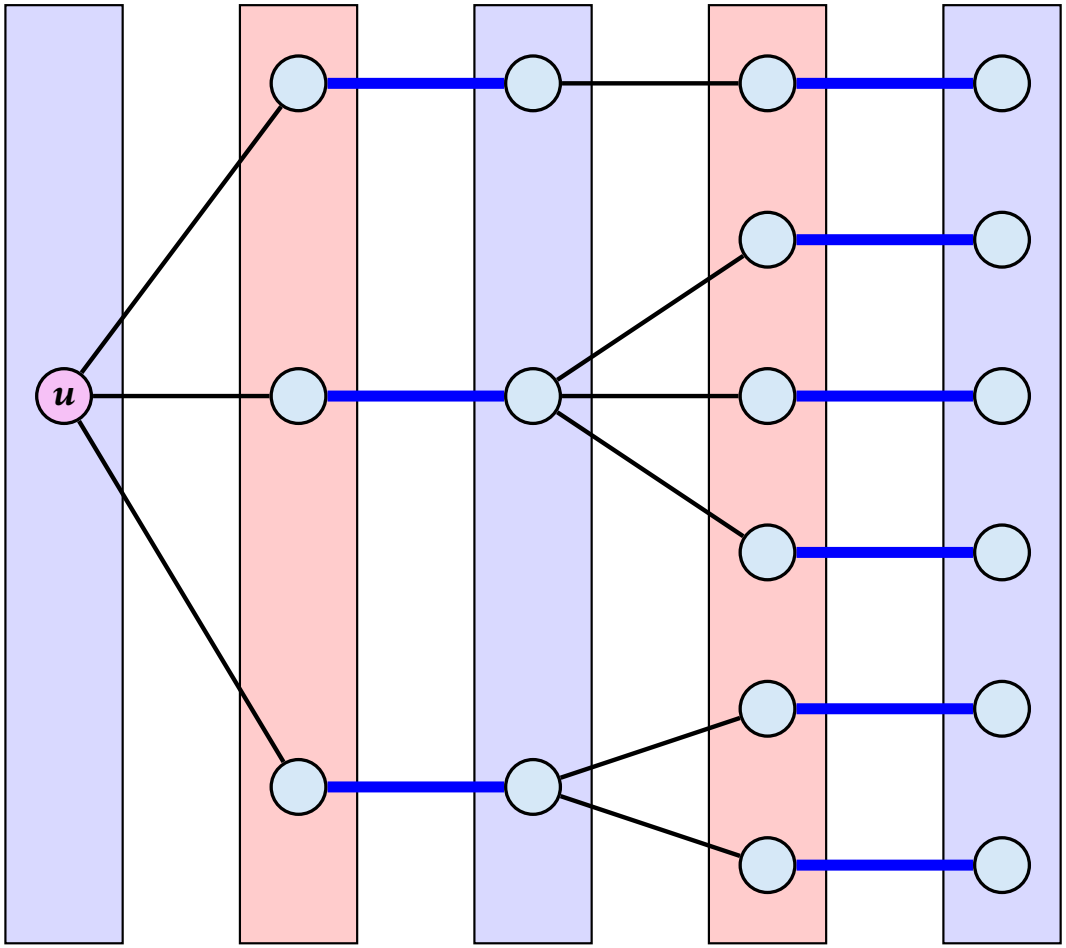
Analysis

- We will show that after at most n reweighting steps the size of the maximum matching can be increased by finding an augmenting path.
- This gives a polynomial running time.

How to find an augmenting path?

Construct an alternating tree.



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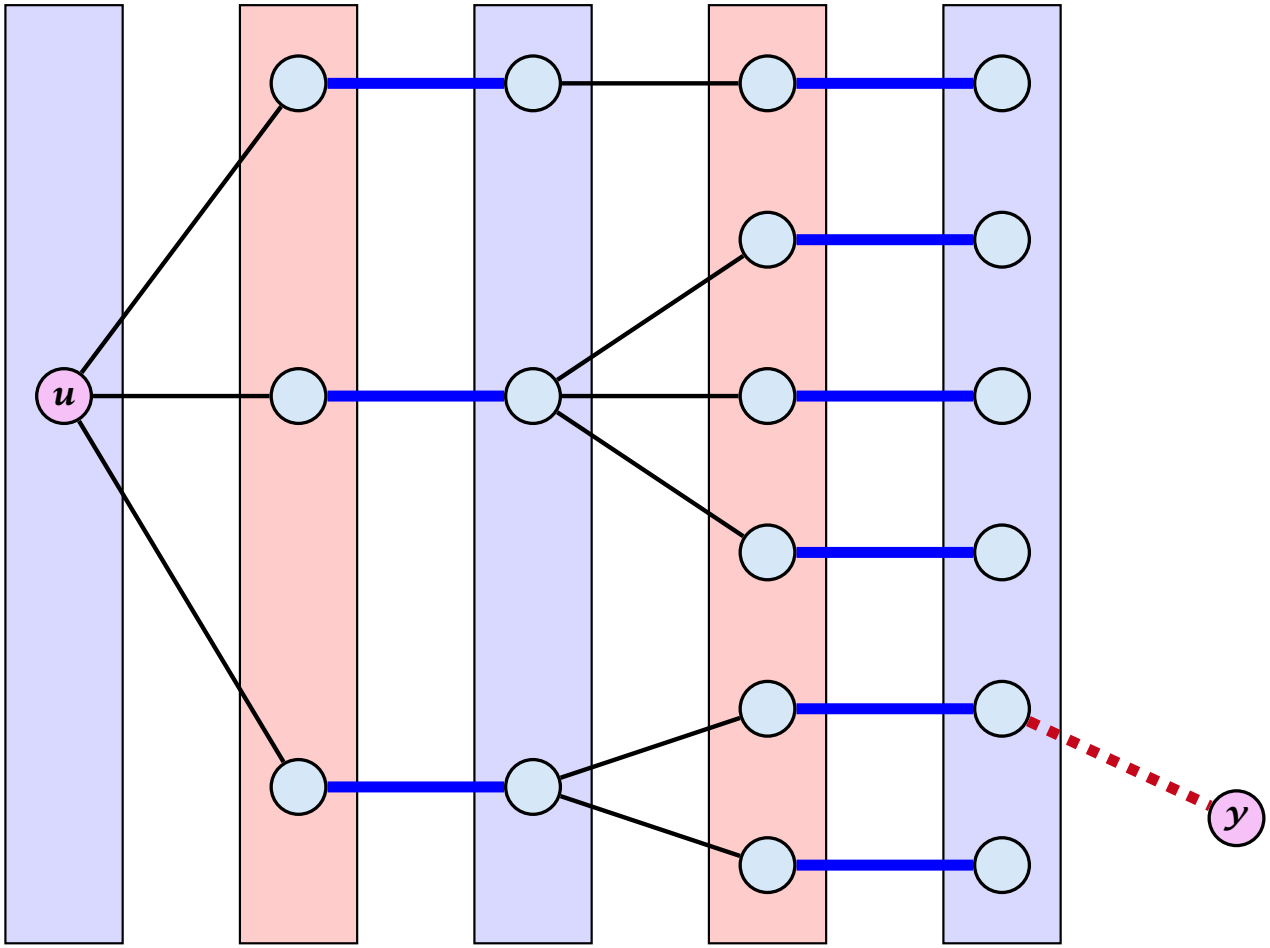

Analysis

How do we find S?

- Start on the left and compute an alternating tree, starting at any free node u.
- If this construction stops, there is no perfect matching in the tight subgraph (because for a perfect matching we need to find an augmenting path starting at u).
- The set of even vertices is on the left and the set of odd vertices is on the right and contains all neighbours of even nodes.
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How do we find S?

- Start on the left and compute an alternating tree, starting at any free node u.
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- The current matching does not have any edges from V_{odd} to outside of $L \setminus V_{\text{even}}$ (edges that may possibly be deleted by changing weights).
- After changing weights, there is at least one more edge connecting V_{even} to a node outside of V_{odd}. After at most n reweights we can do an augmentation.
- A reweighting can be trivially performed in time $\mathcal{O}(n^2)$ (keeping track of the tight edges).
- An augmentation takes at most $\mathcal{O}(n)$ time.
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A Fast Matching Algorithm

Algorithm 53 Bimatch-Hopcroft-Karp(G)

1: $M \leftarrow \emptyset$
2: **repeat**
3: let $\mathcal{P} = \{P_1, \ldots, P_k\}$ be maximal set of
4: vertex-disjoint, shortest augmenting path w.r.t. M.
5: $M \leftarrow M \oplus (P_1 \cup \cdots \cup P_k)$
6: **until** $\mathcal{P} = \emptyset$
7: **return** M

We call one iteration of the repeat-loop a phase of the algorithm.

Analysis

Lemma 95

Given a matching M and a maximal matching M^* there exist $|M^*| - |M|$ vertex-disjoint augmenting path w.r.t. M.

Proof:

- Similar to the proof that a matching is optimal iff it does not contain an augmenting paths.
- Consider the graph $G = (V, M \cup M^*)$ and mark edges in this graph blue if they are in M and red if they are in M^*.
- The connected components of G are cycles and paths.
- The graph contains $|M^*| - |M|$ more red edges than blue edges.
- Hence, there are at least $|M^*| - |M|$ components that contain a path starting and ending with a blue edge. This gives $|M^*| - |M|$ augmenting paths w.r.t. M.

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- The graph contains $k \stackrel{\text{def}}{=} |M^*| - |M|$ more red edges than blue edges.
- Hence, there are at least k components that form a path starting and ending with a blue edge. These are augmenting paths w.r.t. M.

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Analysis

- Let $P_1, \ldots, P_k$ be a maximal collection of vertex-disjoint, shortest augmenting paths w.r.t. M (let $\ell = |P_i|$).
- $M' \stackrel{\text{def}}{=} M \oplus (P_1 \cup \cdots \cup P_k) = M \oplus P_1 \oplus \cdots \oplus P_k$.
- Let P be an augmenting path in M'.

Lemma 96

The set $A \stackrel{\text{def}}{=} M \oplus (M' \oplus P) = (P_1 \cup \cdots \cup P_k) \oplus P$ contains at least $(k+1)\ell$ edges.

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- The set describes exactly the symmetric difference between matchings M and $M' \oplus P$.
- Hence, the set contains at least $k+1$ vertex-disjoint augmenting paths w.r.t. M as $|M'| = |M| + k + 1$.
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P is of length at least $\ell + 1$. This shows that the length of a shortest augmenting path increases between two phases of the Hopcroft-Karp algorithm.

Proof.

- If P does not intersect any of the P_i then this follows from the maximality of the set $\{P_1, \ldots, P_k\}$.
- Otherwise, at least one edge from P coincides with an edge from paths $P_1, \ldots, P_k$.
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- Otherwise, at least one edge from P coincides with an edge from paths $\{P_1, \ldots, P_k\}$.
- This edge is not contained in A.
- Hence, $|A| \leq k\ell + |P| - 1$.
- The lower bound on $|A|$ gives $(k+1)\ell \leq |A| \leq k\ell + |P| - 1$, and hence $|P| \geq \ell + 1$.

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If the shortest augmenting path w.r.t. a matching M has ℓ edges then the cardinality of the maximum matching is of size at most $|M| + \frac{|V|}{\ell+1}$.

Proof.

The symmetric difference between M and M^* contains $|M^*| - |M|$ vertex-disjoint augmenting paths. Each of these paths contains at least $\ell + 1$ vertices. Hence, there can be at most $\frac{|V|}{\ell+1}$ of them.

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Lemma 98

The Hopcroft-Karp algorithm requires at most $2\sqrt{|V|}$ phases.

Proof.

- After iteration $\lfloor \sqrt{|V|} \rfloor$ the length of a shortest augmenting path must be at least $\lfloor \sqrt{|V|} \rfloor + 1 \geq \sqrt{|V|}$.
- Hence, there can be at most $|V|/(\sqrt{|V|}+1) \leq \sqrt{|V|}$ additional augmentations.

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Analysis

Lemma 99

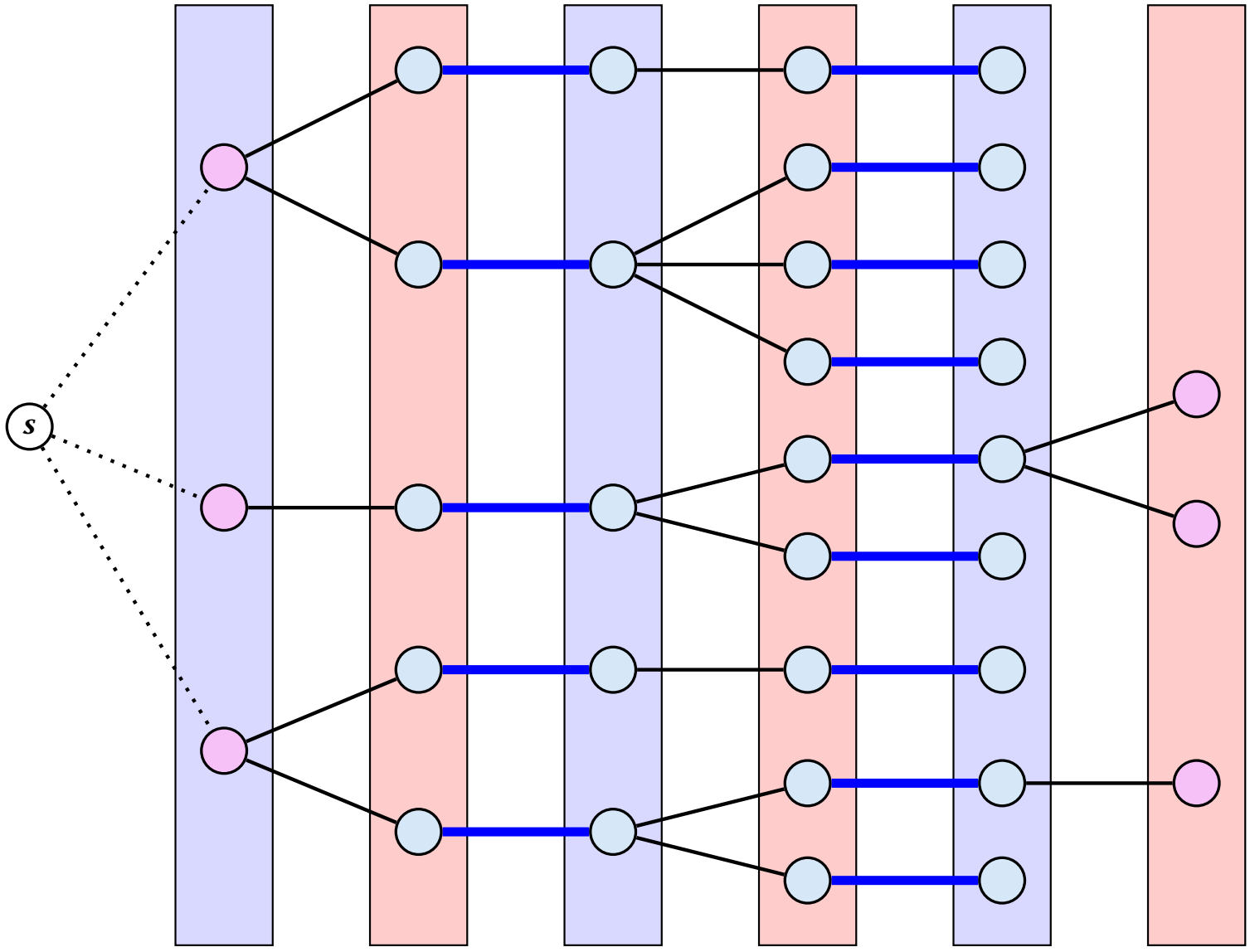
One phase of the Hopcroft-Karp algorithm can be implemented in time $\mathcal{O}(m)$.

- Do a breadth first search starting at all free vertices in the left side L.

  (alternatively add a super-startnode; connect it to all free vertices in L and start breadth first search from there)
- The search stops when reaching a free vertex. However, the current level of the BFS tree is still finished in order to find a set F of free vertices (on the right side) that can be reached via shortest augmenting paths.

Analysis

- Then a maxim**al** set of shortest path from the leftmost layer of the tree construction to nodes in F needs to be computed.
- Any such path must visit the layers of the BFS-tree from left to right.
- To go from an odd layer to an even layer it must use a matching edge.
- To go from an even layer to an odd layer edge it can use edges in the BFS-tree **or** edges that have been ignored during BFS-tree construction.
- We direct all edges btw. an even node in some layer ℓ to an odd node in layer $\ell + 1$ from left to right.
- A DFS search in the resulting graph gives us a maximal set of vertex disjoint path from left to right in the resulting graph.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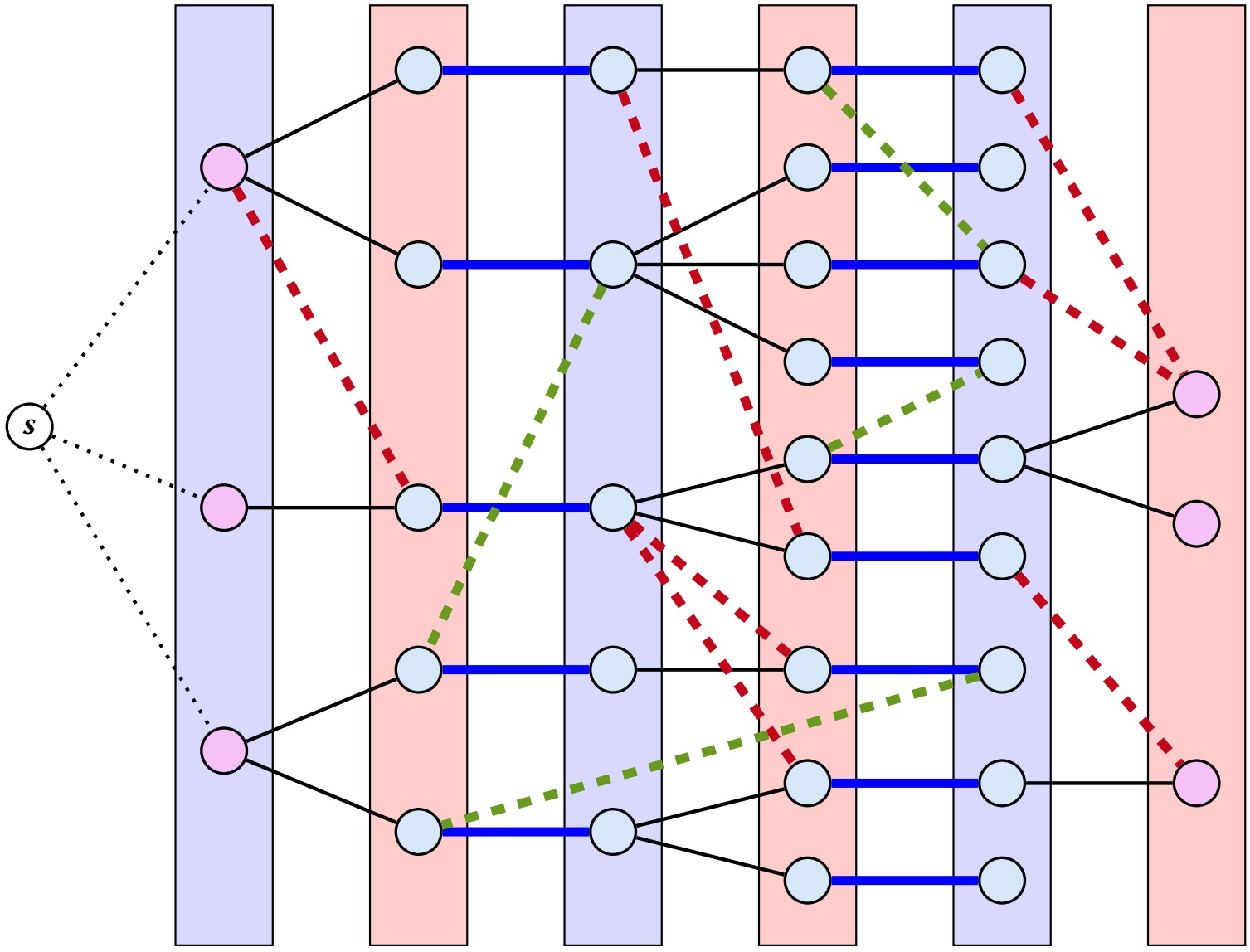
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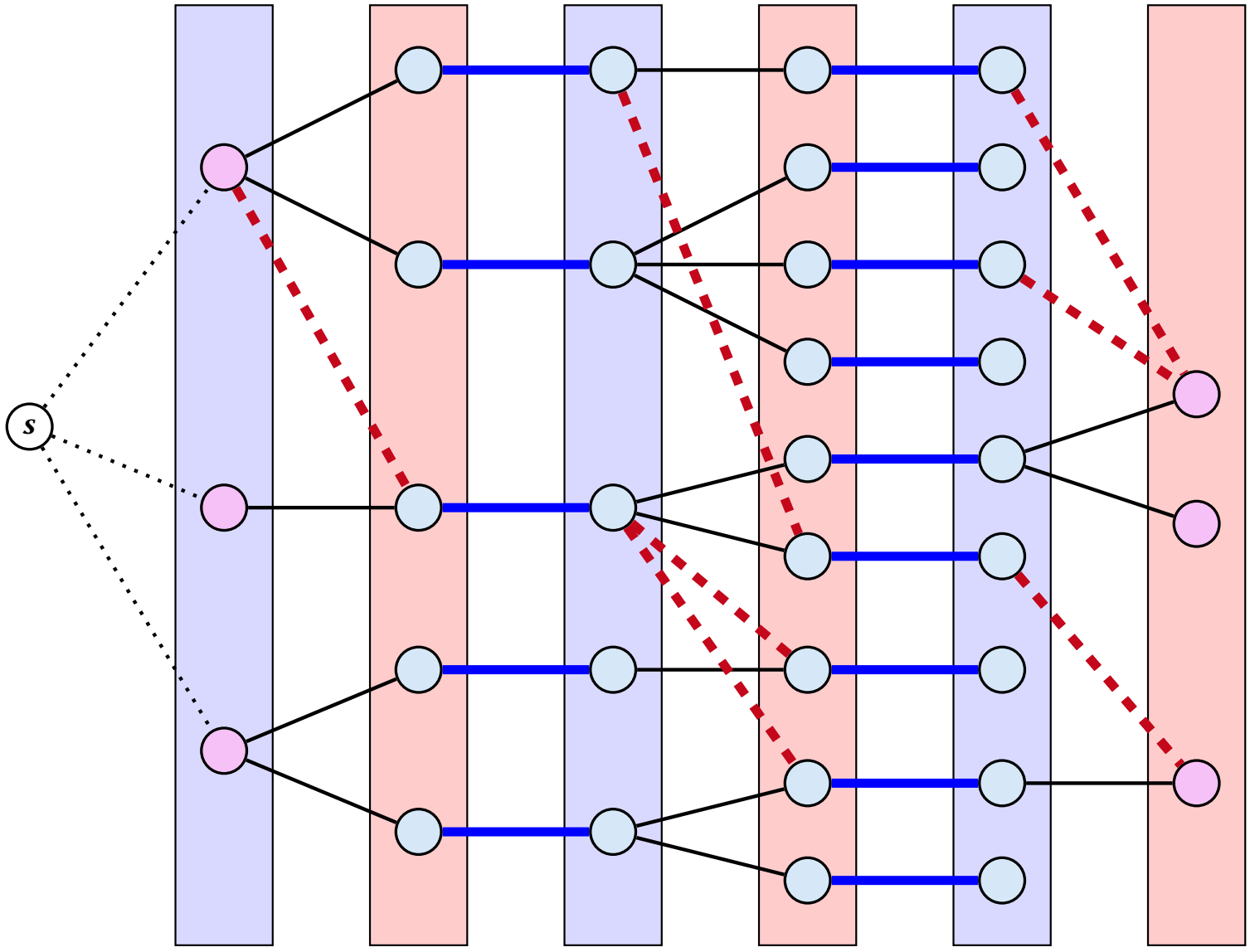
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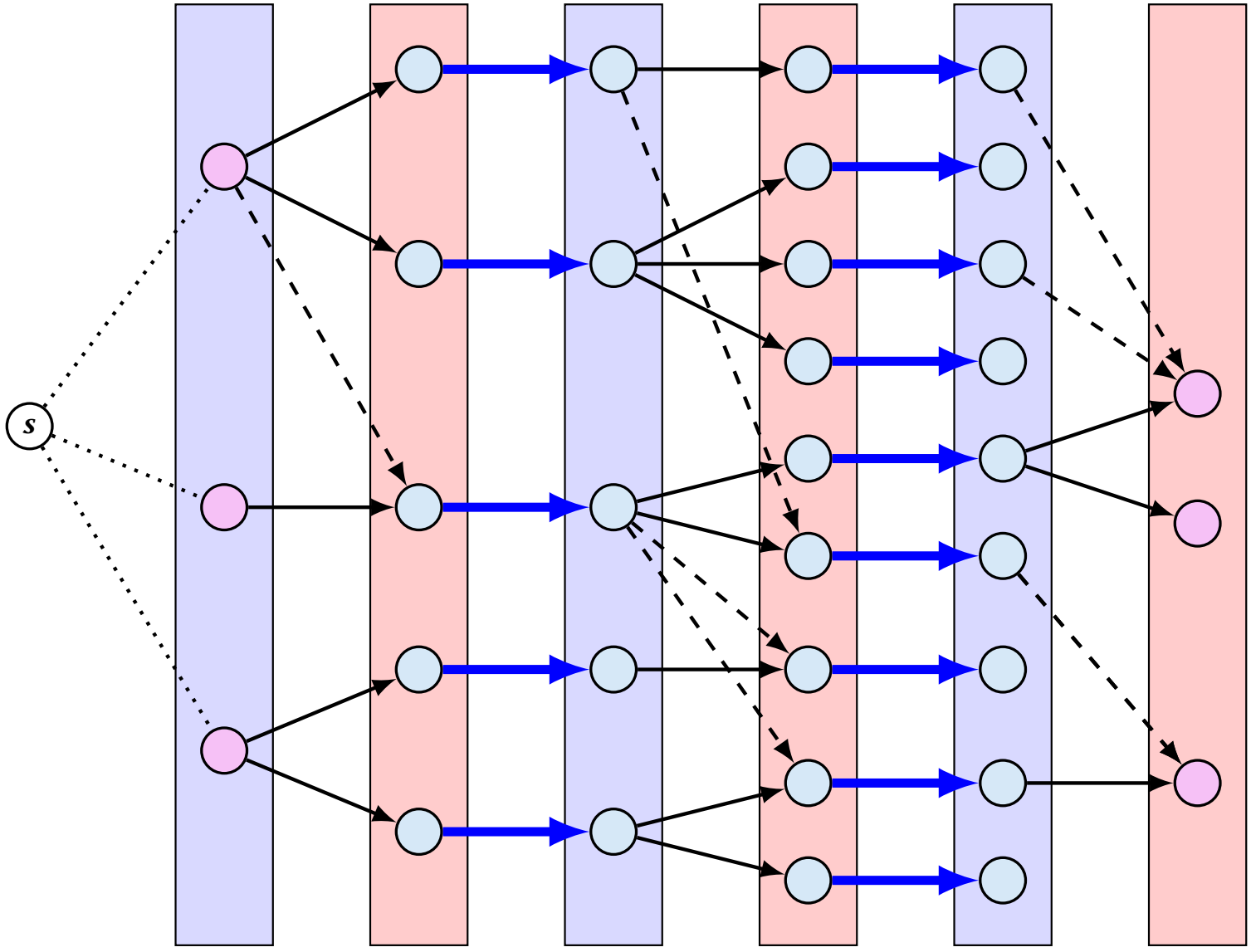
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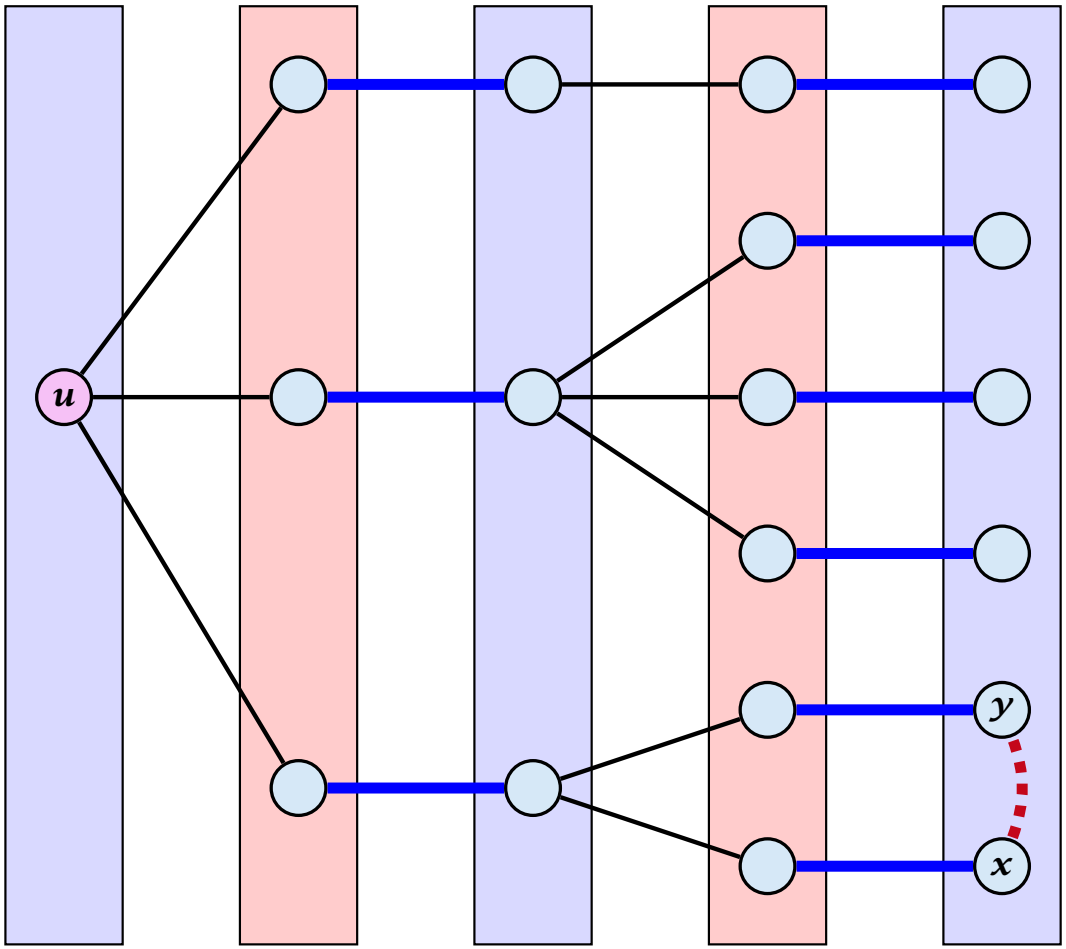
How to find an augmenting path?

Construct an alternating tree.



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Case 4:

y is already contained in T as an even vertex

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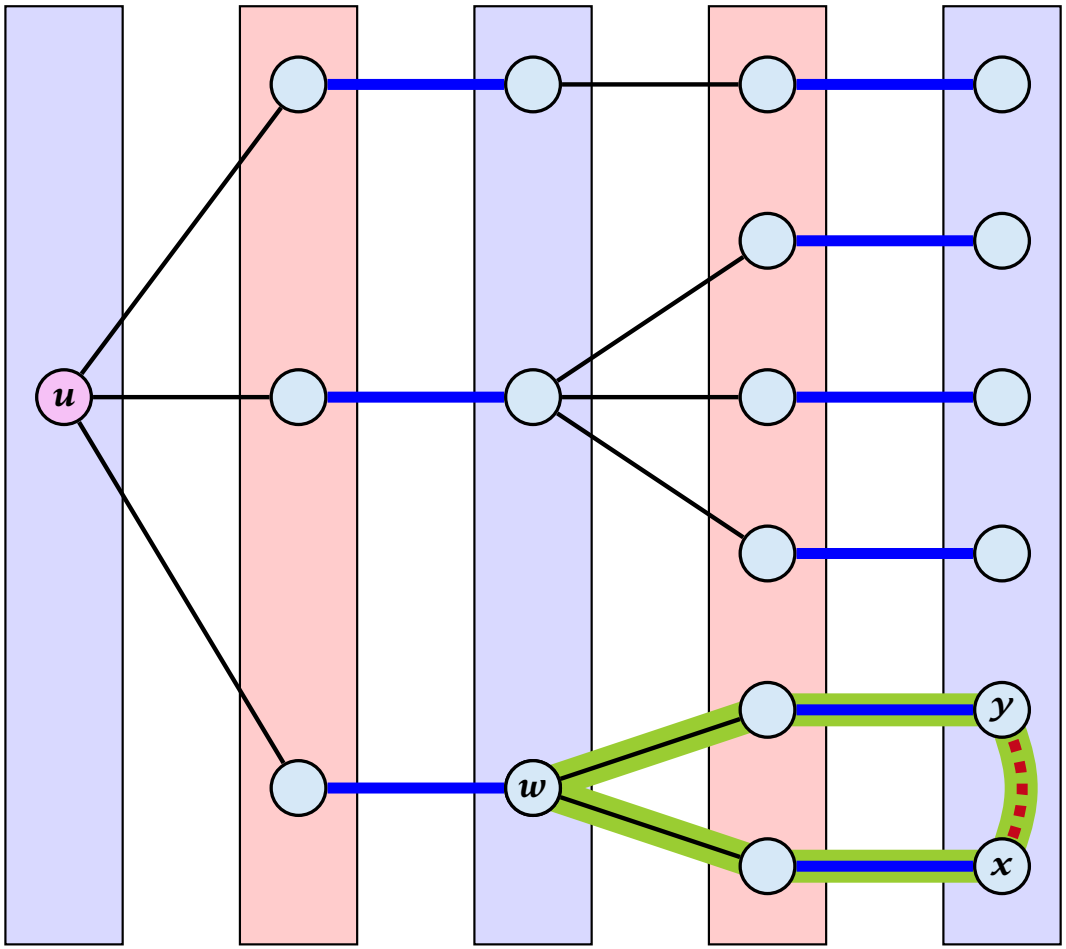

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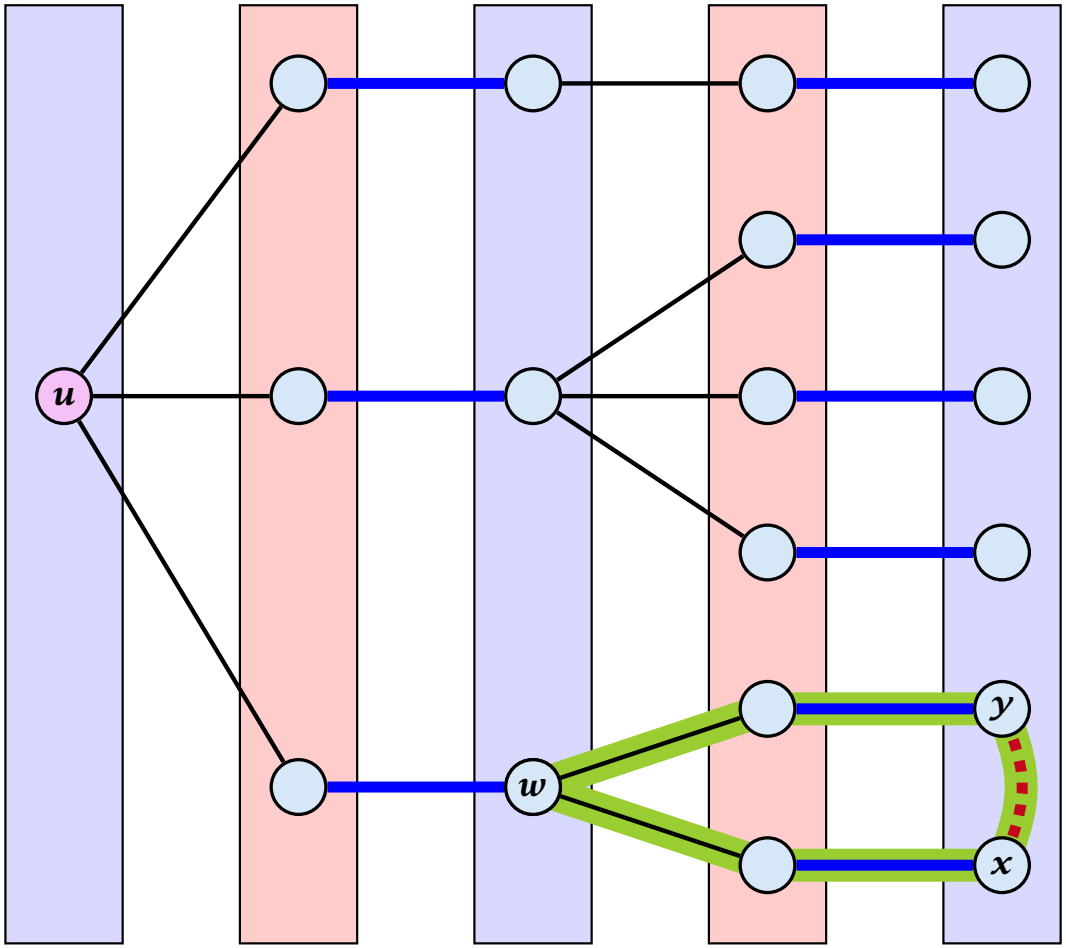

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The cycle $w \leftrightarrow y - x \leftrightarrow w$ is called a blossom.
w is called the base of the blossom (even node!!!).
The path u-w path is called the stem of the blossom.

Flowers and Blossoms

Definition 100

A flower in a graph $G = (V, E)$ w.r.t. a matching M and a (free) root node r, is a subgraph with two components:

- A stem is an even length alternating path that starts at the root node r and terminates at some node w. We permit the possibility that $r = w$ (empty stem).
- A blossom is an odd length alternating cycle that starts and terminates at the terminal node w of a stem and has no other node in common with the stem. w is called the base of the blossom.

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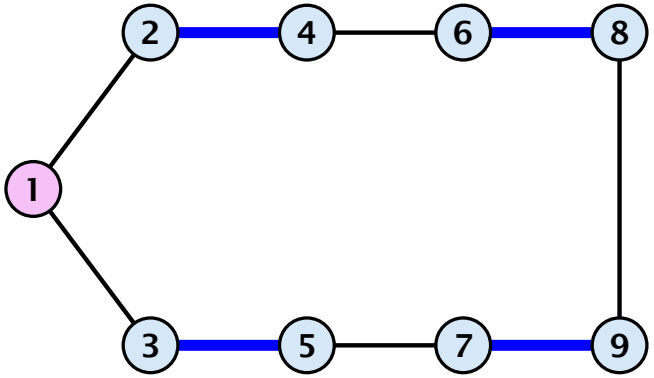

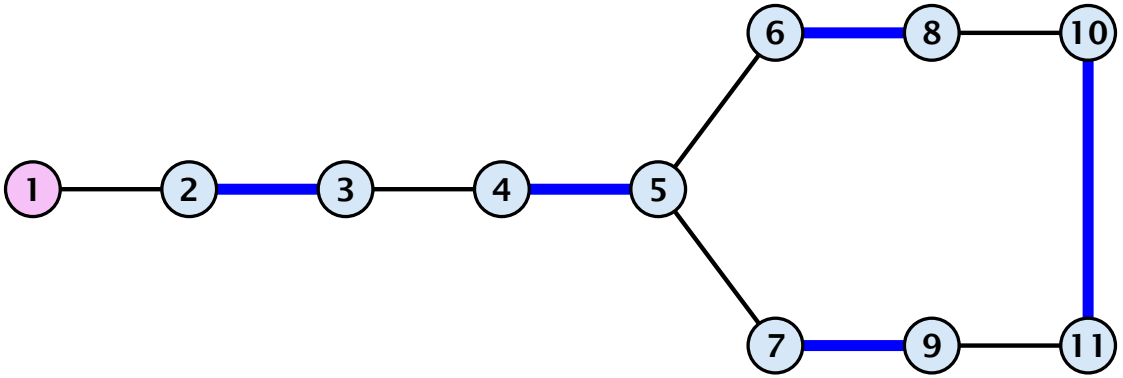

Flowers and Blossoms

Properties:

1. A stem spans $2\ell + 1$ nodes and contains ℓ matched edges for some integer $\ell \geq 0$.
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3. The base of a blossom is an even node (if the stem is part of an alternating tree starting at r).

Flowers and Blossoms

Properties:

4. Every node x in the blossom (except its base) is reachable from the root (or from the base of the blossom) through two distinct alternating paths; one with even and one with odd length.
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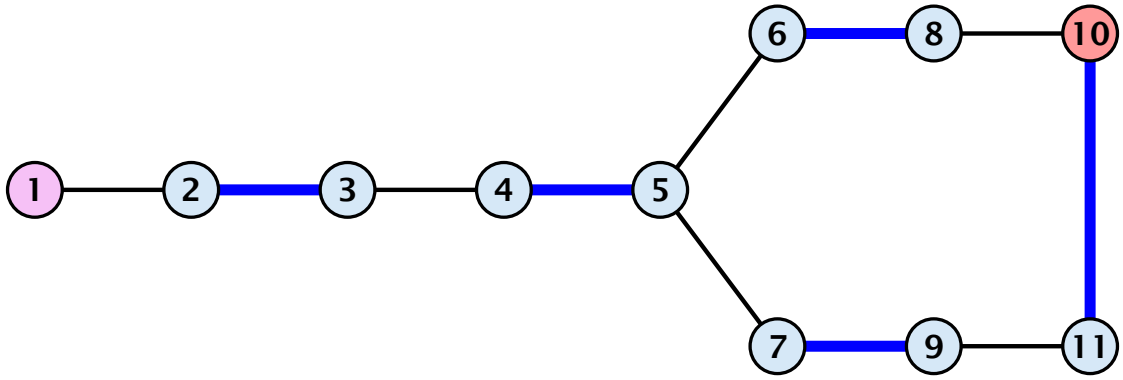

Shrinking Blossoms

When during the alternating tree construction we discover a blossom B we replace the graph G by $G' = G/B$, which is obtained from G by contracting the blossom B.

- Delete all vertices in B (and its incident edges) from G.
- Add a new (pseudo-)vertex b. The new vertex b is connected to all vertices in $V \setminus B$ that had at least one edge to a vertex from B.

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- Edges of T that connect a node u not in B to a node in B become tree edges in T' connecting u to b.
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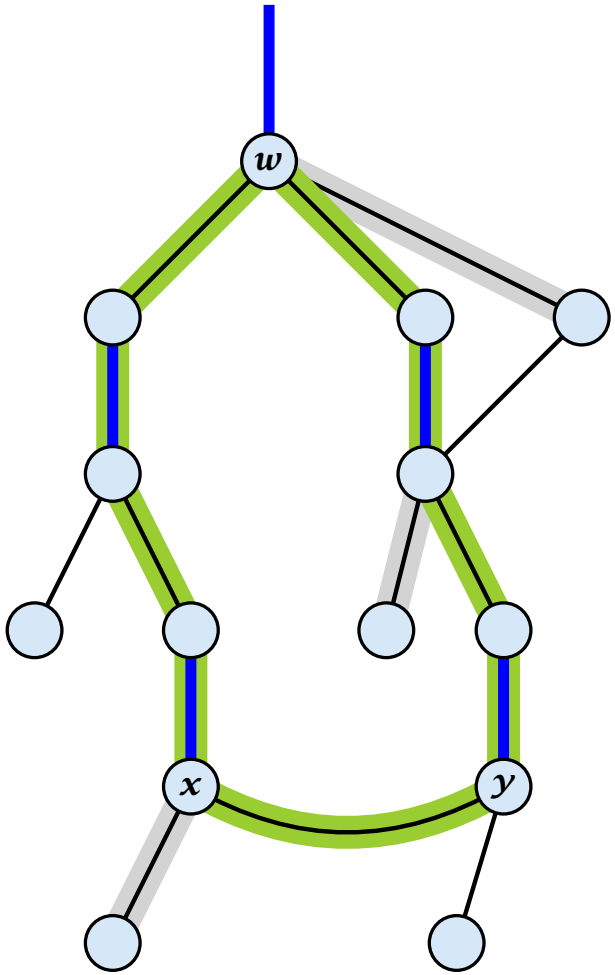

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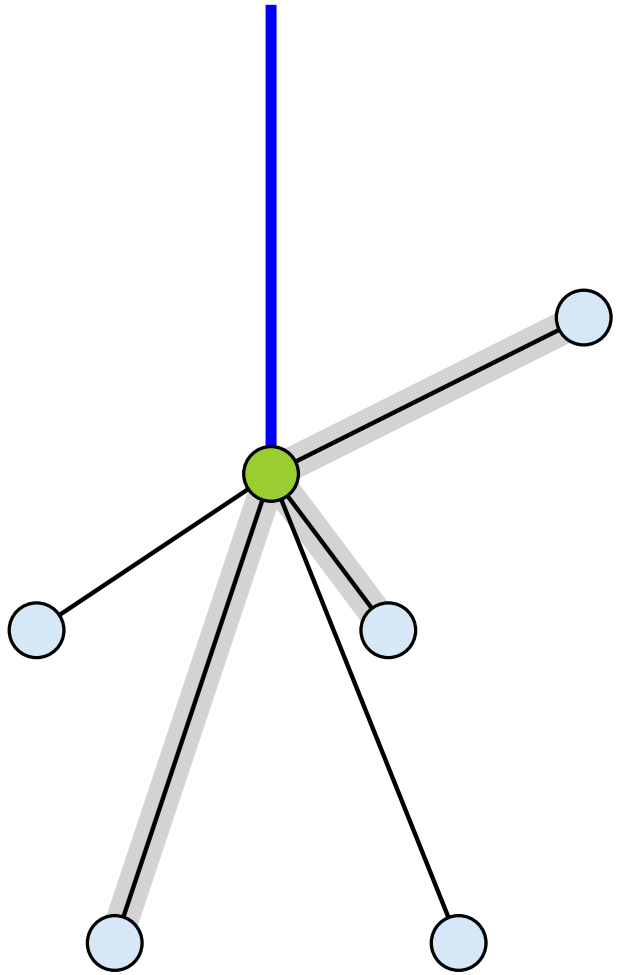
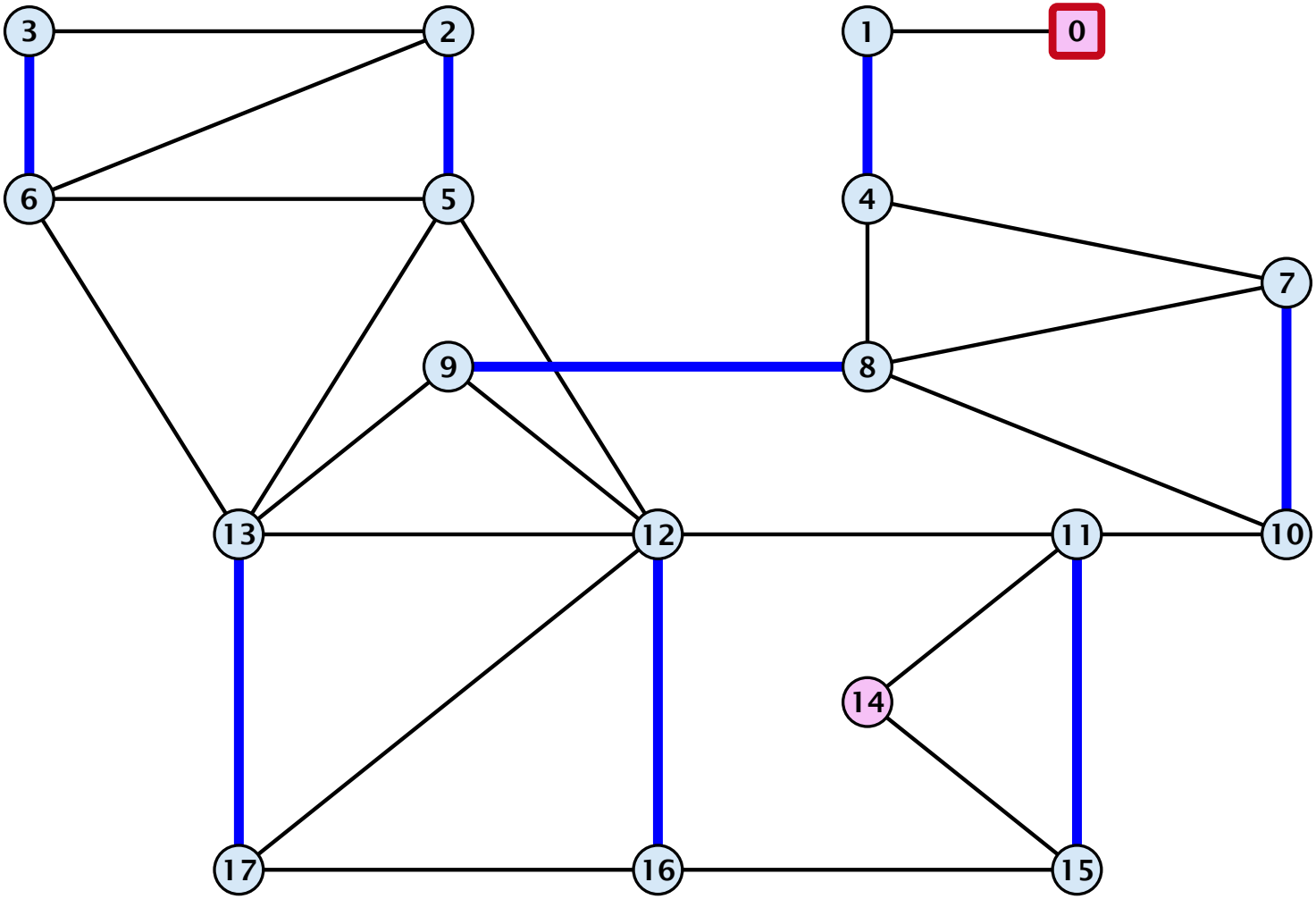


Example: Blossom Algorithm



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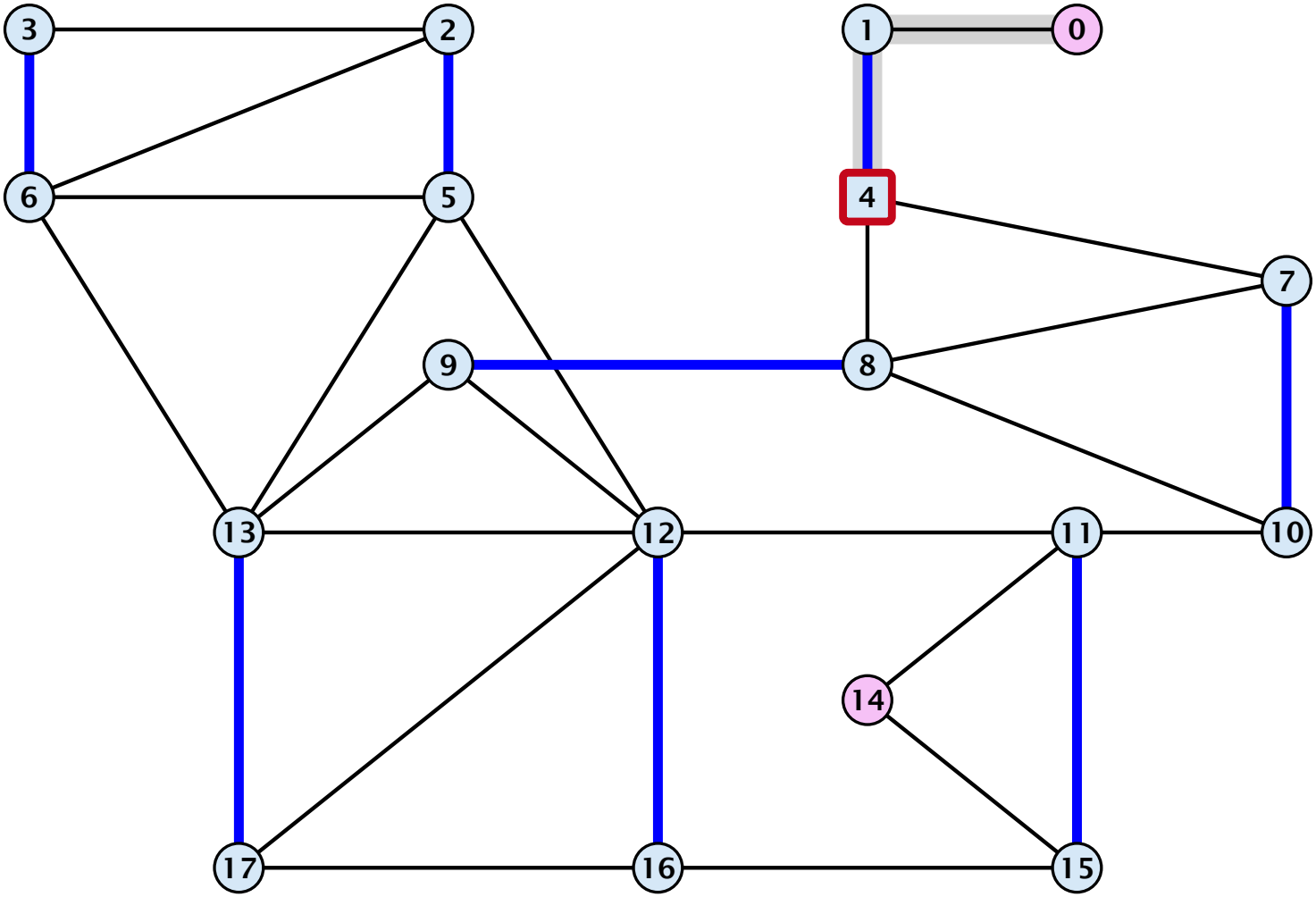

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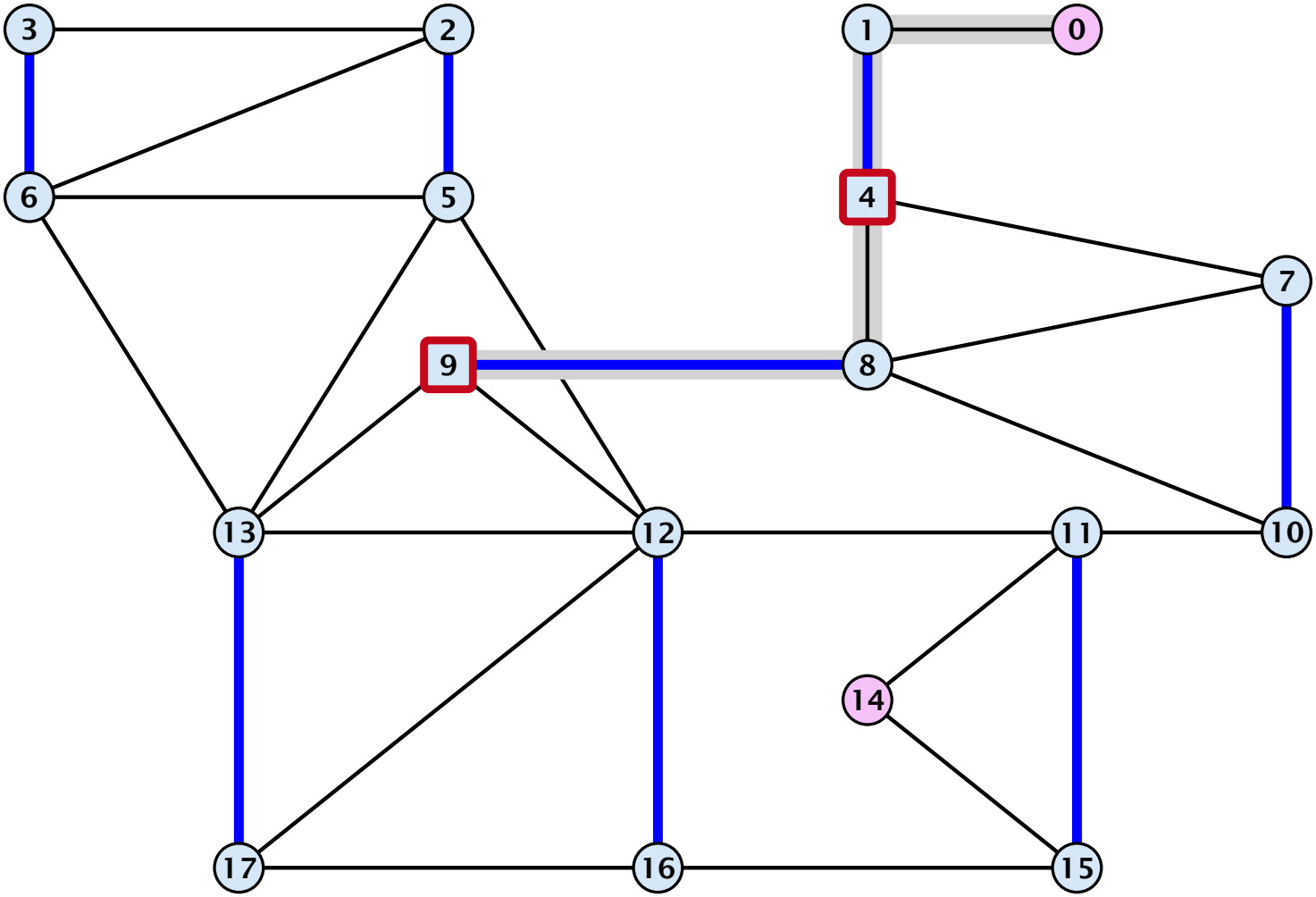

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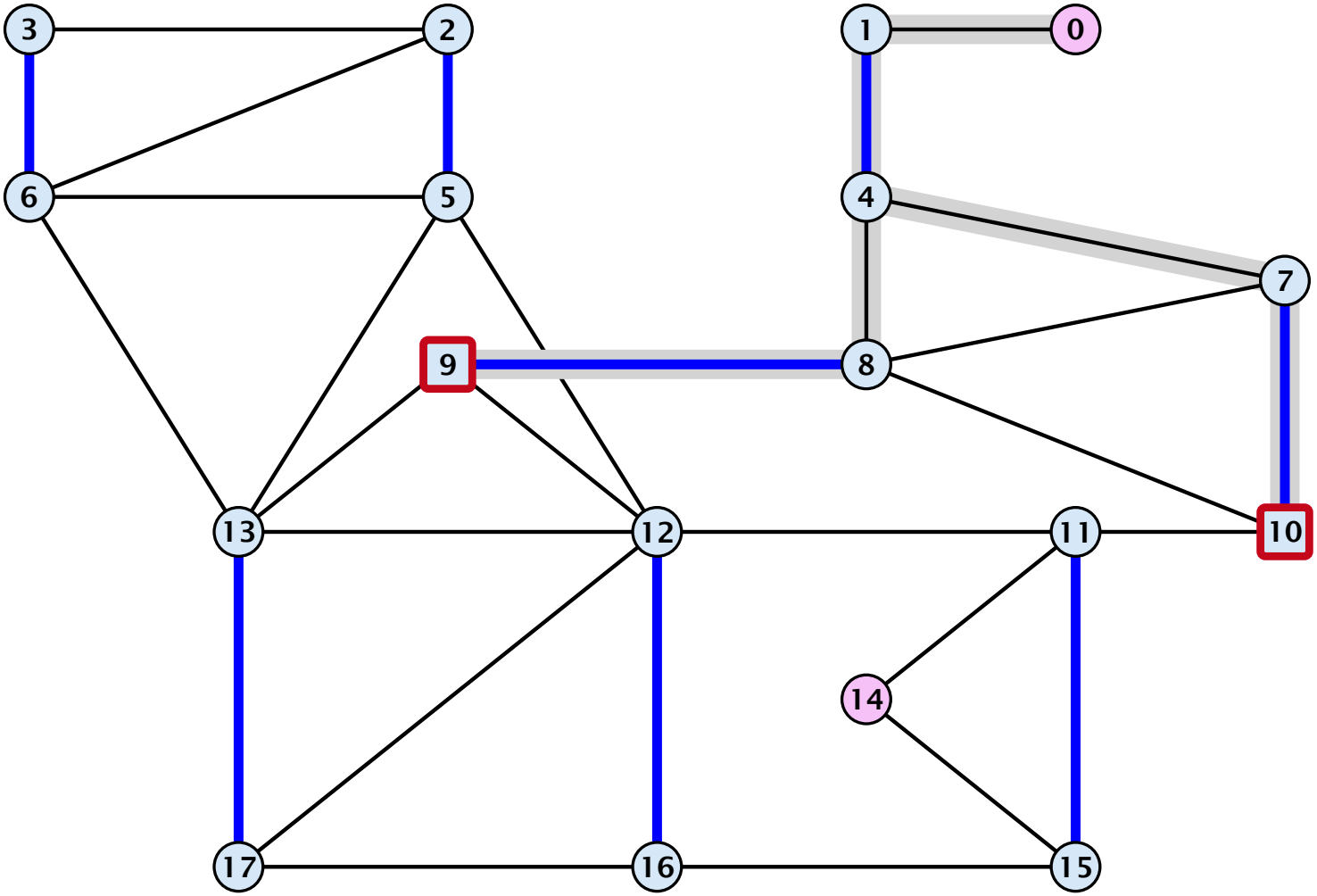

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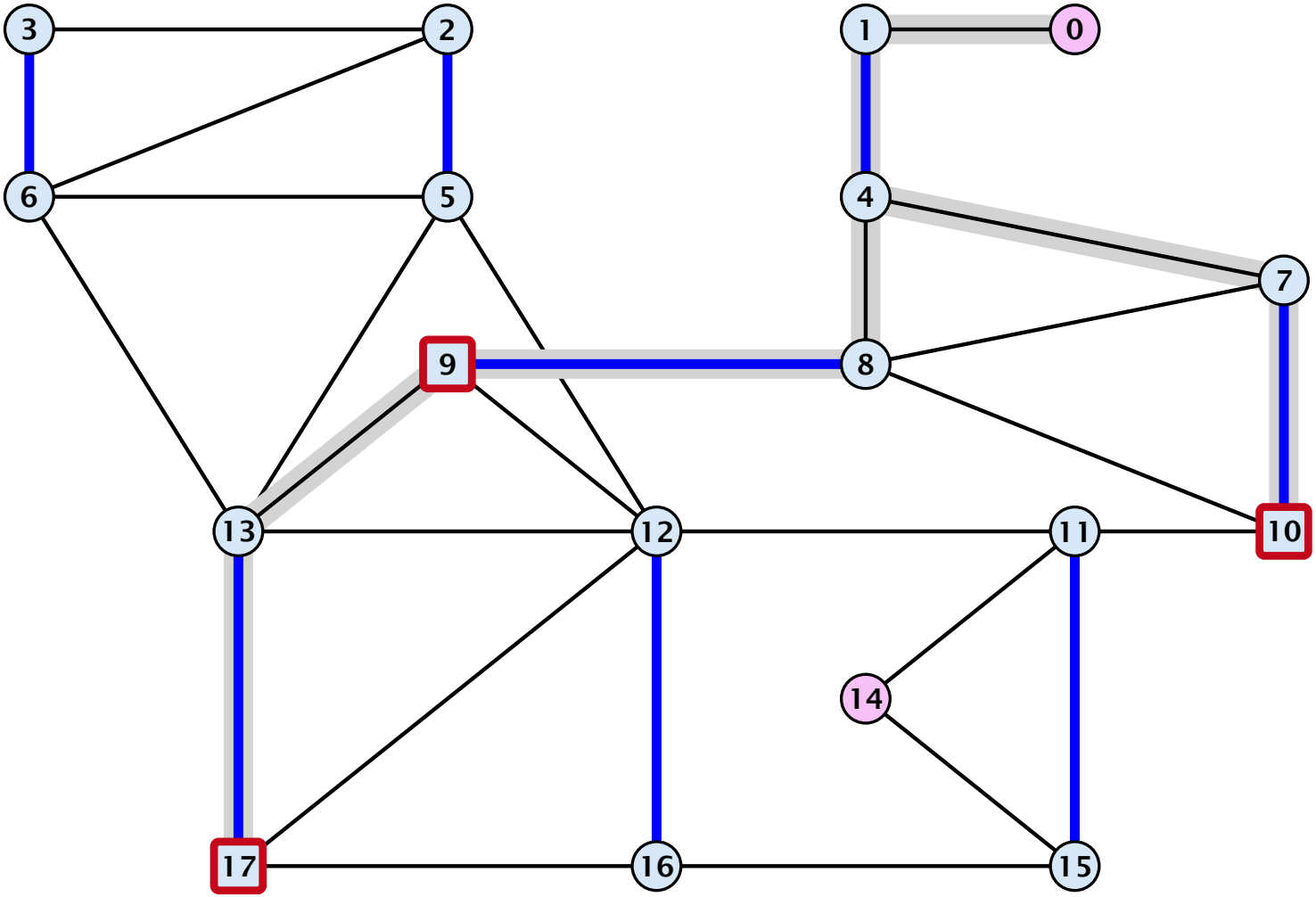

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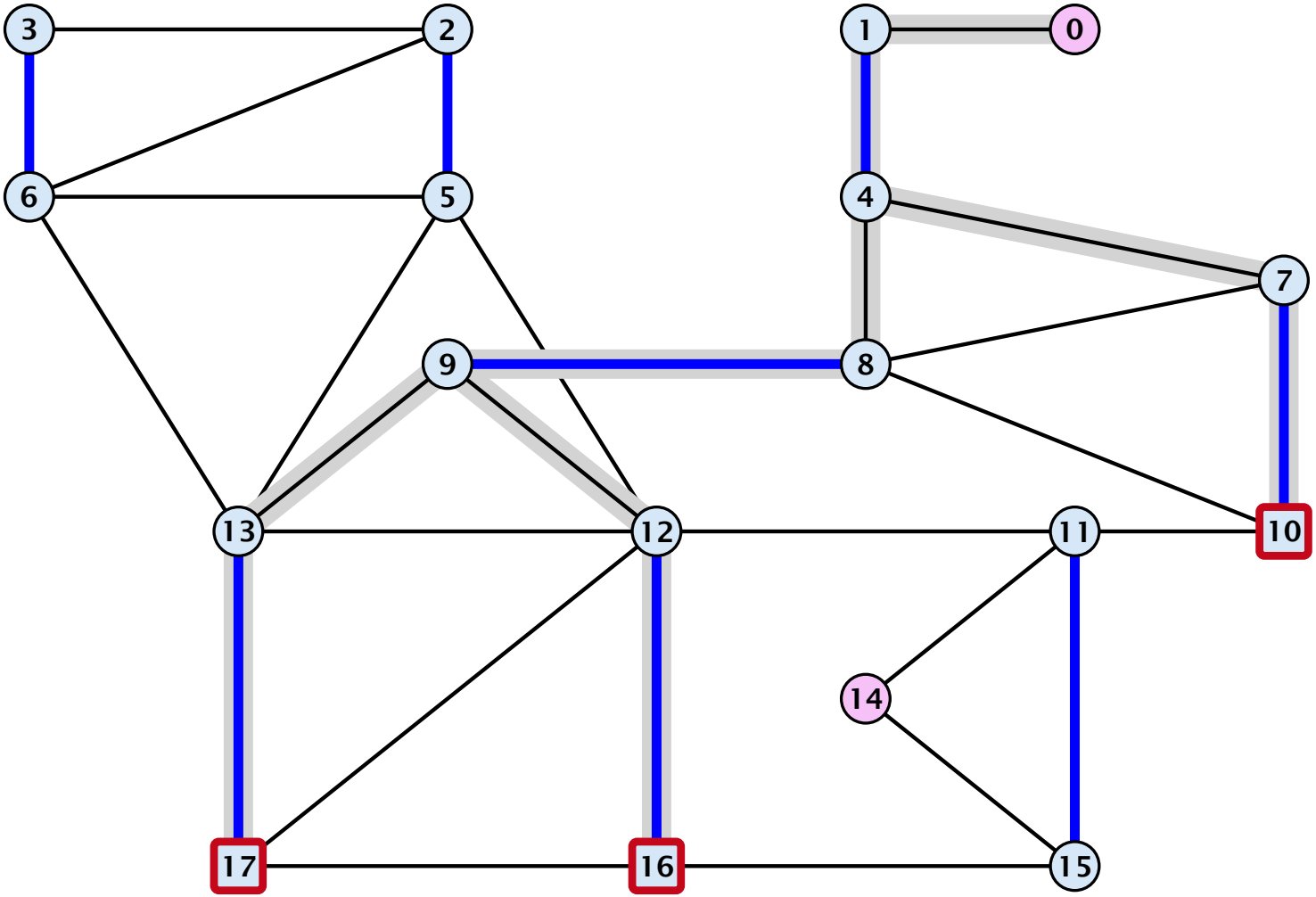

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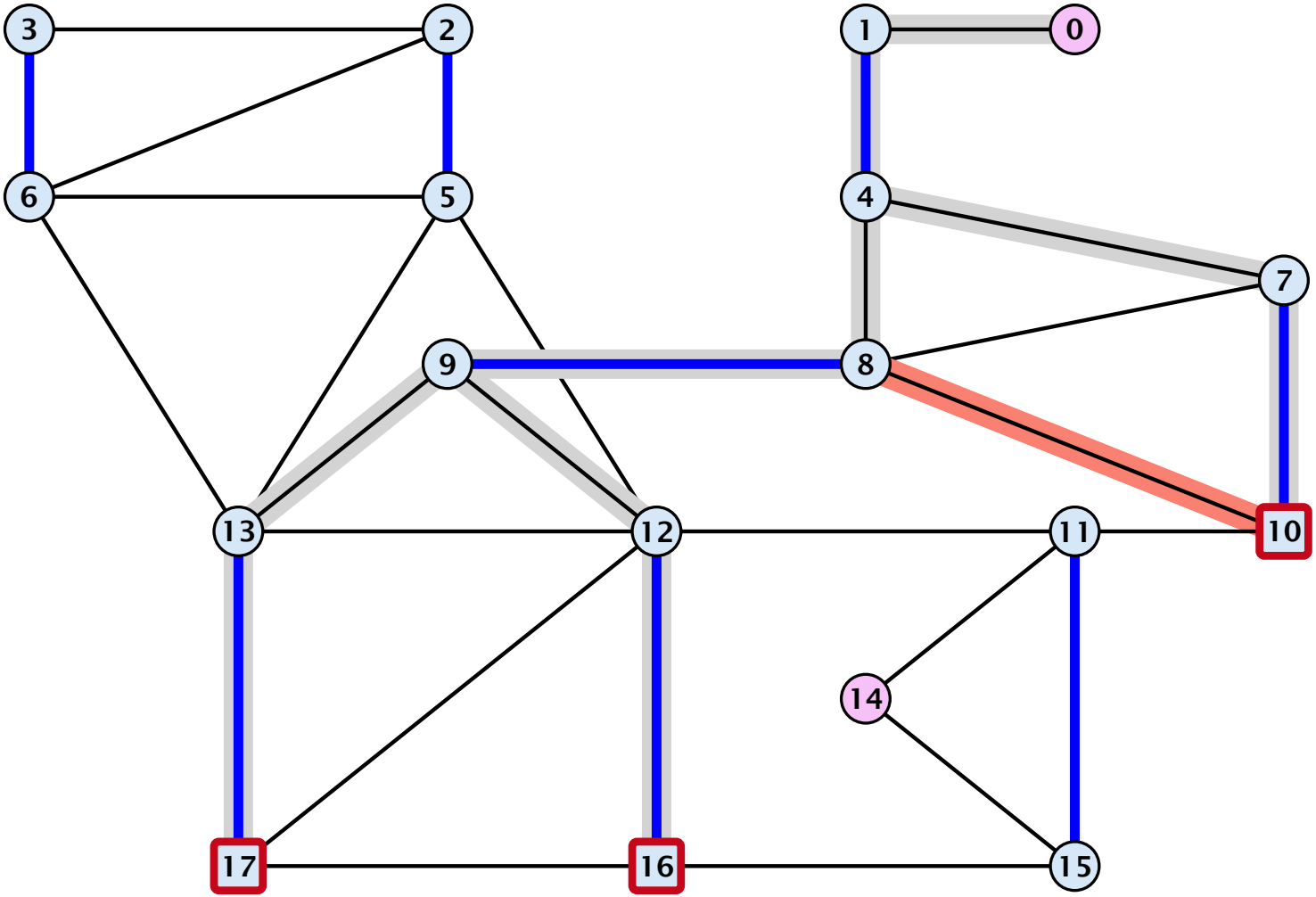

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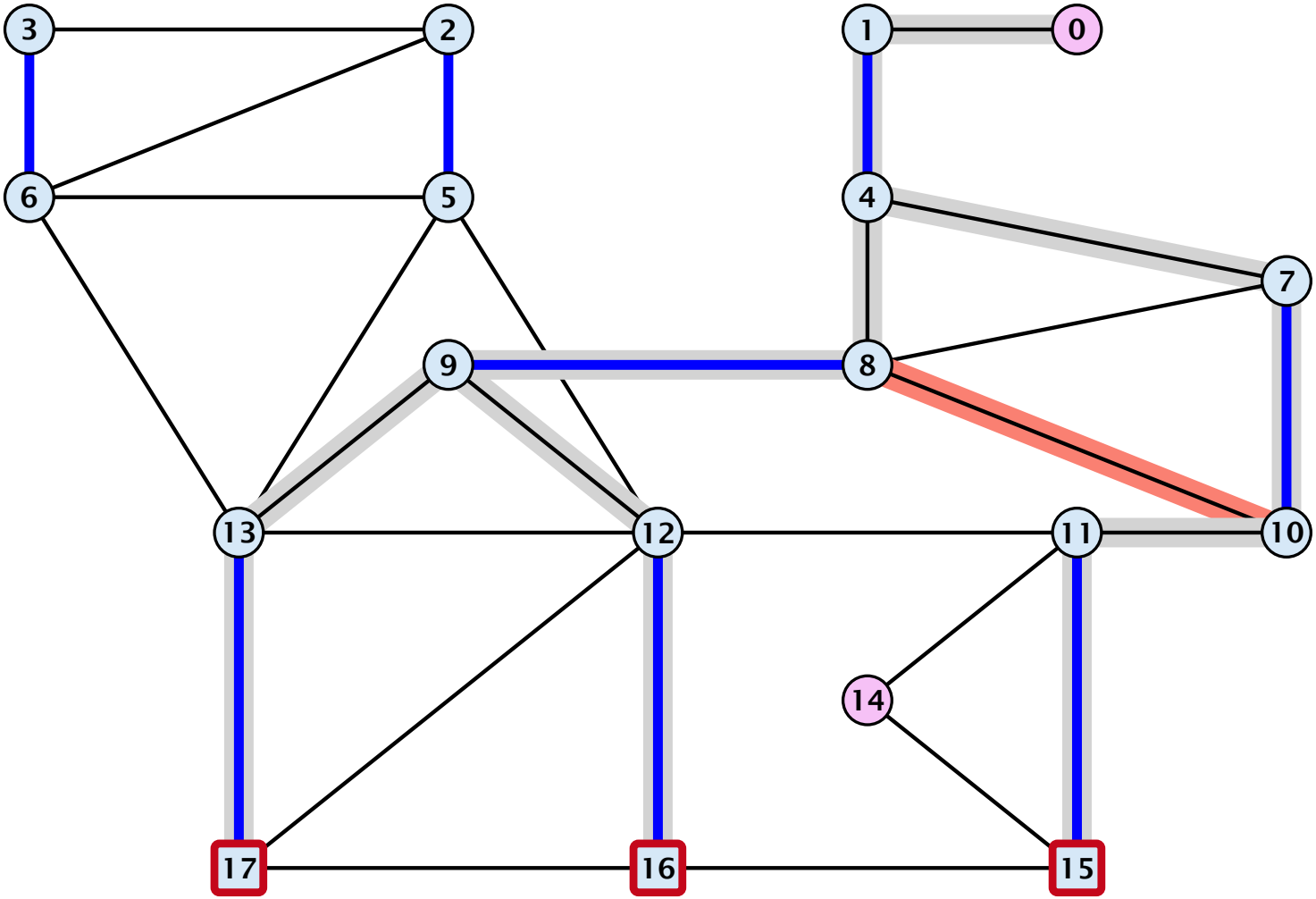

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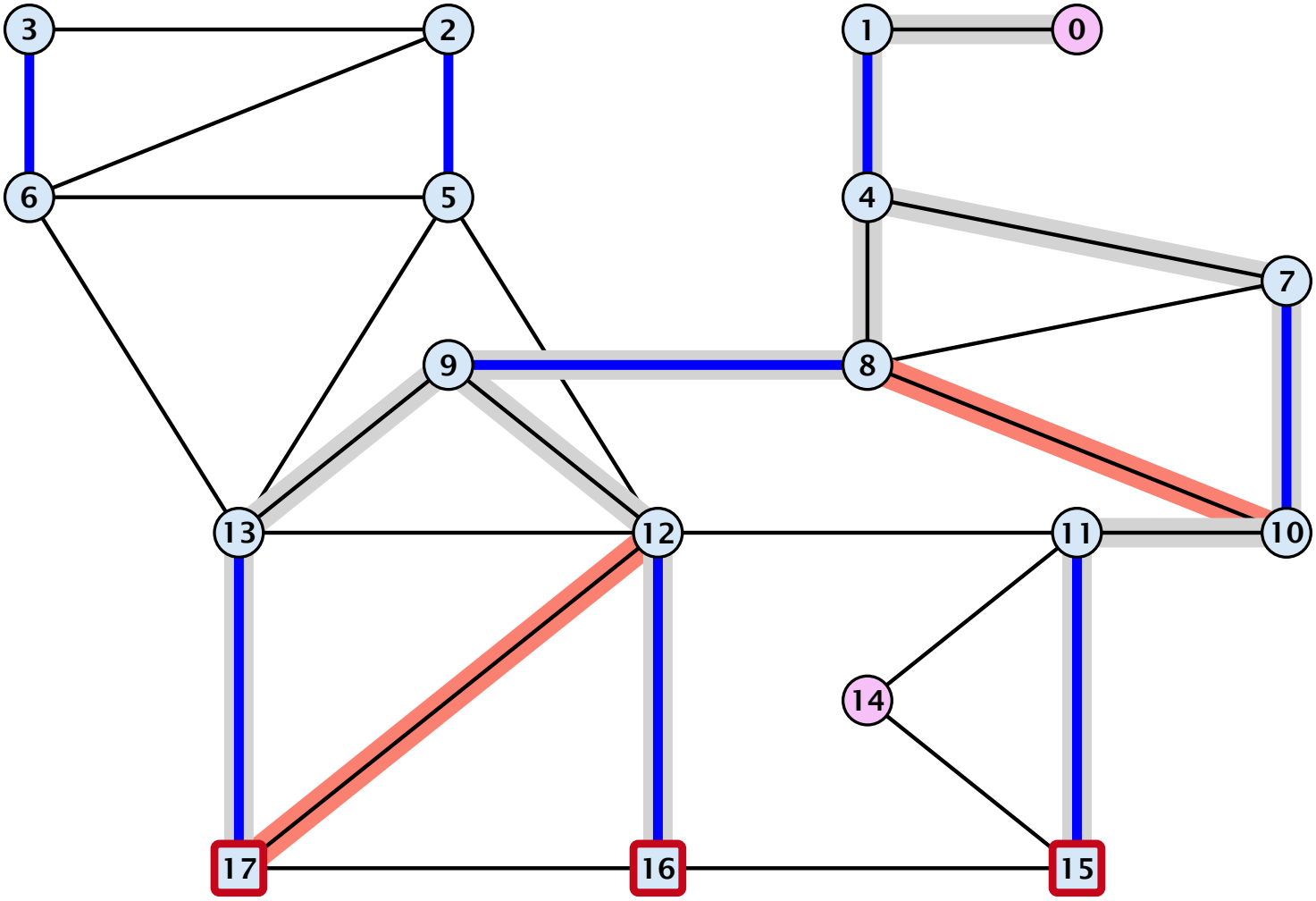

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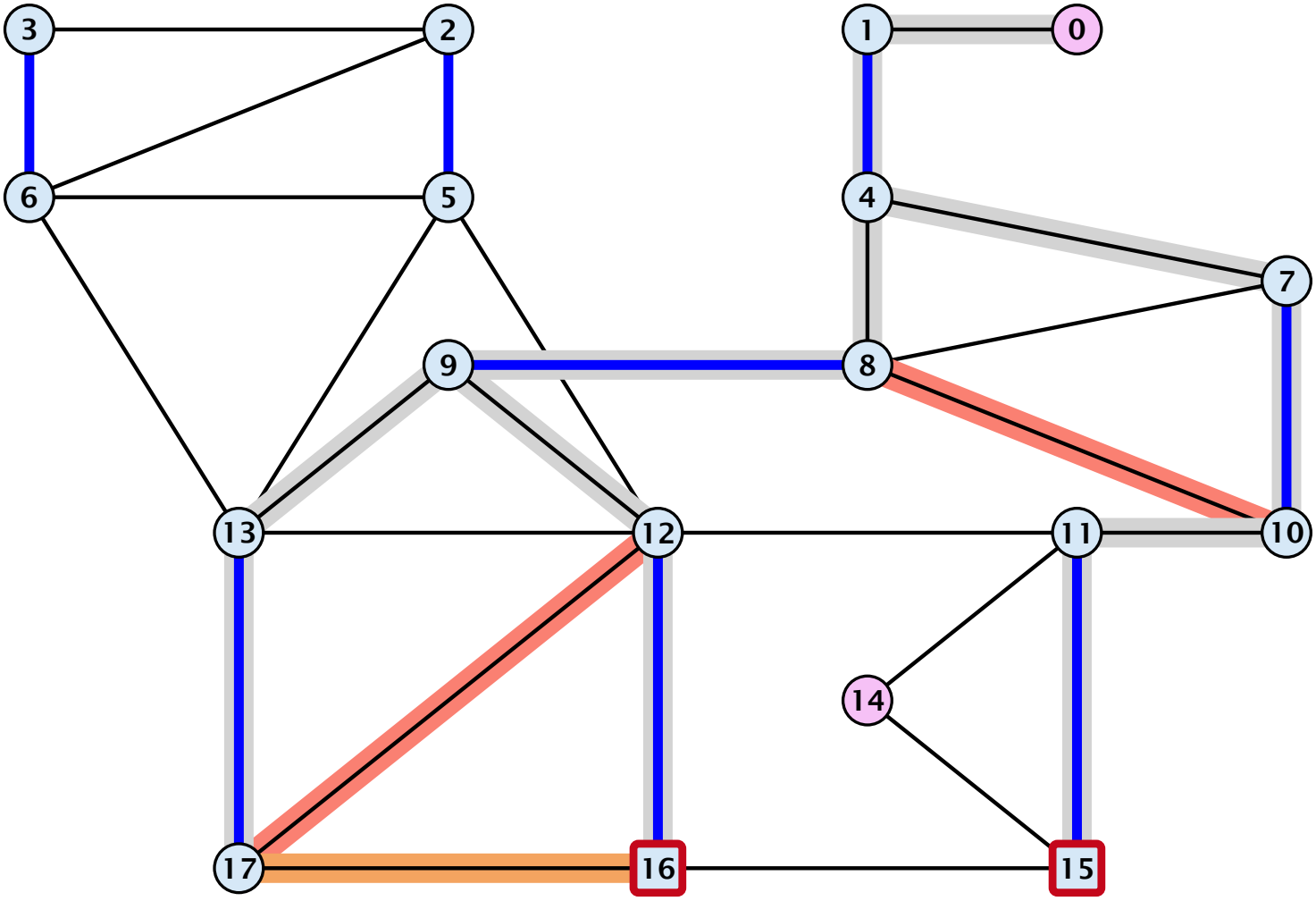

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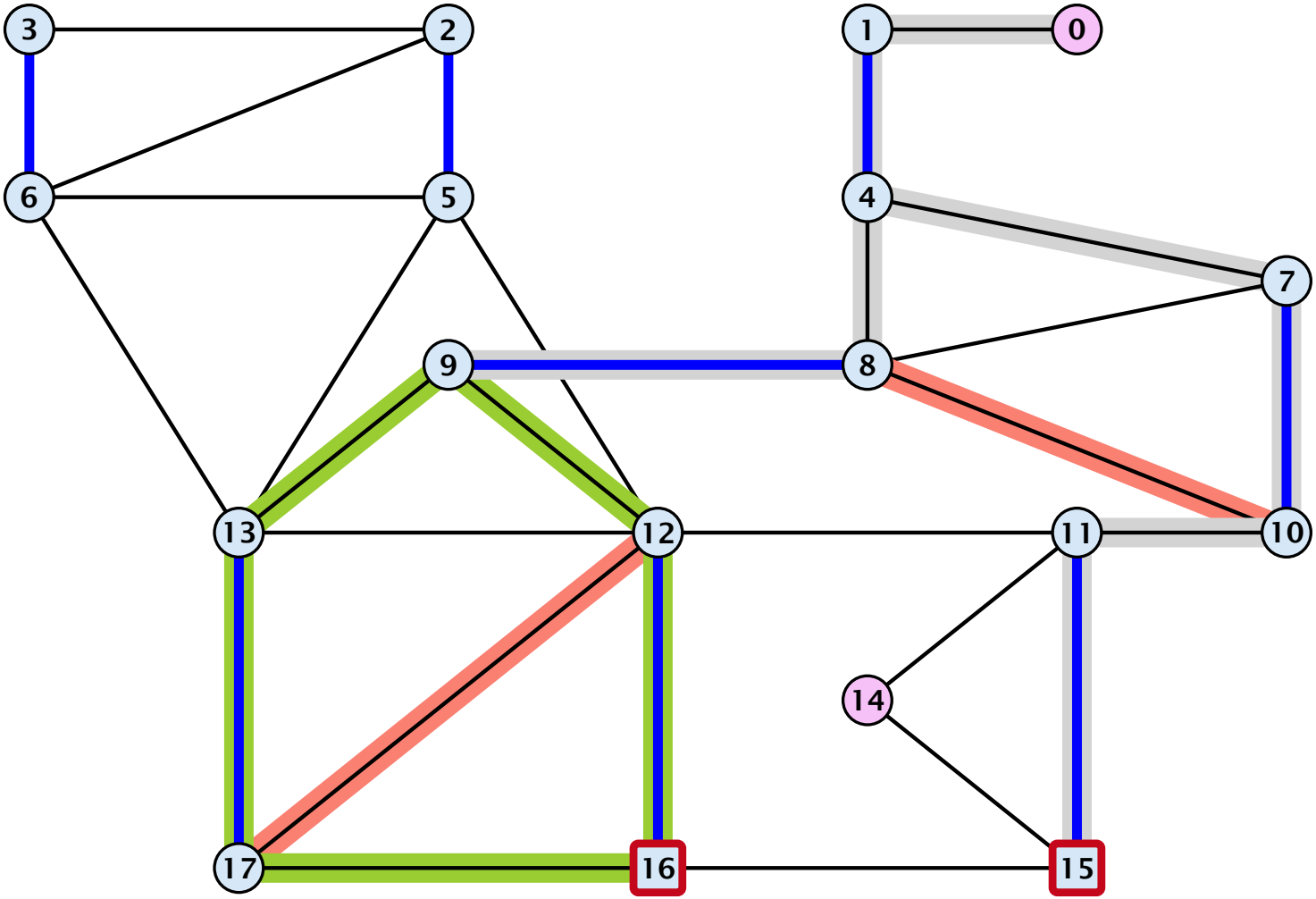

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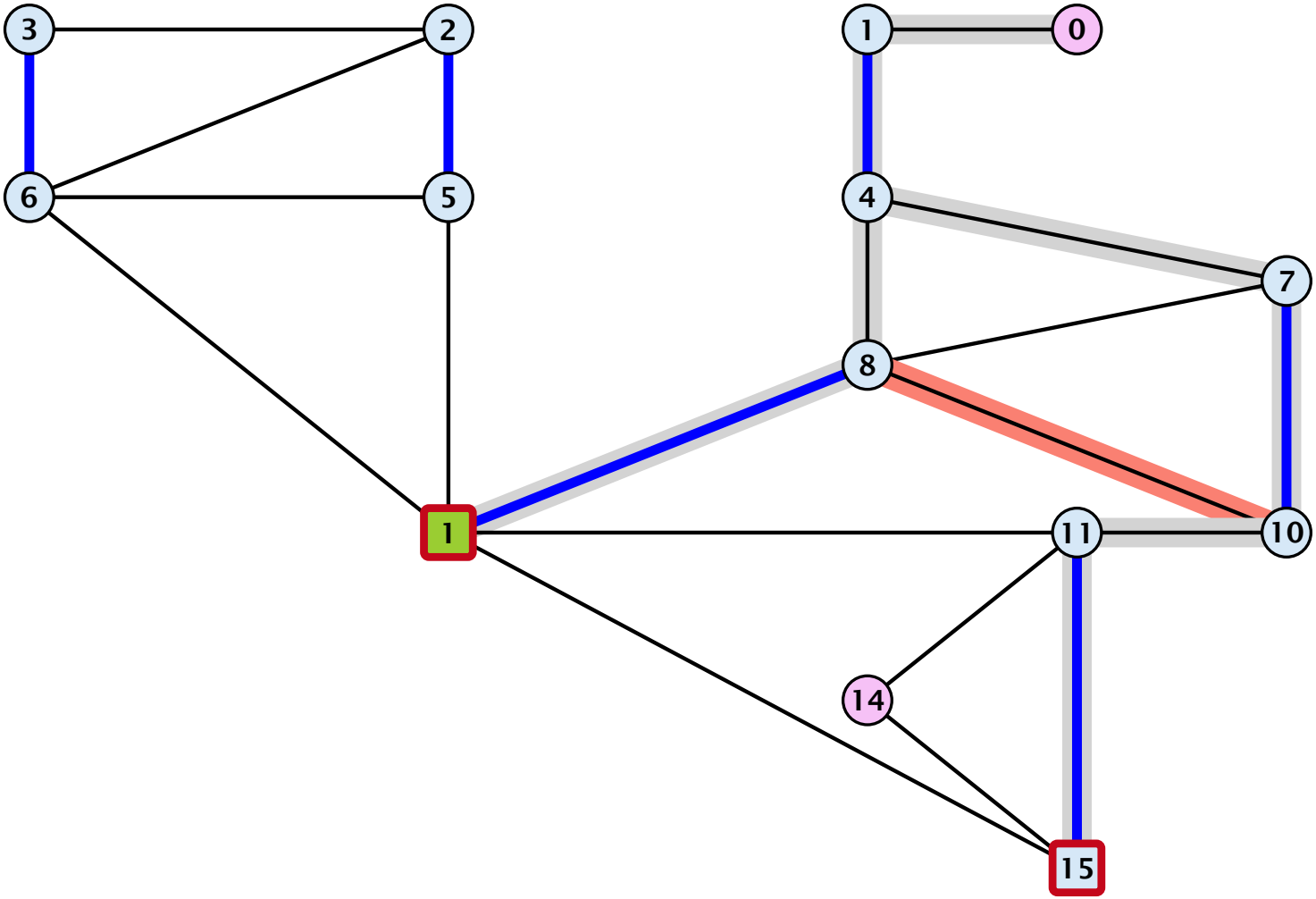

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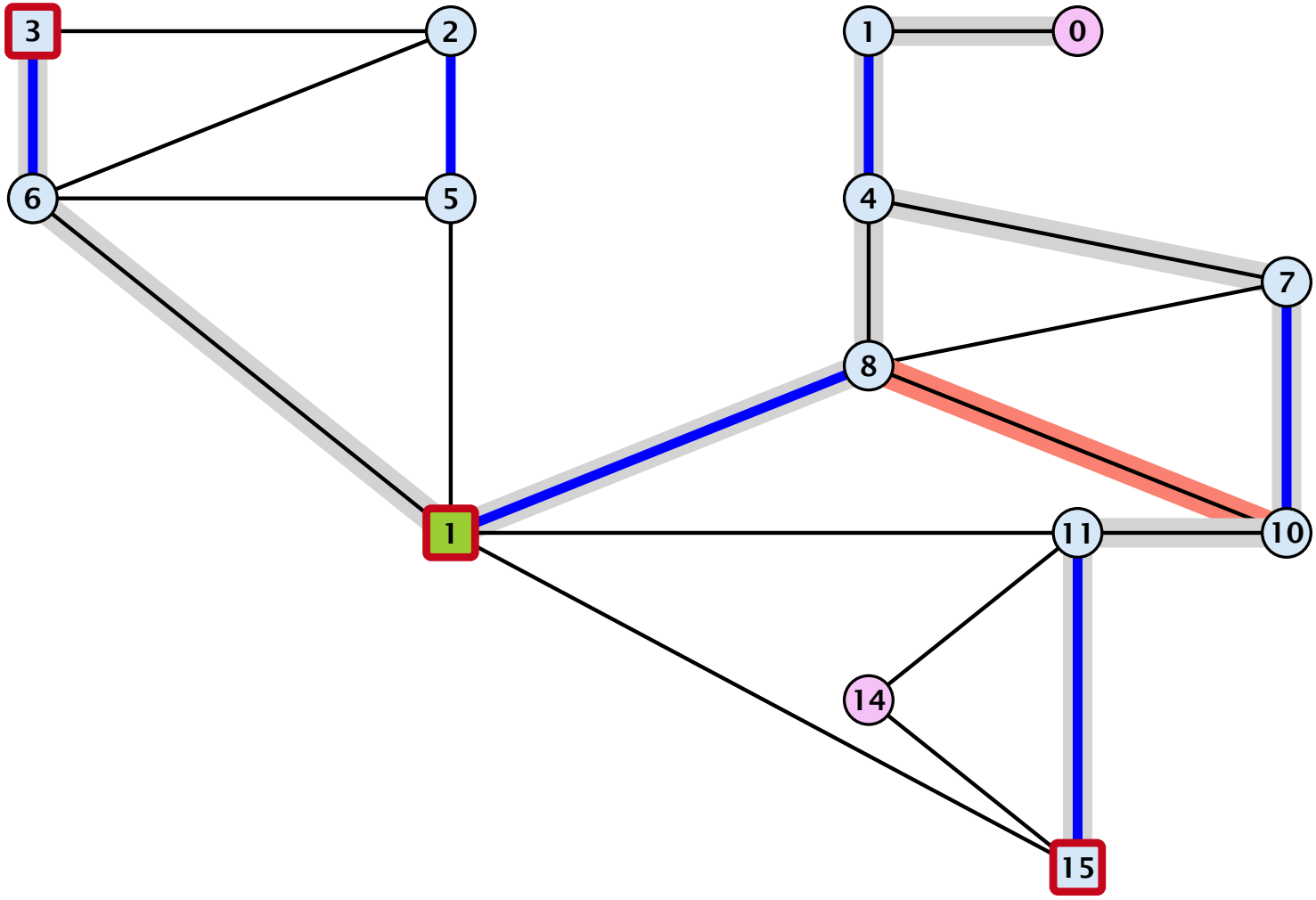

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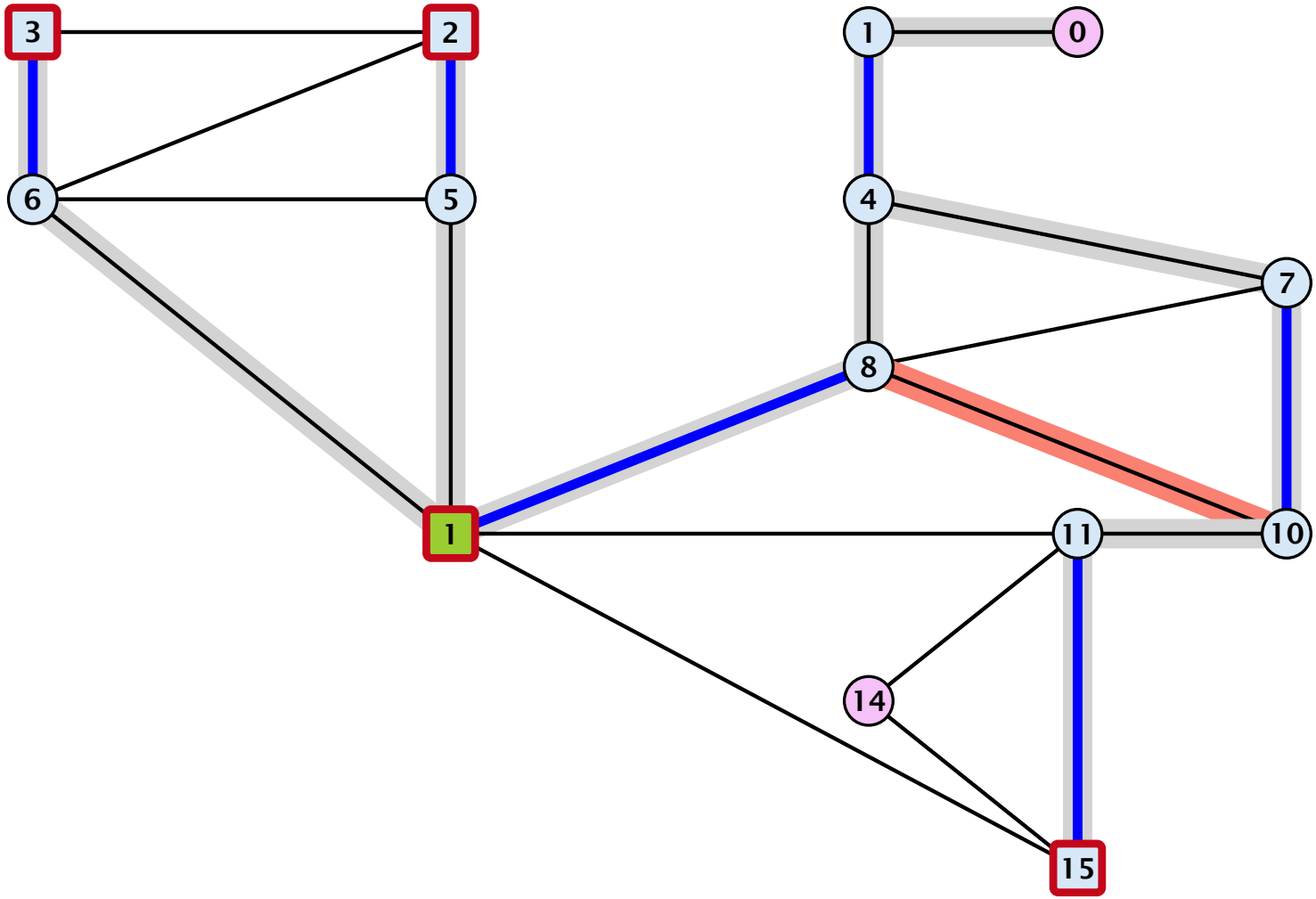

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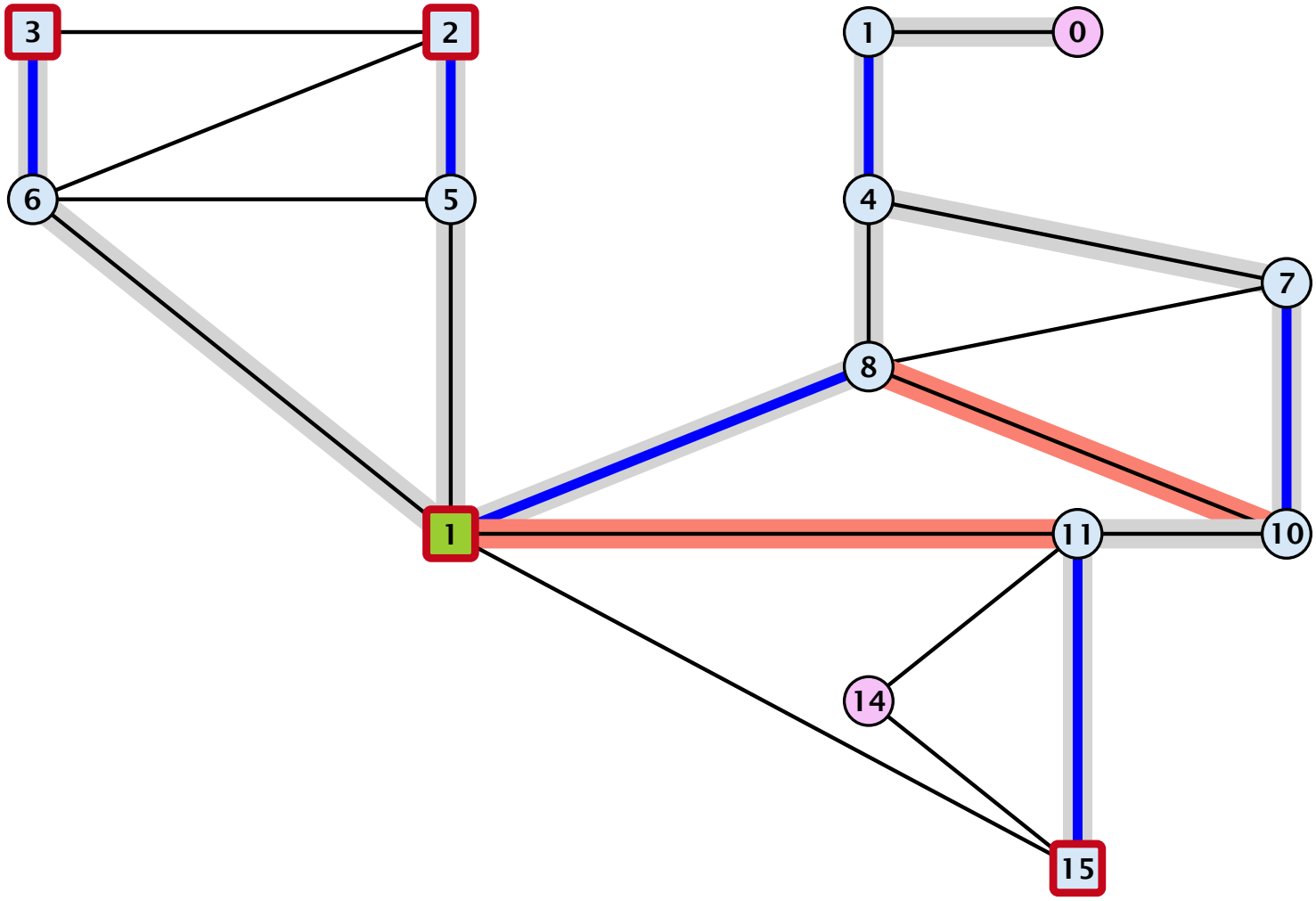

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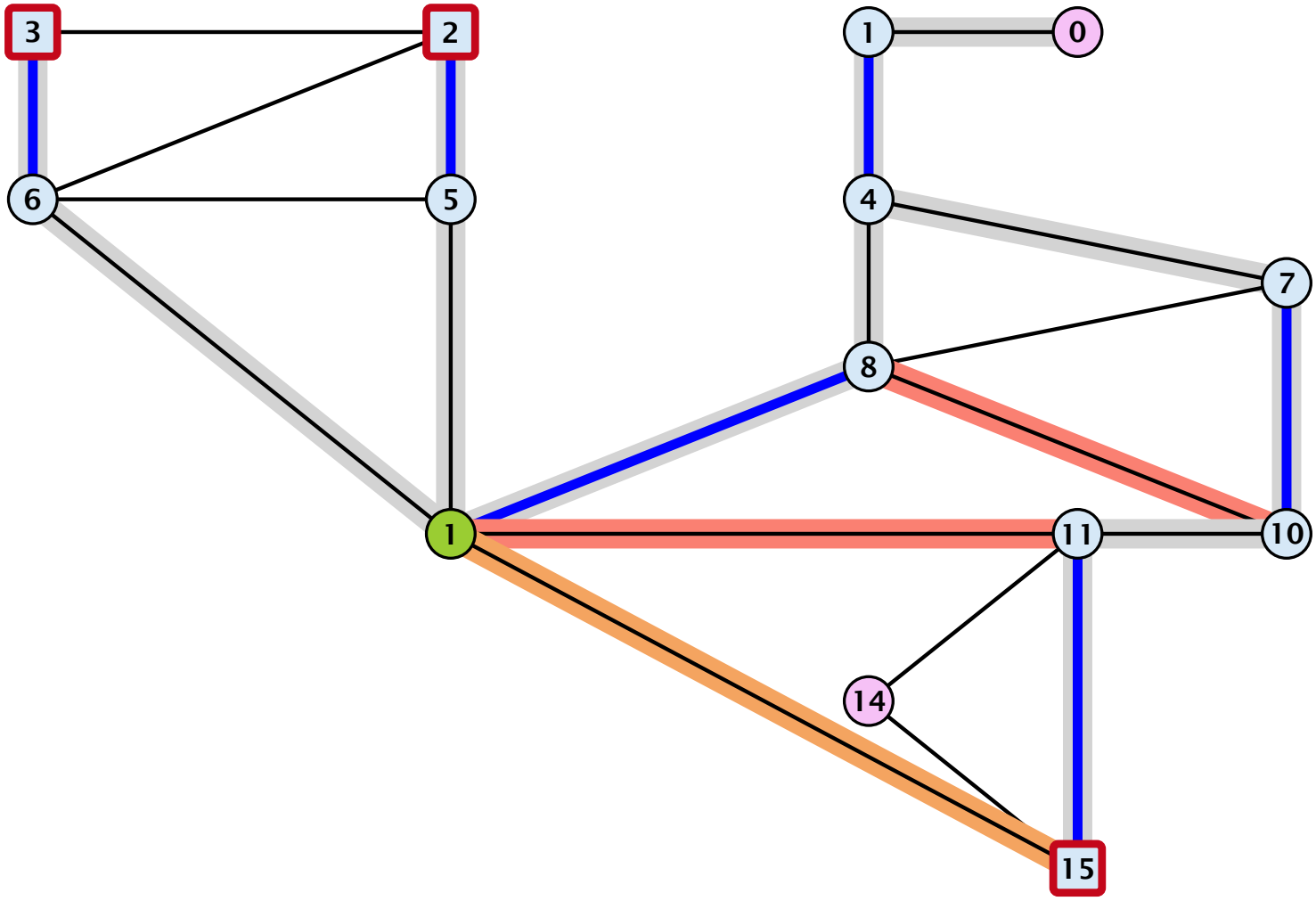

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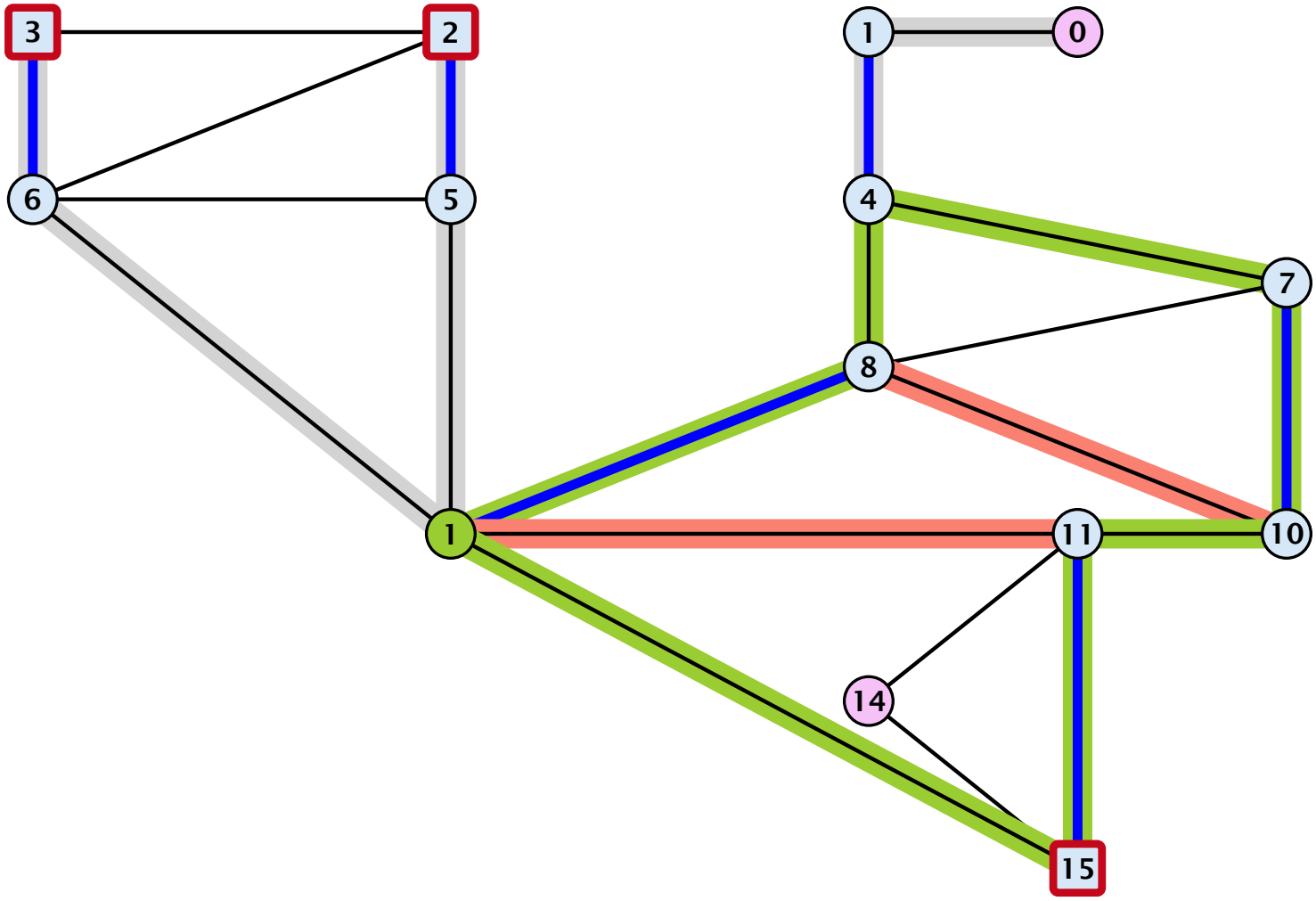

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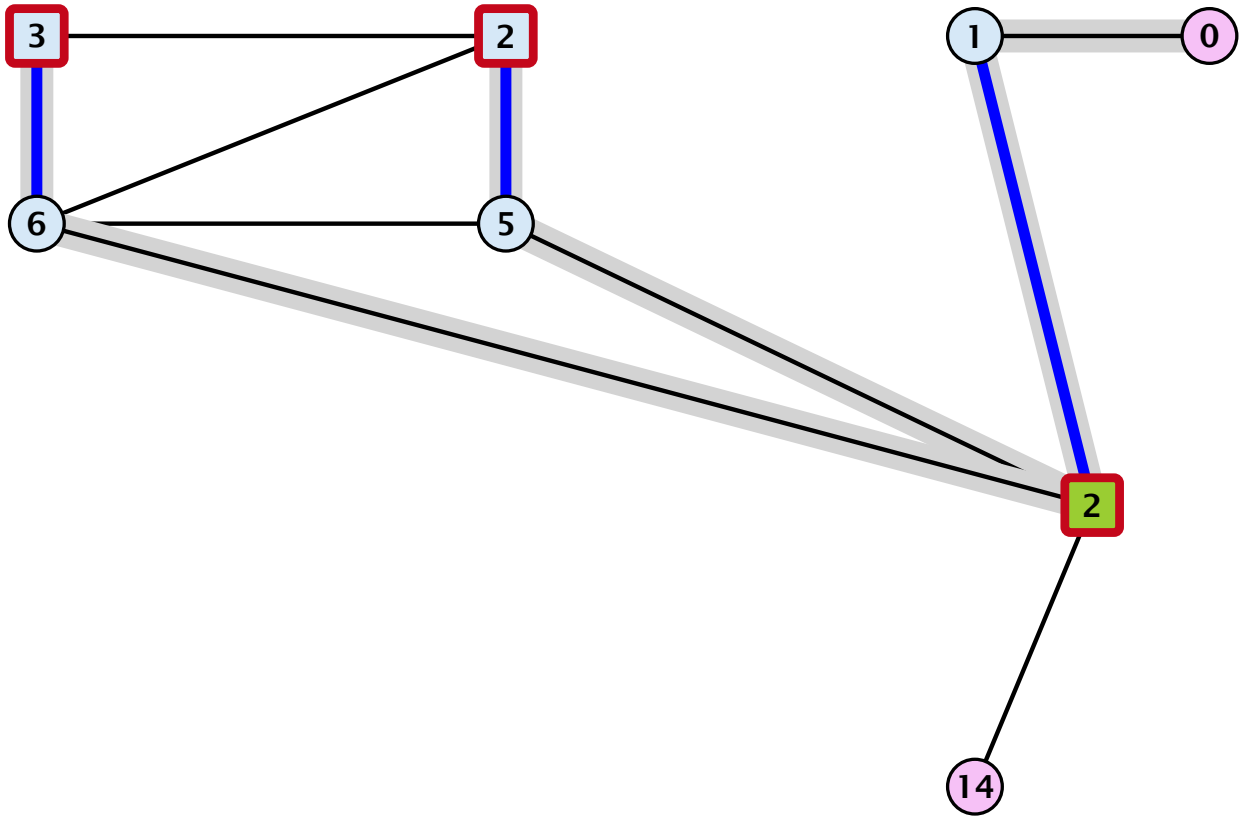

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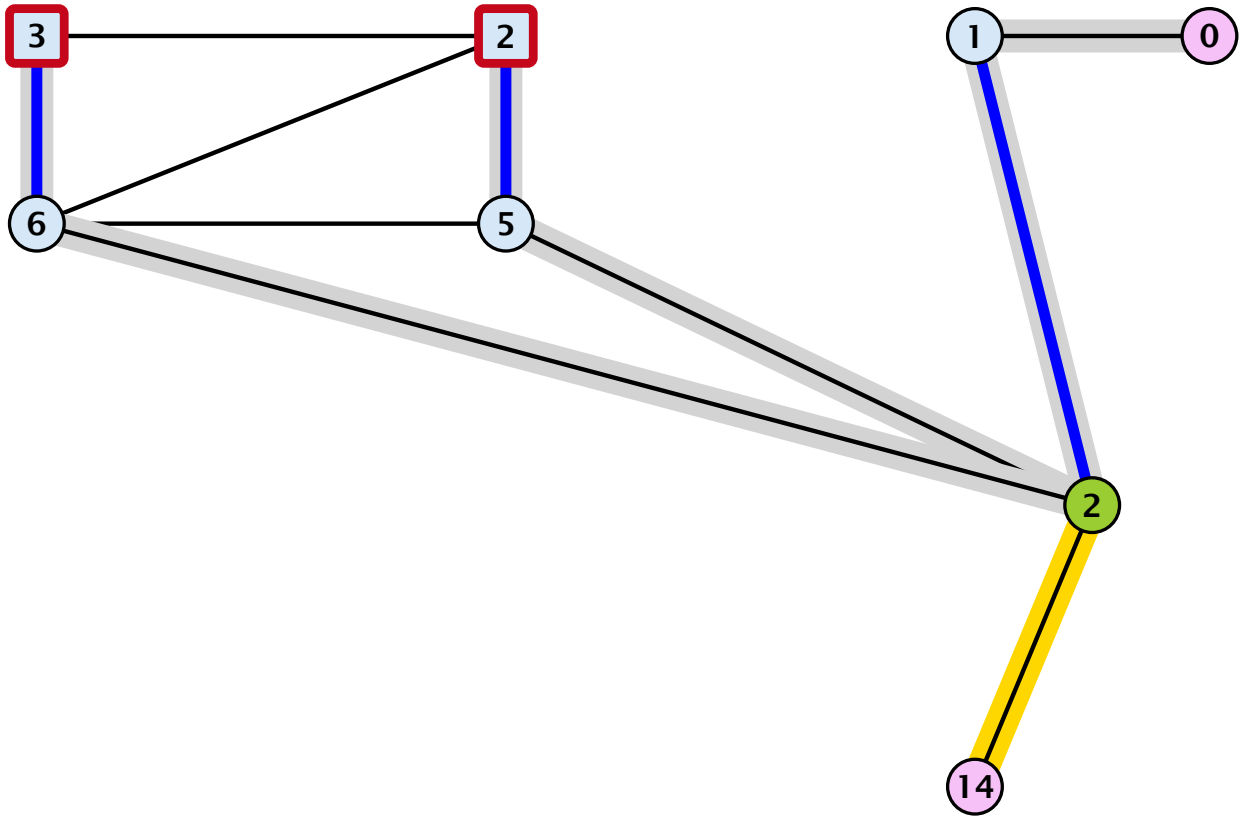

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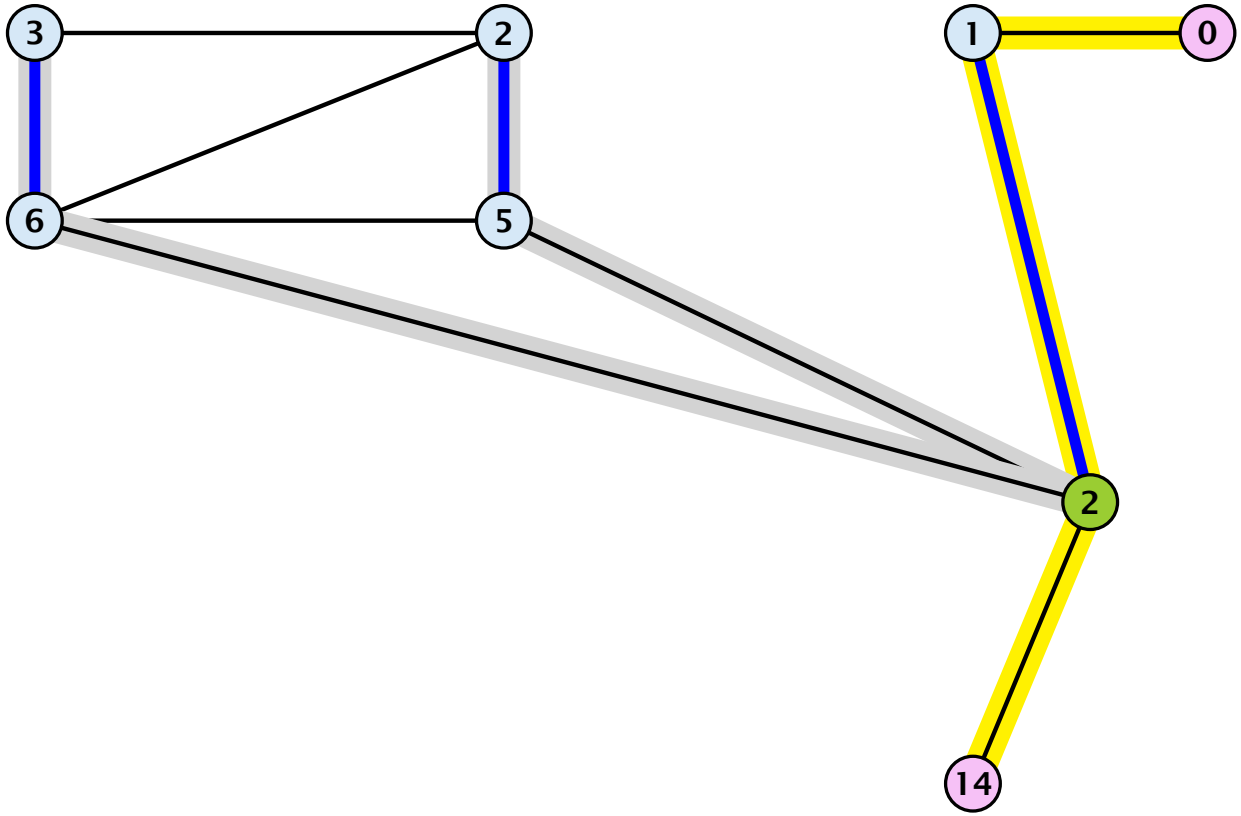

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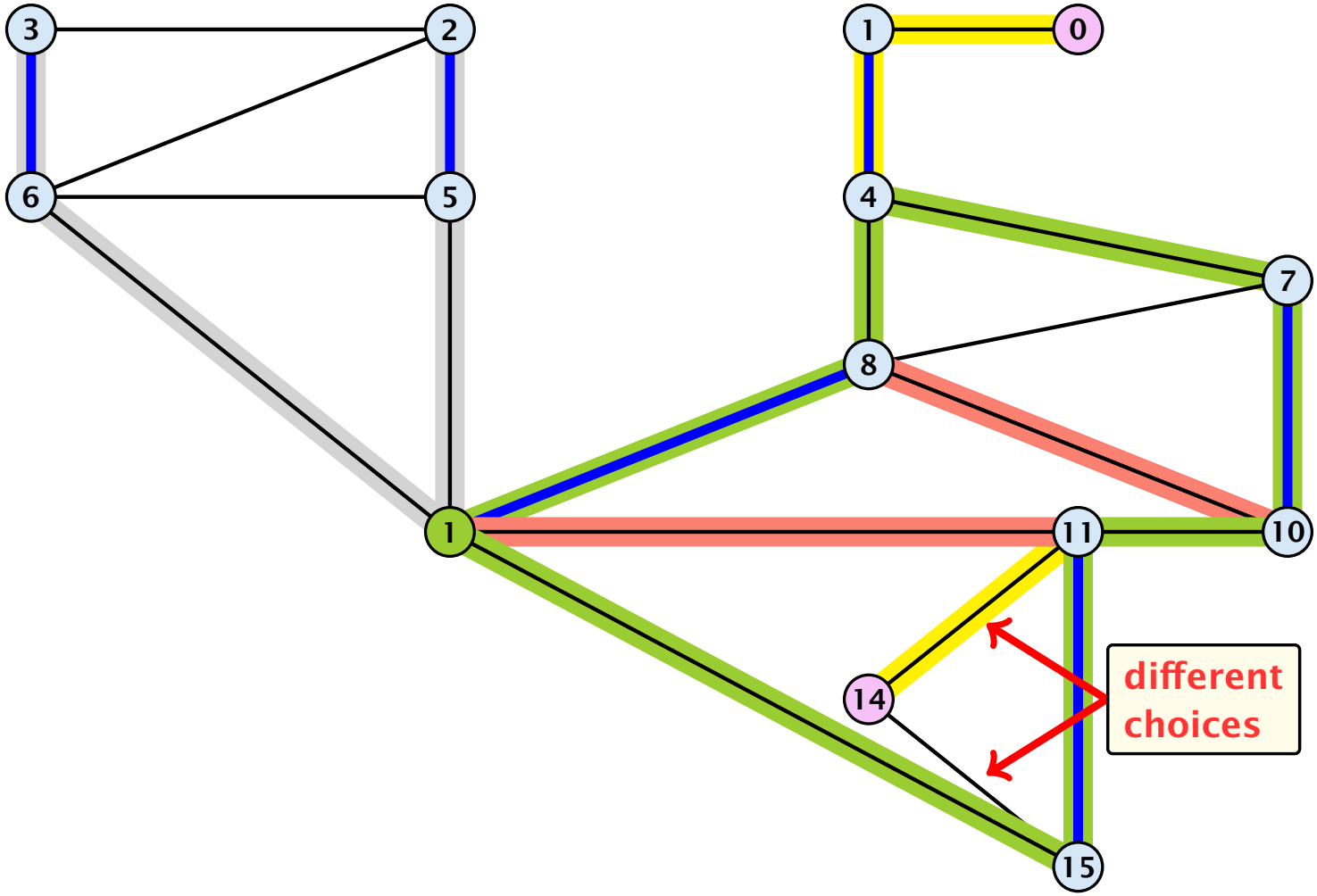

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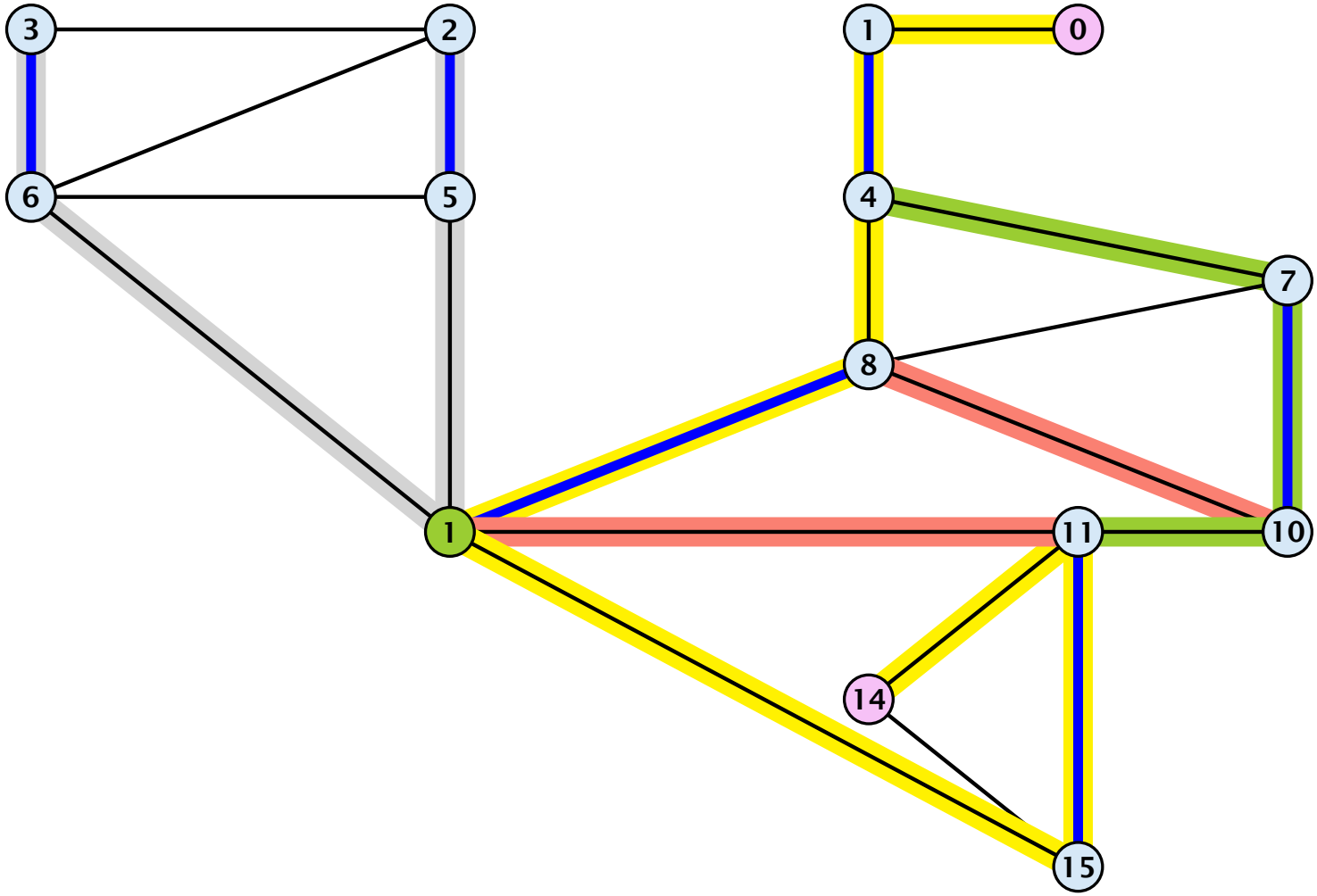

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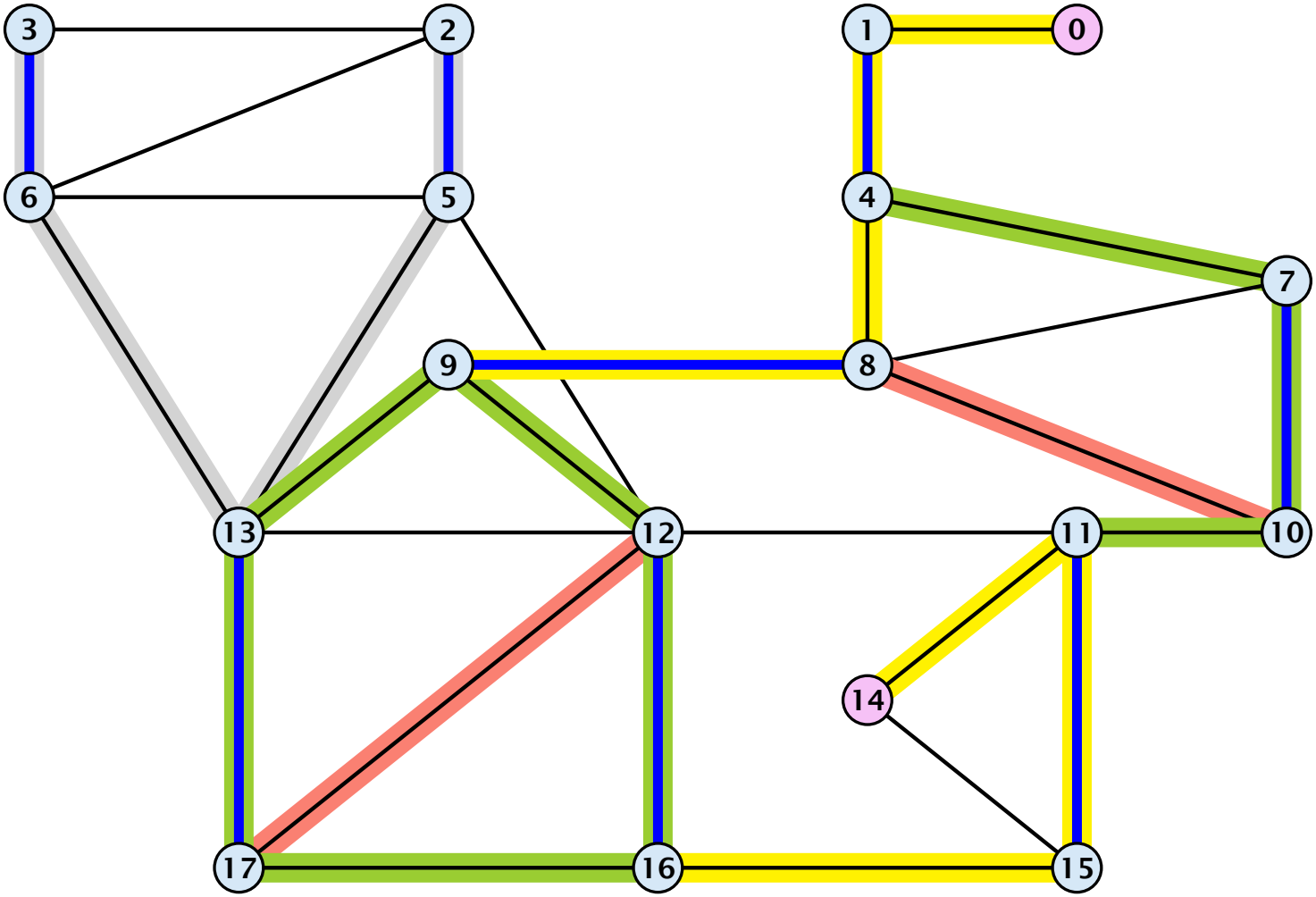

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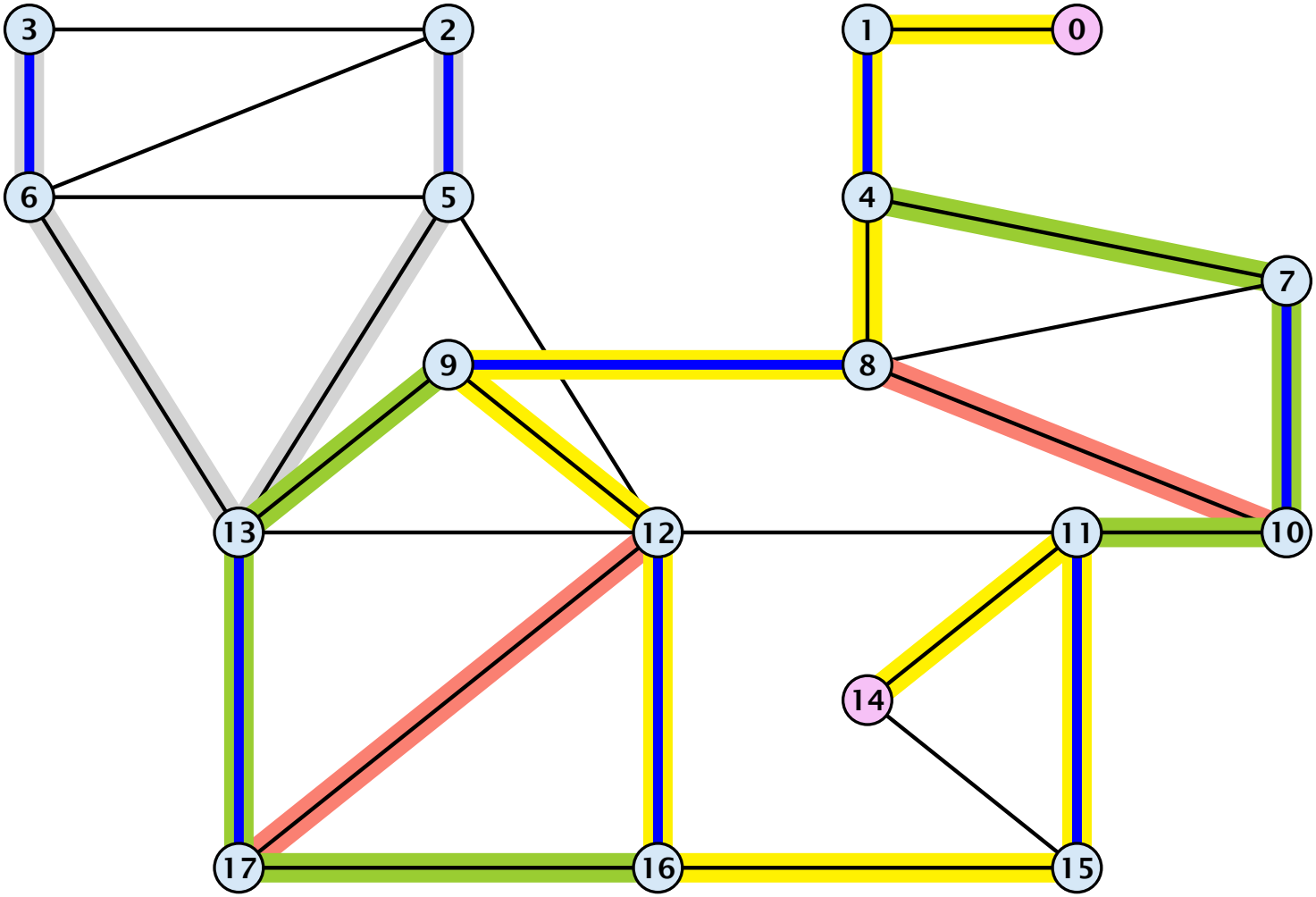

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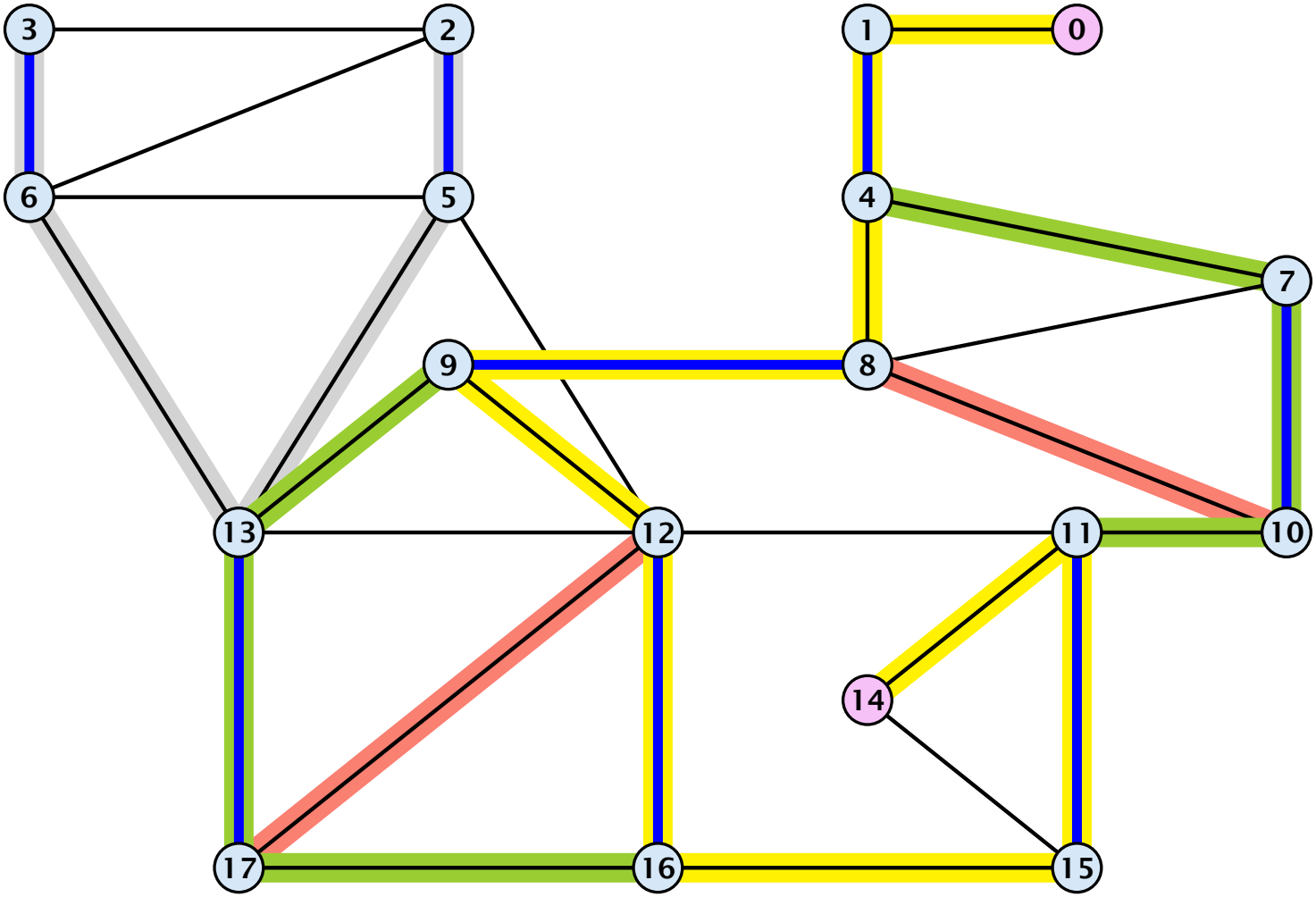

Correctness

Assume that in G we have a flower w.r.t. matching M. Let r be the root, B the blossom, and w the base. Let graph $G' = G/B$ with pseudonode b. Let M' be the matching in the contracted graph.

Lemma 101

If G' contains an augmenting path P' starting at r (or the pseudo-node containing r) w.r.t. the matching M' then G contains an augmenting path starting at r w.r.t. matching M.

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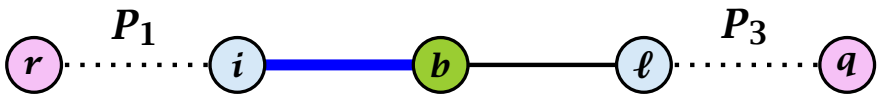

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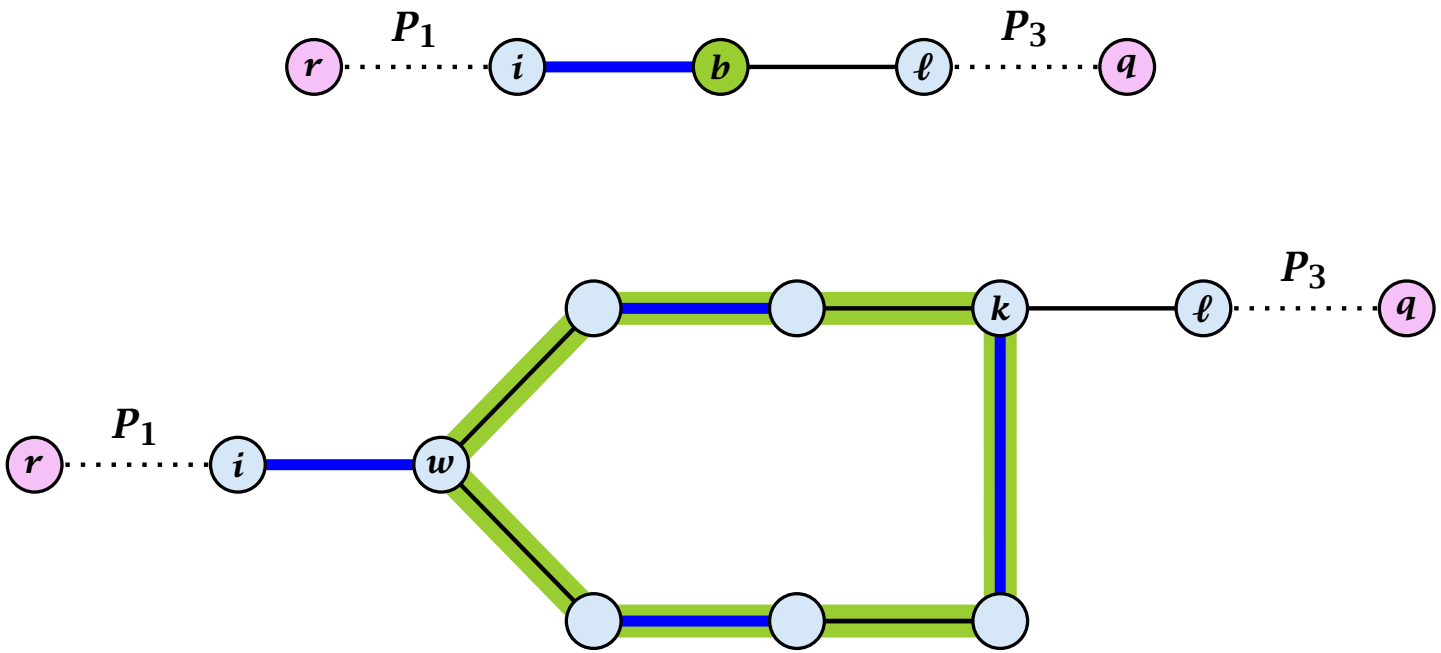

Correctness

- After the expansion ℓ must be incident to some node in the blossom. Let this node be k.
- If $k \neq w$ there is an alternating path P_2 from w to k that ends in a matching edge.
- $P_1 \circ (i,w) \circ P_2 \circ (k,\ell) \circ P_3$ is an alternating path.
- If $k = w$ then $P_1 \circ (i,w) \circ (w,\ell) \circ P_3$ is an alternating path.

Correctness

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Case 2: empty stem

- If the stem is empty then after expanding the blossom, $w = r$.

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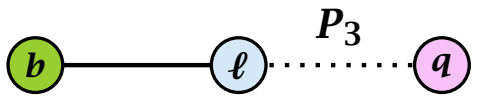

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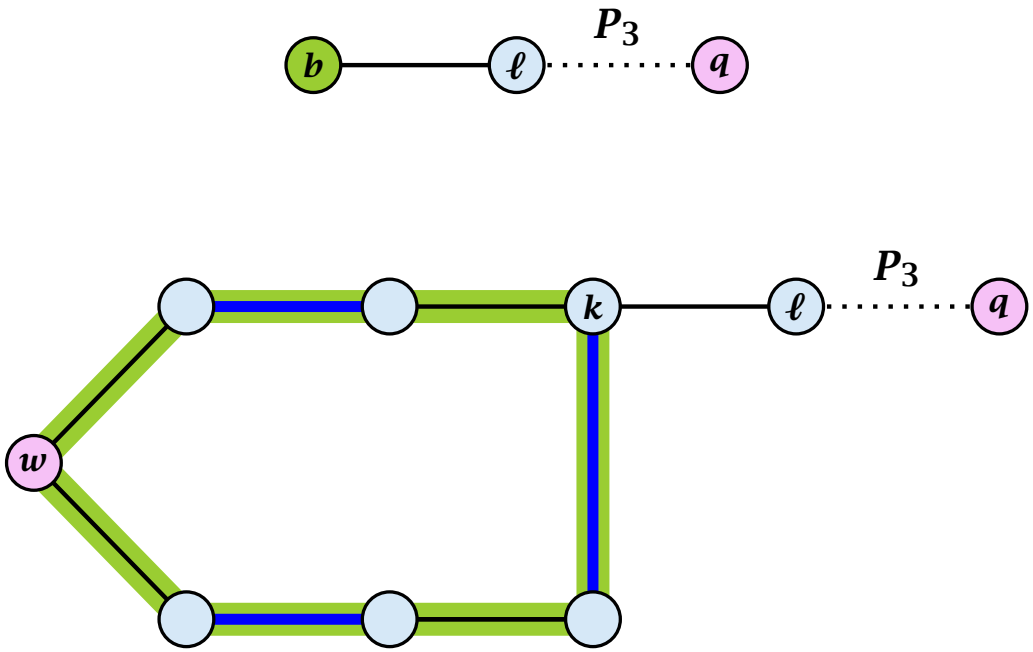

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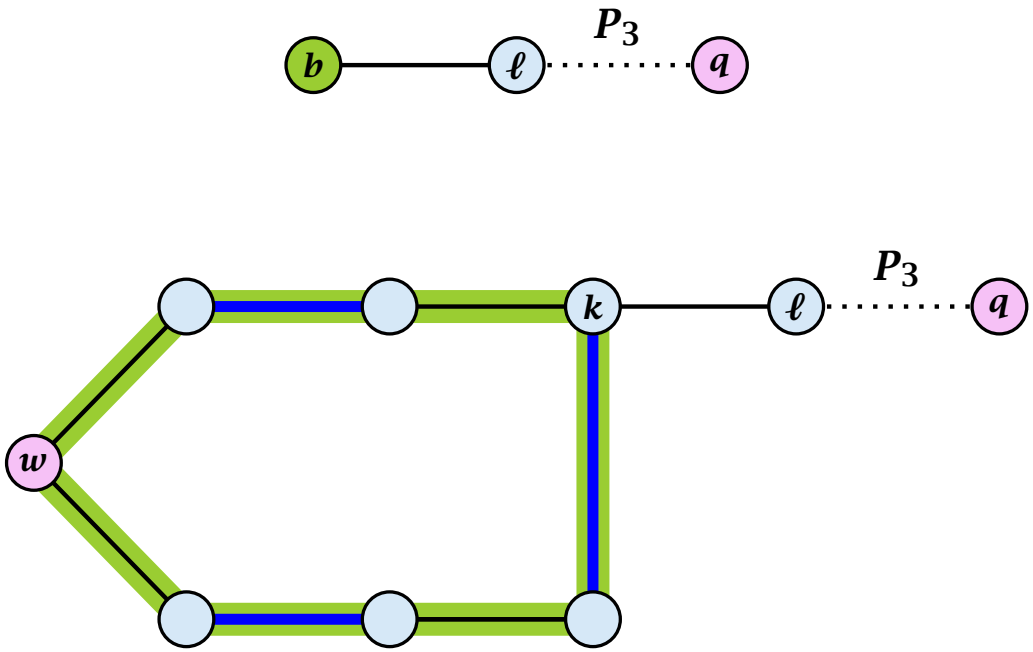

- The path $r \circ P_2 \circ (k, \ell) \circ P_3$ is an alternating path.

Correctness

Lemma 102

If G contains an augmenting path P from r to q w.r.t. matching M then G' contains an augmenting path from r (or the pseudo-node containing r) to q w.r.t. M'.

Correctness

Proof.

- If P does not contain a node from B there is nothing to prove.
- We can assume that r and q are the only free nodes in G.

Case 1: empty stem

Let i be the last node on the path P that is part of the blossom.

P is of the form $P_1 \circ (i,j) \circ P_2$, for some node j and (i,j) is unmatched.

$(b,j) \circ P_2$ is an augmenting path in the contracted network.

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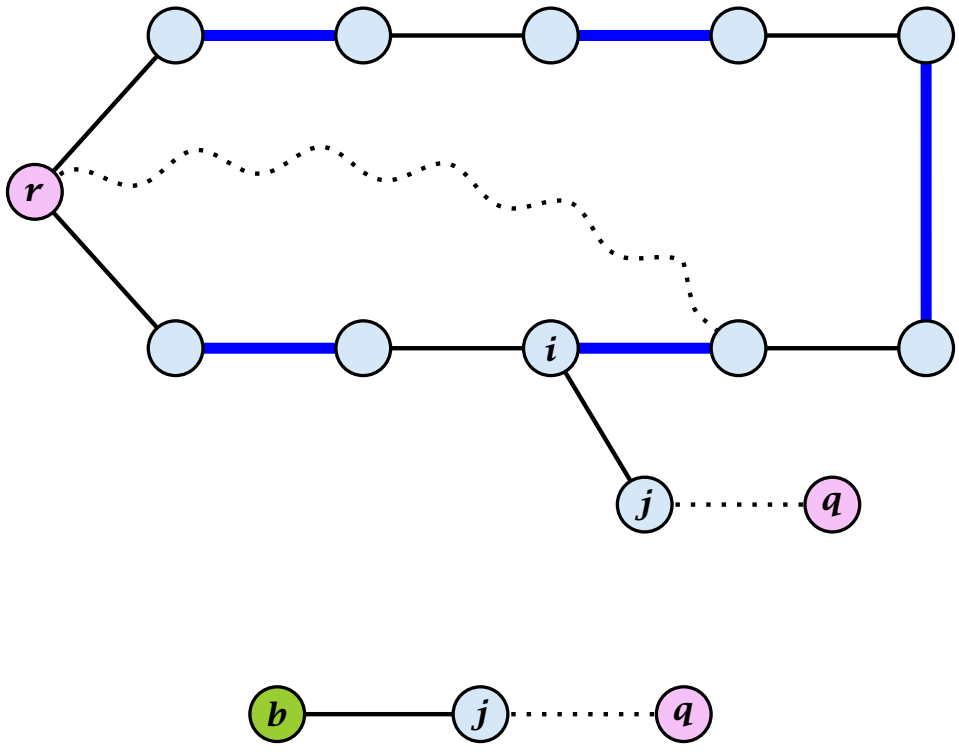
Illustration for Case 1:



Correctness

Case 2: non-empty stem

Let P_3 be alternating path from r to w; this exists because r and w are root and base of a blossom. Define $M_+ = M \oplus P_3$.

In M_+, r is matched and w is unmatched.

G must contain an augmenting path w.r.t. matching M_+, since M and M_+ have same cardinality.

This path must go between w and q as these are the only unmatched vertices w.r.t. M_+.

For M'_+ the blossom has an empty stem. Case 1 applies.

G' has an augmenting path w.r.t. M'_+. It must also have an augmenting path w.r.t. M', as both matchings have the same cardinality.

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For M'_+ the blossom has an empty stem. Case 1 applies.

G' has an augmenting path w.r.t. M'_+. It must also have an augmenting path w.r.t. M', as both matchings have the same cardinality.

This path must go between r and q.

Correctness

Case 2: non-empty stem

Let P_3 be alternating path from r to w; this exists because r and w are root and base of a blossom. Define $M_+ = M \oplus P_3$.

In M_+, r is matched and w is unmatched.

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This path must go between r and q.

Algorithm 54 search(r, *found*)

1: set $\bar{A}(i) \leftarrow A(i)$ for all nodes i
2: $found \leftarrow$ false
3: unlabel all nodes;
4: give an even label to r and initialize $list \leftarrow \{r\}$
5: **while** $list \neq \emptyset$ **do**
6: delete a node i from $list$
7: examine(i, *found*)
8: **if** $found =$ true **then return**

Search for an augmenting path
starting at r.

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$A(i)$ contains neighbours of node i.

We create a copy $\bar{A}(i)$ so that we later can shrink blossoms.

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found is just a Boolean that allows
to abort the search process...

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In the beginning no node is in the tree.

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7: examine(i, *found*)
8: **if** *found* = true **then return**

Put the root in the tree.

list could also be a set or a stack.

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As long as there are nodes with unexamined neighbours...

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7: examine(i, *found*)
8: **if** $found =$ true **then return**

...examine the next one

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5: **while** *list* $\neq \emptyset$ **do**
6: delete a node i from *list*
7: examine(i, *found*)
8: **if** *found* = true **then return**

If you found augmenting path
abort and start from next root.

Algorithm 55 examine$(i, found)$

```
 1: for all j ∈ Ā(i) do
 2:     if j is even then contract(i, j) and return
 3:     if j is unmatched then
 4:         q ← j;
 5:         pred(q) ← i;
 6:         found ← true;
 7:         return
 8:     if j is matched and unlabeled then
 9:         pred(j) ← i;
10:         pred(mate(j)) ← j;
11:         add mate(j) to list
```

Examine the neighbours of a node i

Algorithm 55 examine$(i, found)$

```
1: for all j ∈ Ā(i) do
2:     if j is even then contract(i, j) and return
3:     if j is unmatched then
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For all neighbours j do...

Algorithm 55 examine(i, *found*)

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You have found a blossom...

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```
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You have found a free node which gives you an augmenting path.

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```

If you find a matched node that is not in the tree you grow...

Algorithm 55 examine$(i, found)$

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 3:     if j is unmatched then
 4:         q ← j;
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 7:         return
 8:     if j is matched and unlabeled then
 9:         pred(j) ← i;
10:         pred(mate(j)) ← j;
11:         add mate(j) to list
```

mate(j) is a new node from which you can grow further.

Algorithm 56 contract(i, j)

1: trace pred-indices of i and j to identify a blossom B
2: create new node b and set $\bar{A}(b) \leftarrow \cup_{x \in B} \bar{A}(x)$
3: label b even and add to *list*
4: update $\bar{A}(j) \leftarrow \bar{A}(j) \cup \{b\}$ for each $j \in \bar{A}(b)$
5: form a circular double linked list of nodes in B
6: delete nodes in B from the graph

Contract blossom identified by
nodes i and j

Algorithm 56 contract(i, j)

1: trace pred-indices of i and j to identify a blossom B
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6: delete nodes in B from the graph

Get all nodes of the blossom.

Time: $\mathcal{O}(m)$

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Identify all neighbours of b.

Time: $\mathcal{O}(m)$ (how?)

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b will be an even node, and it has unexamined neighbours.

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5: form a circular double linked list of nodes in B
6: delete nodes in B from the graph

Every node that was adjacent to a node in B is now adjacent to b

Algorithm 56 contract(i, j)

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Only for making a blossom expansion easier.

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5: form a circular double linked list of nodes in B
6: delete nodes in B from the graph

Only delete links from nodes not in B to B.

When expanding the blossom again we can recreate these links in time $\mathcal{O}(m)$.

Analysis

- A contraction operation can be performed in time $\mathcal{O}(m)$. Note, that any graph created will have at most m edges.
- The time between two contraction-operation is basically a BFS/DFS on a graph. Hence takes time $\mathcal{O}(m)$.
- There are at most n contractions as each contraction reduces the number of vertices.
- The expansion can trivially be done in the same time as needed for all contractions.
- An augmentation requires time $\mathcal{O}(n)$. There are at most n of them.
- In total the running time is at most

$$n \cdot (\mathcal{O}(mn) + \mathcal{O}(n)) = \mathcal{O}(mn^2) \ .$$

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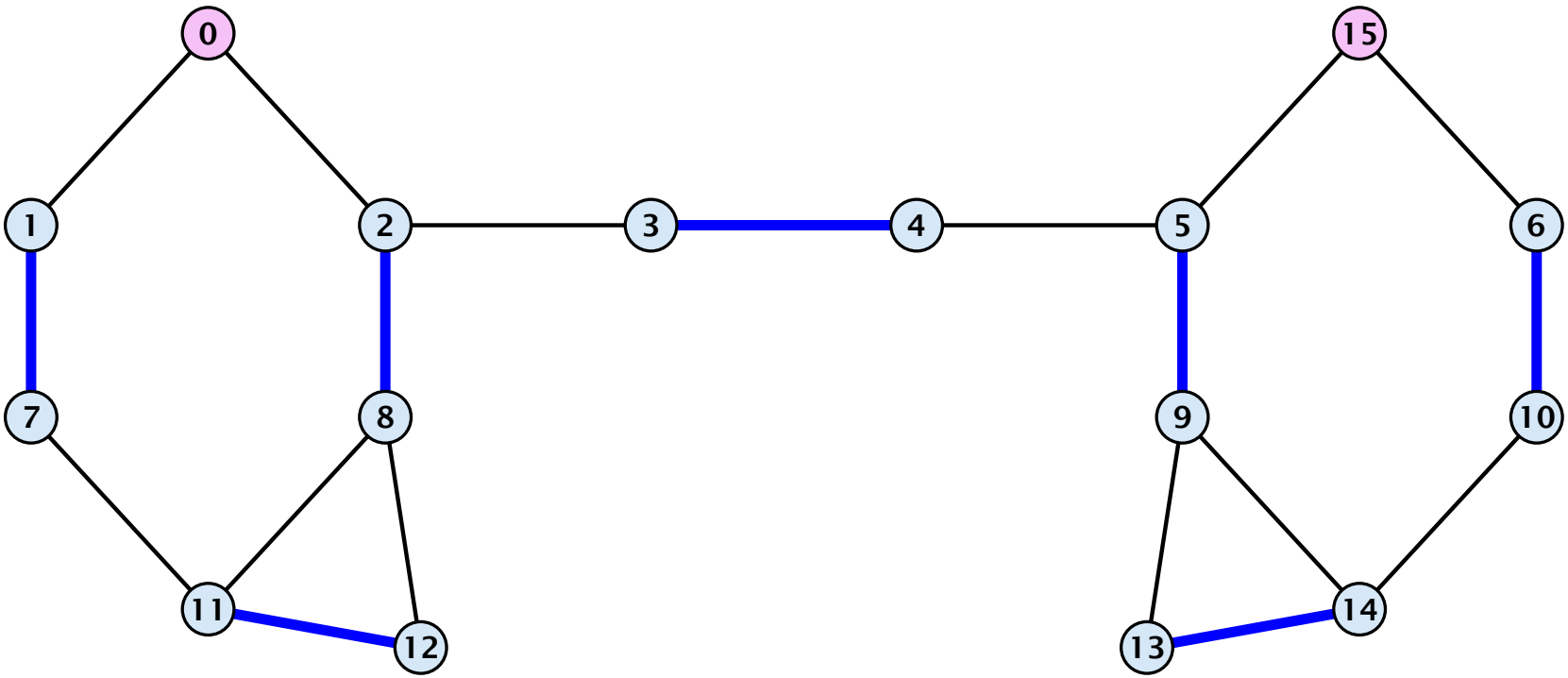
Example: Blossom Algorithm



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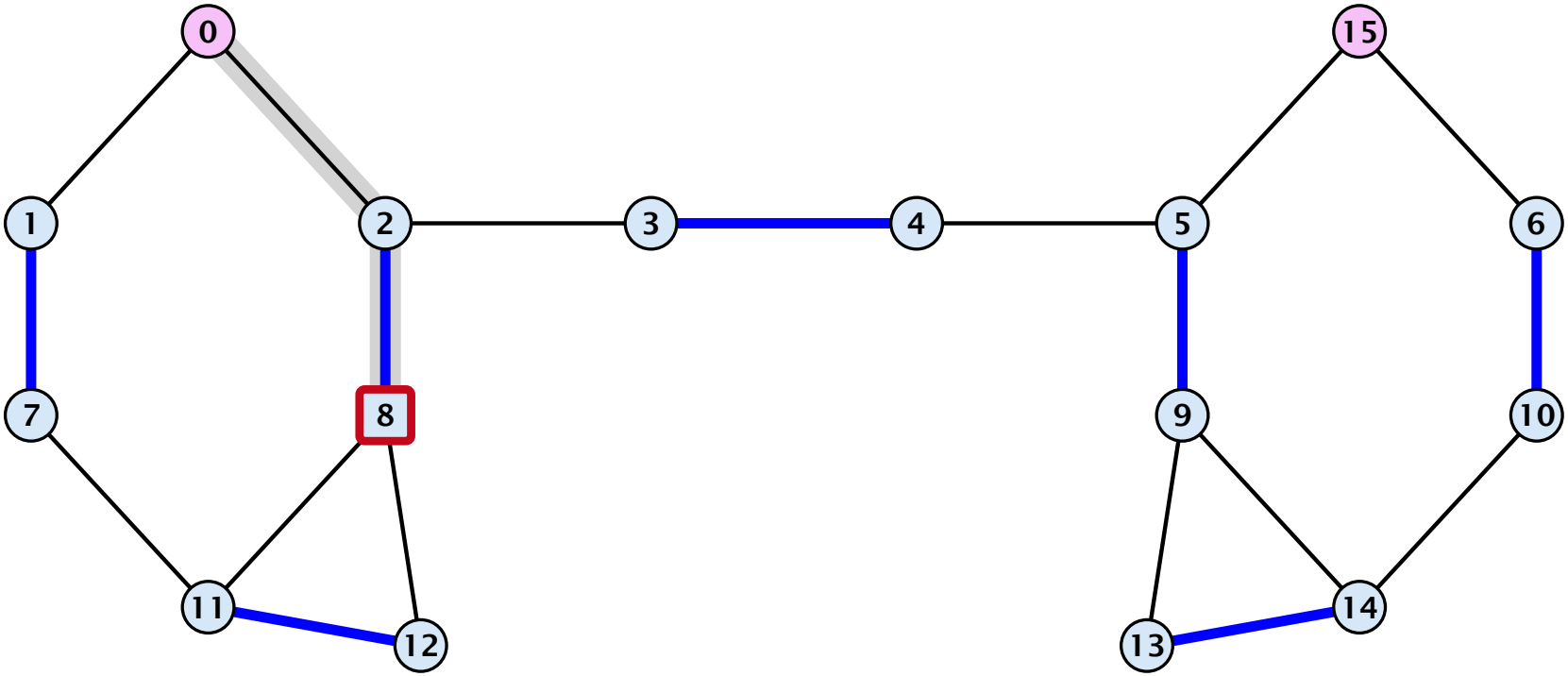

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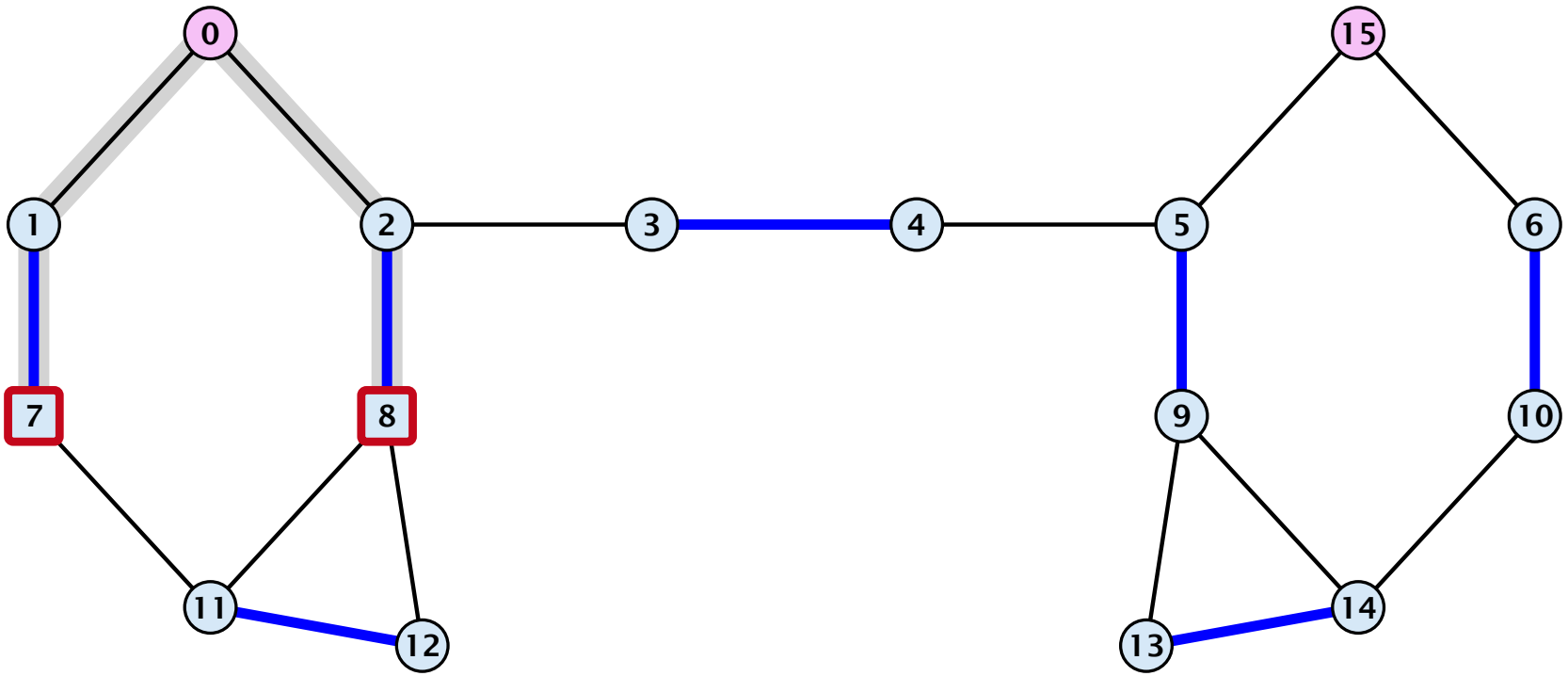

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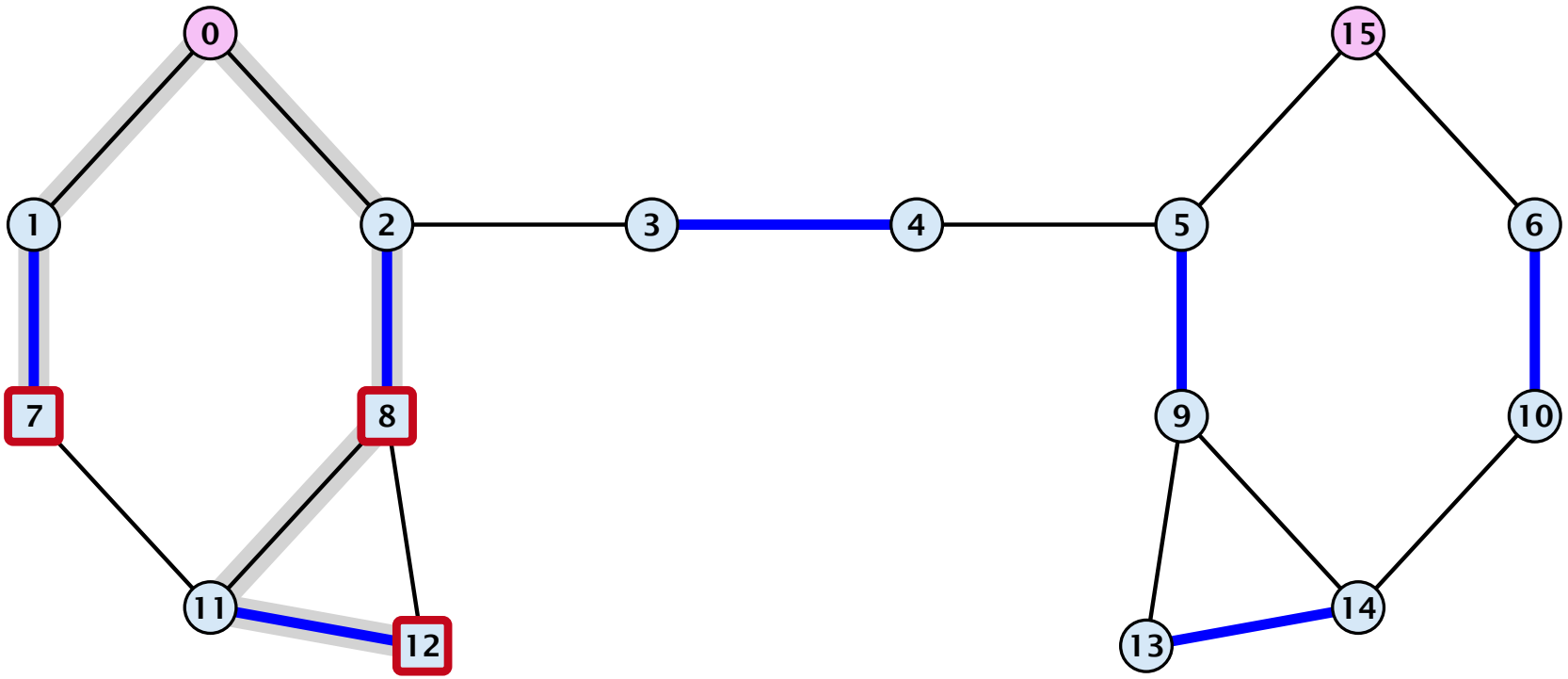

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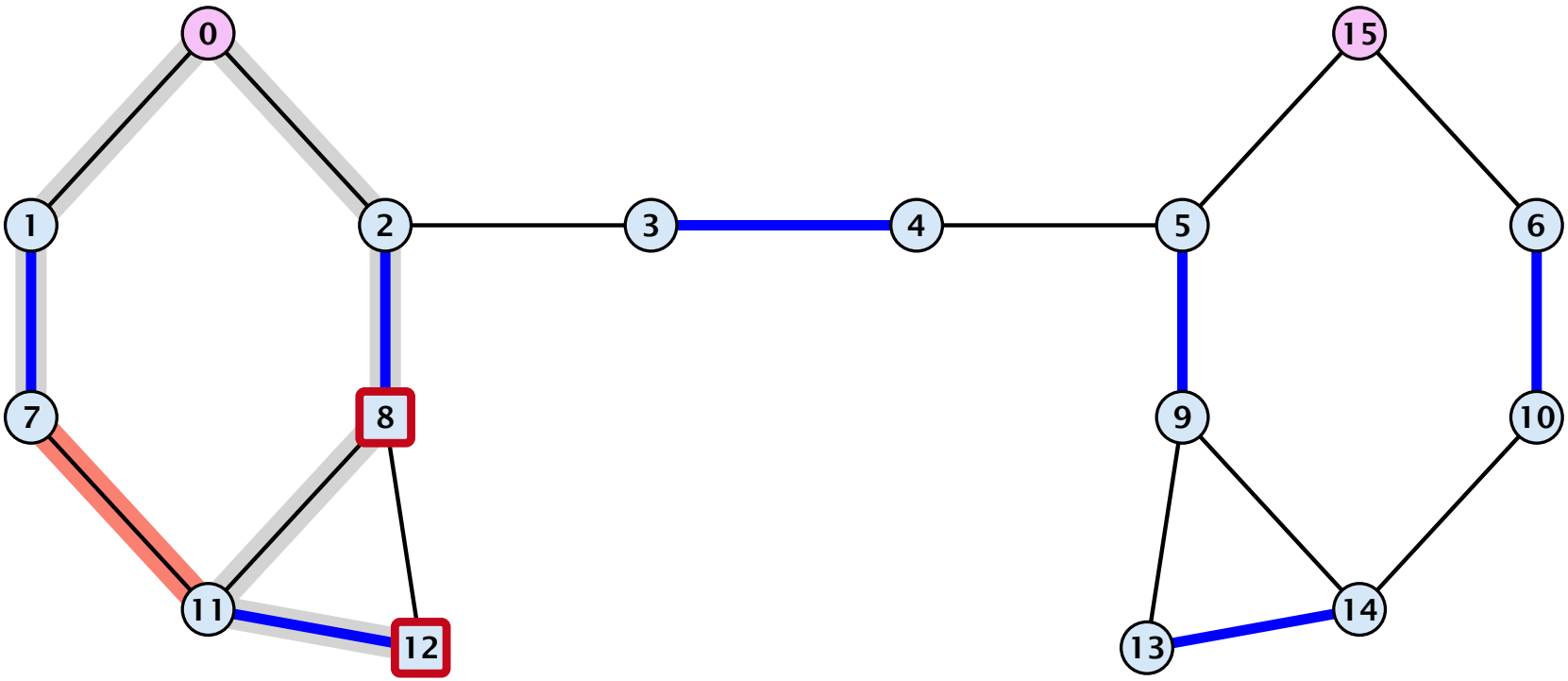

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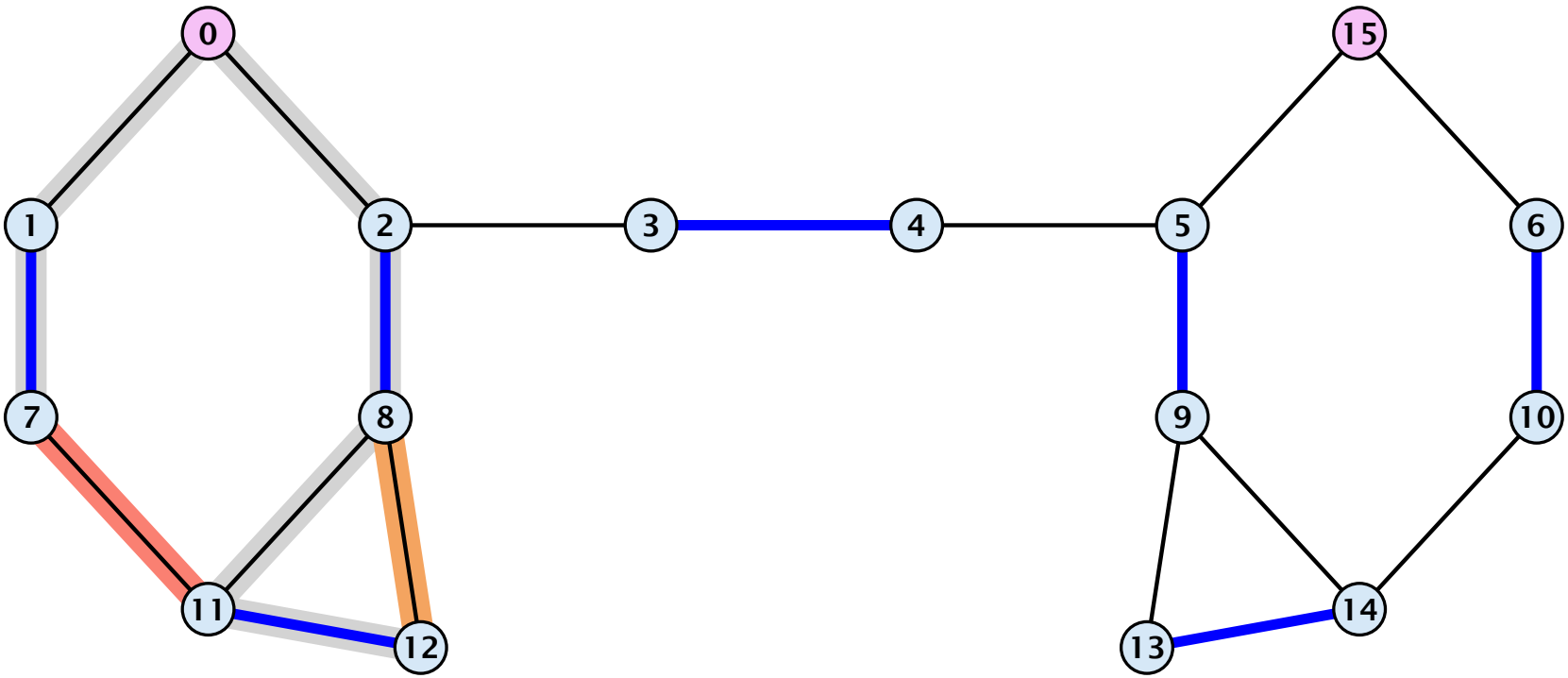

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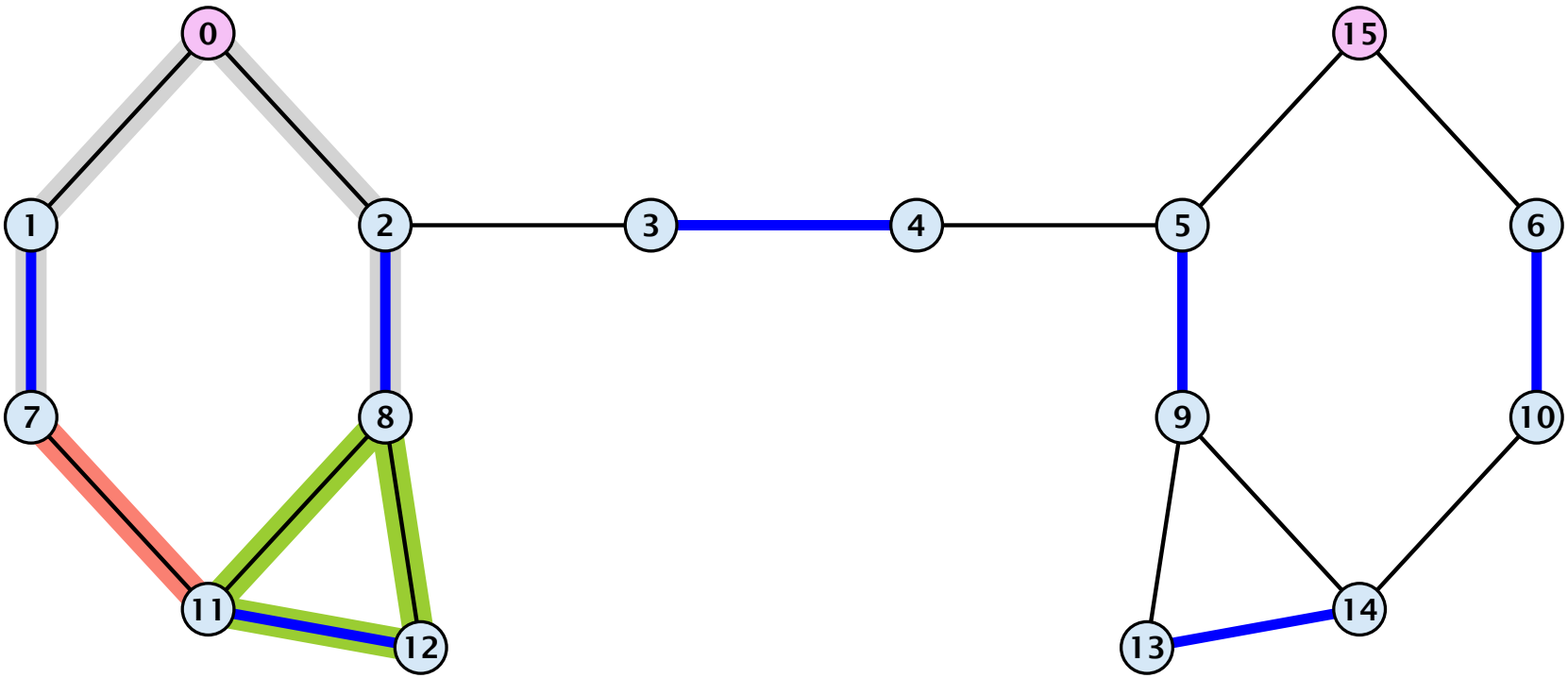

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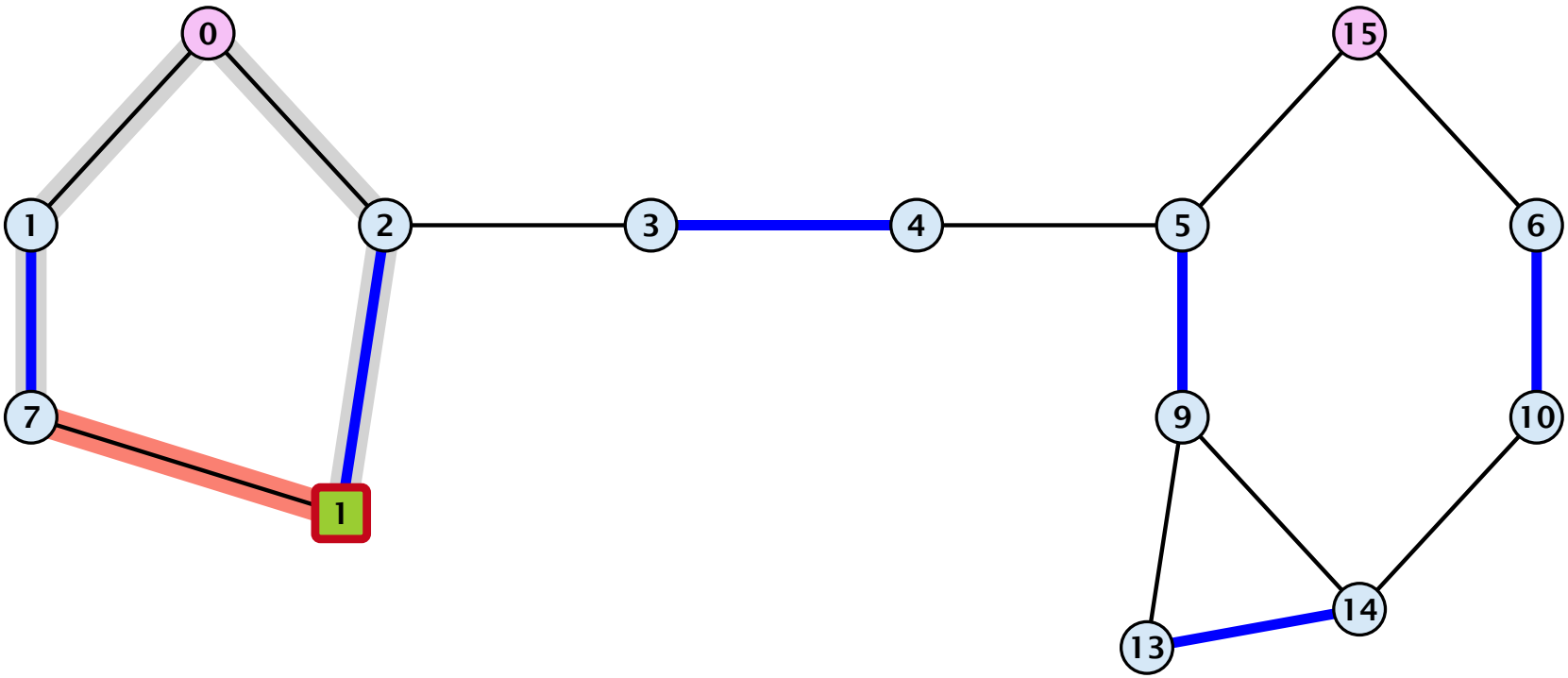

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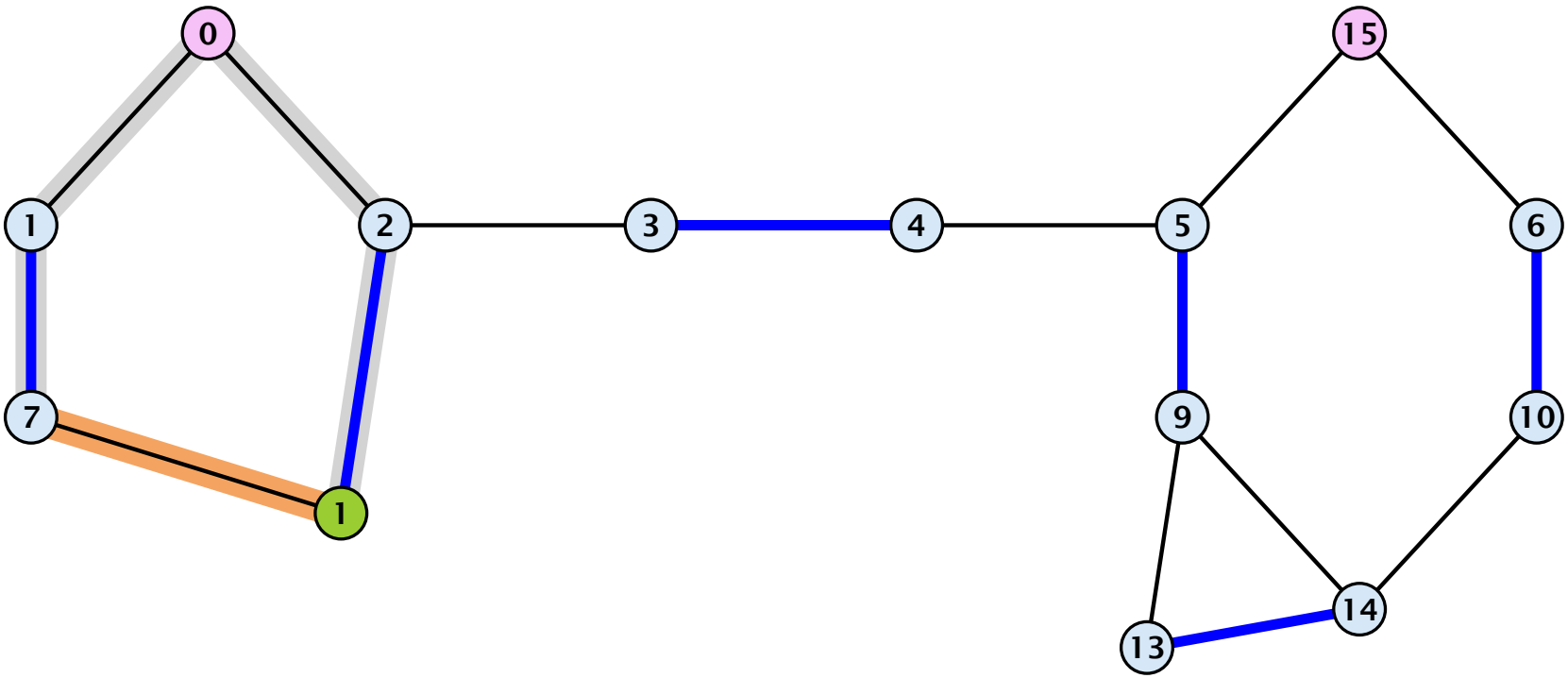

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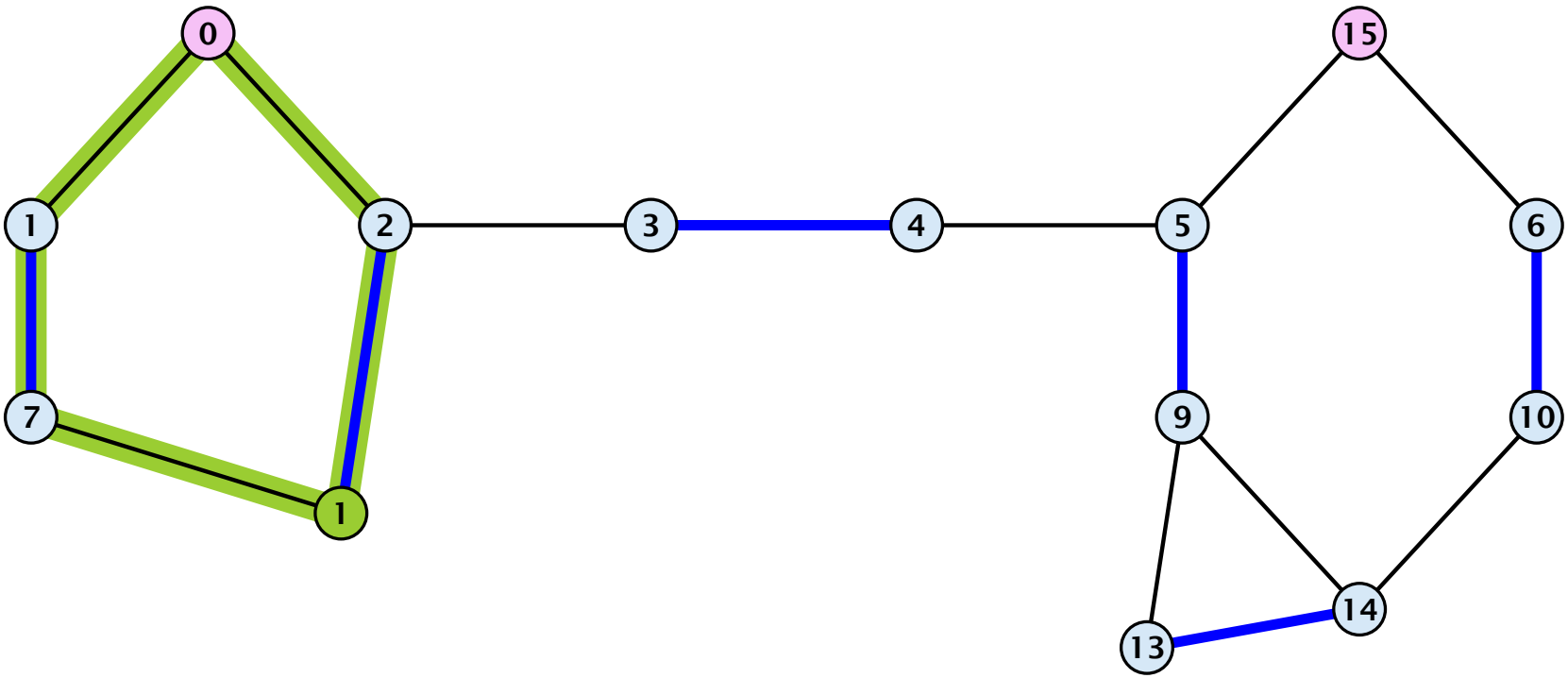

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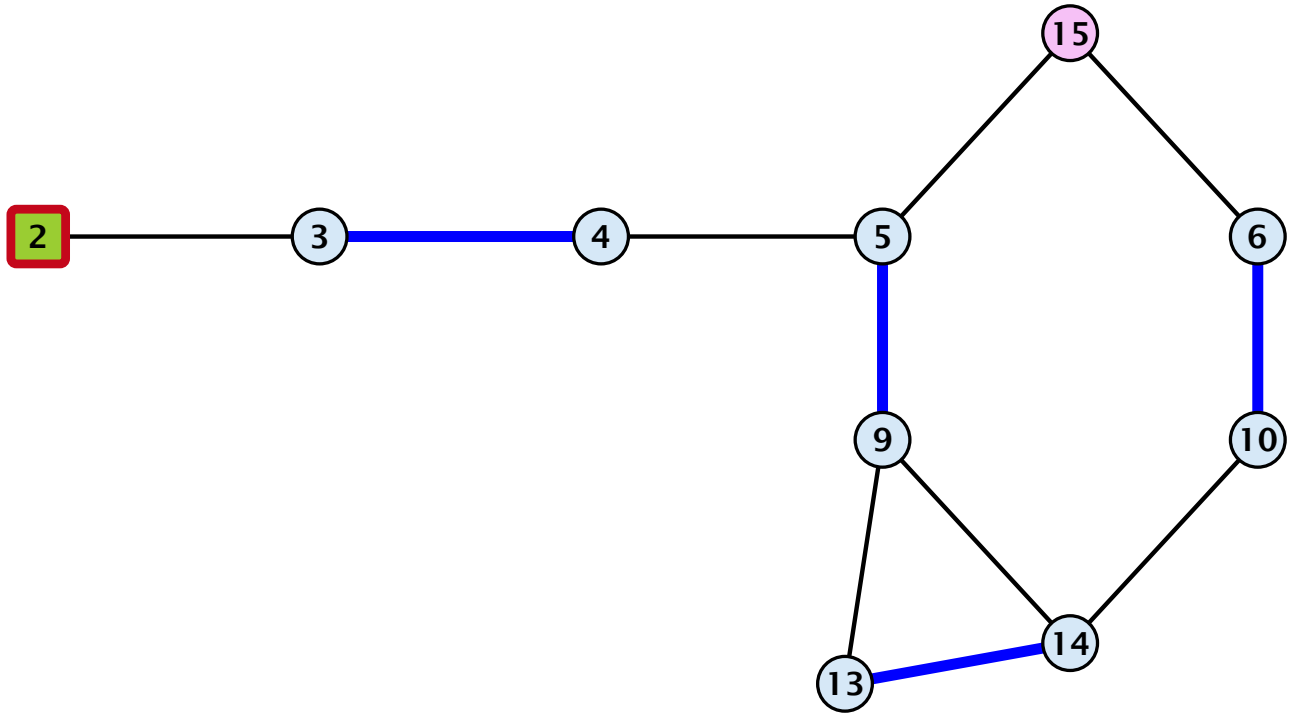

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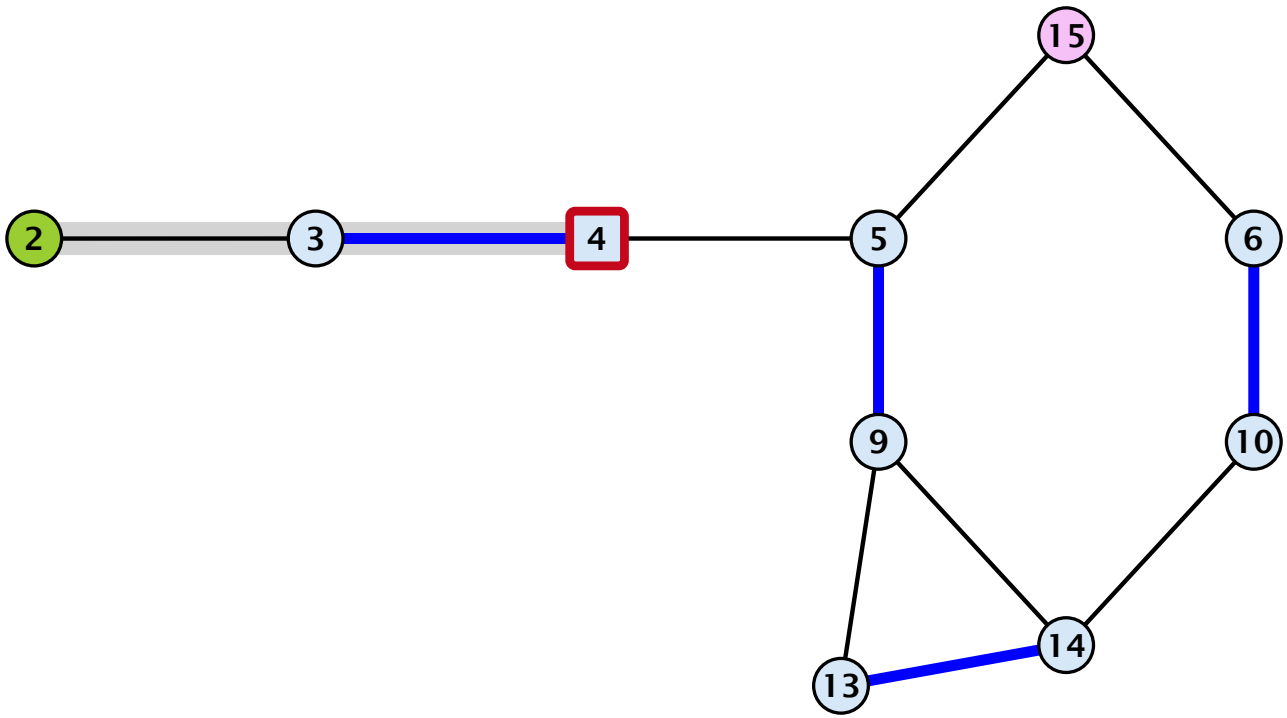

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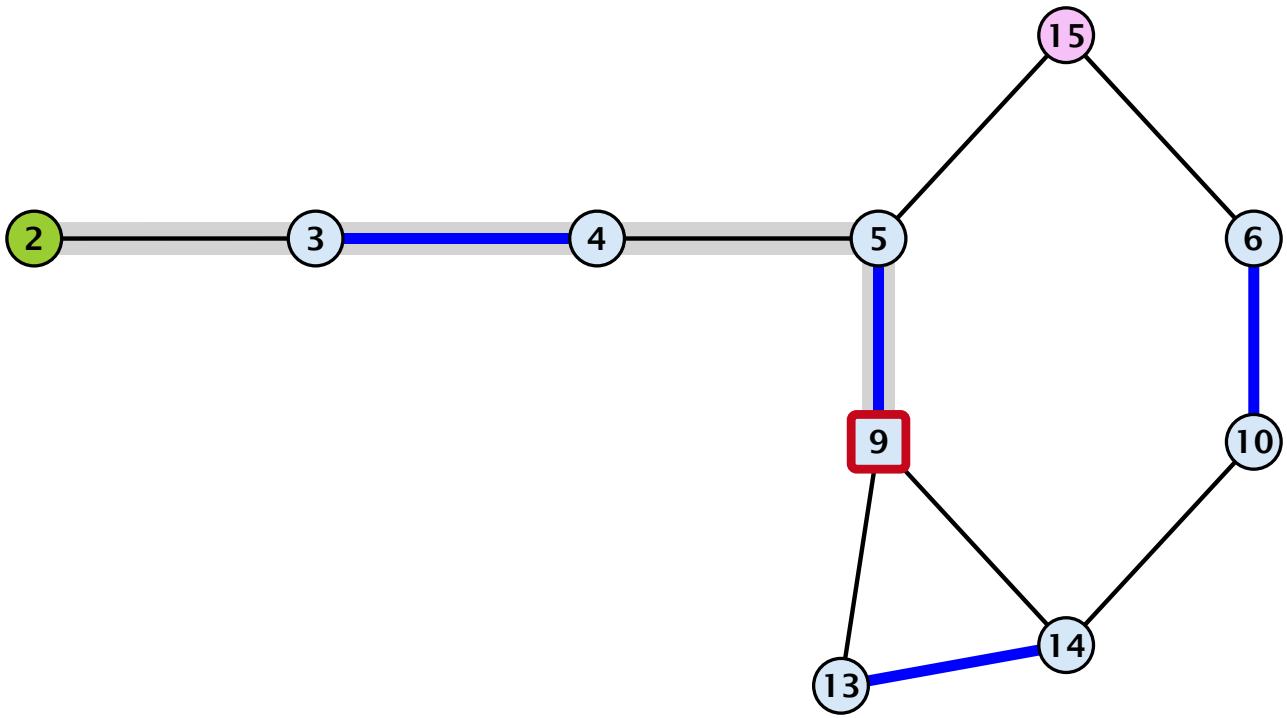

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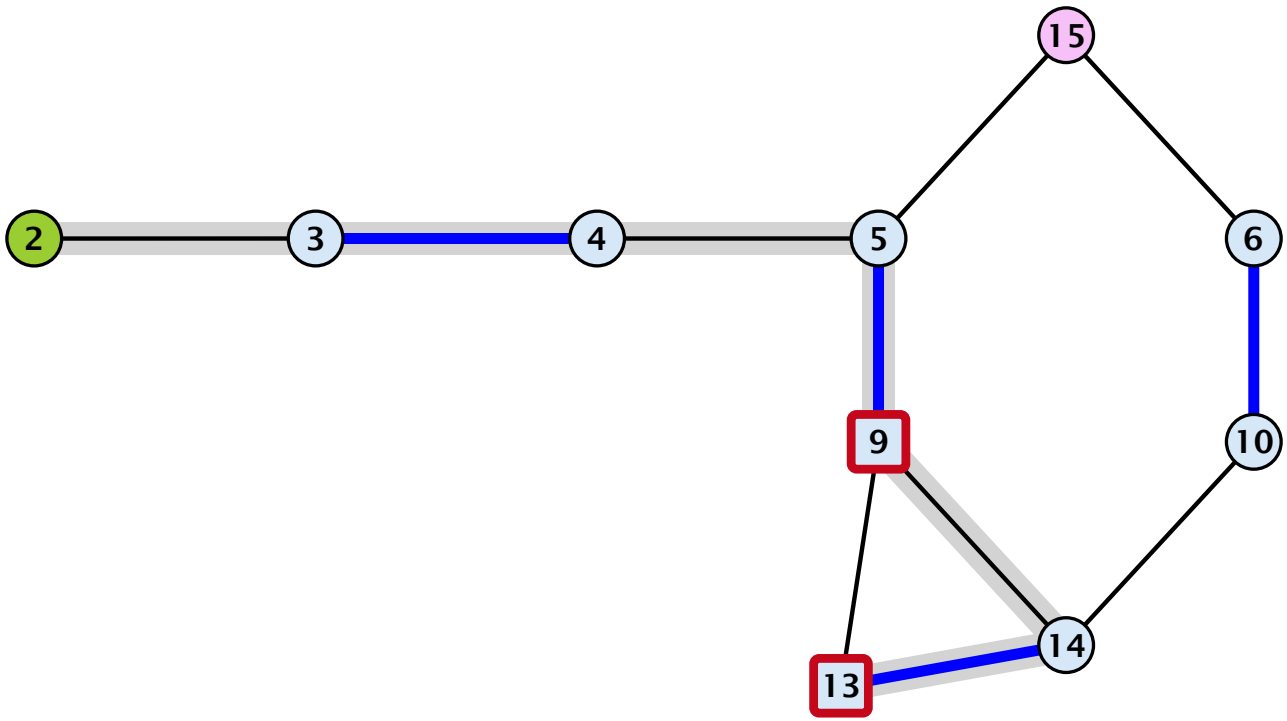

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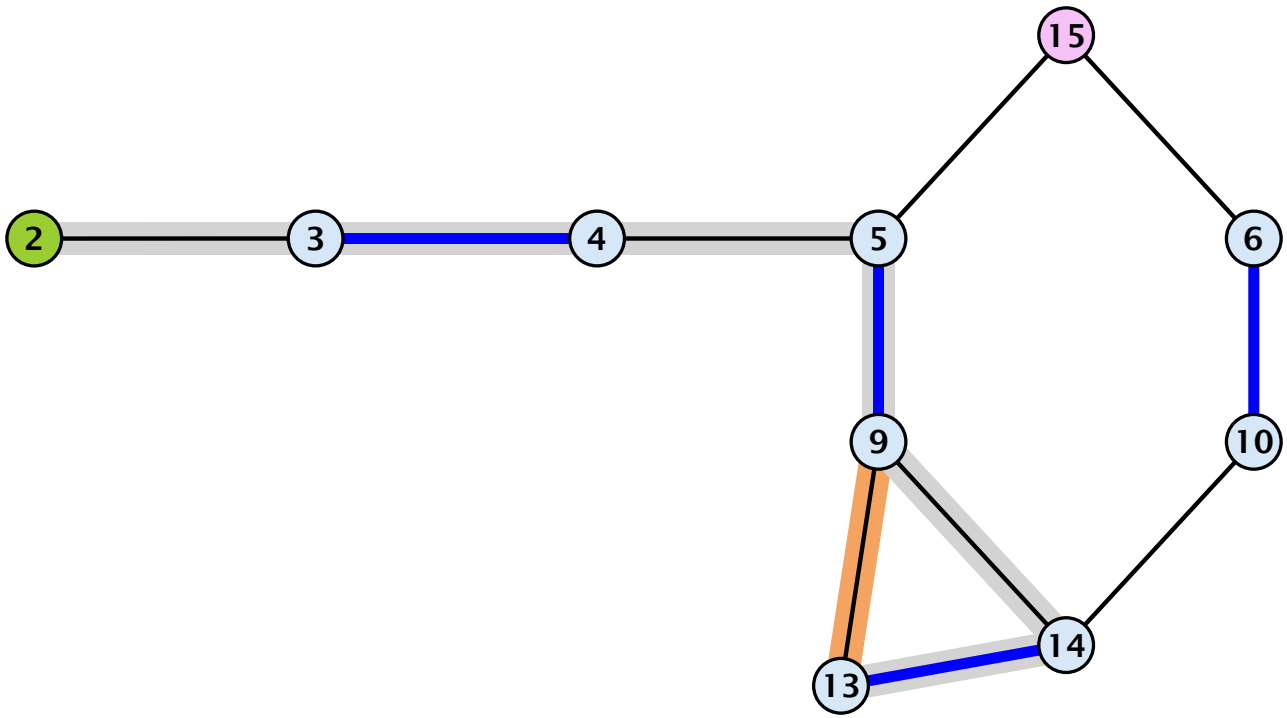
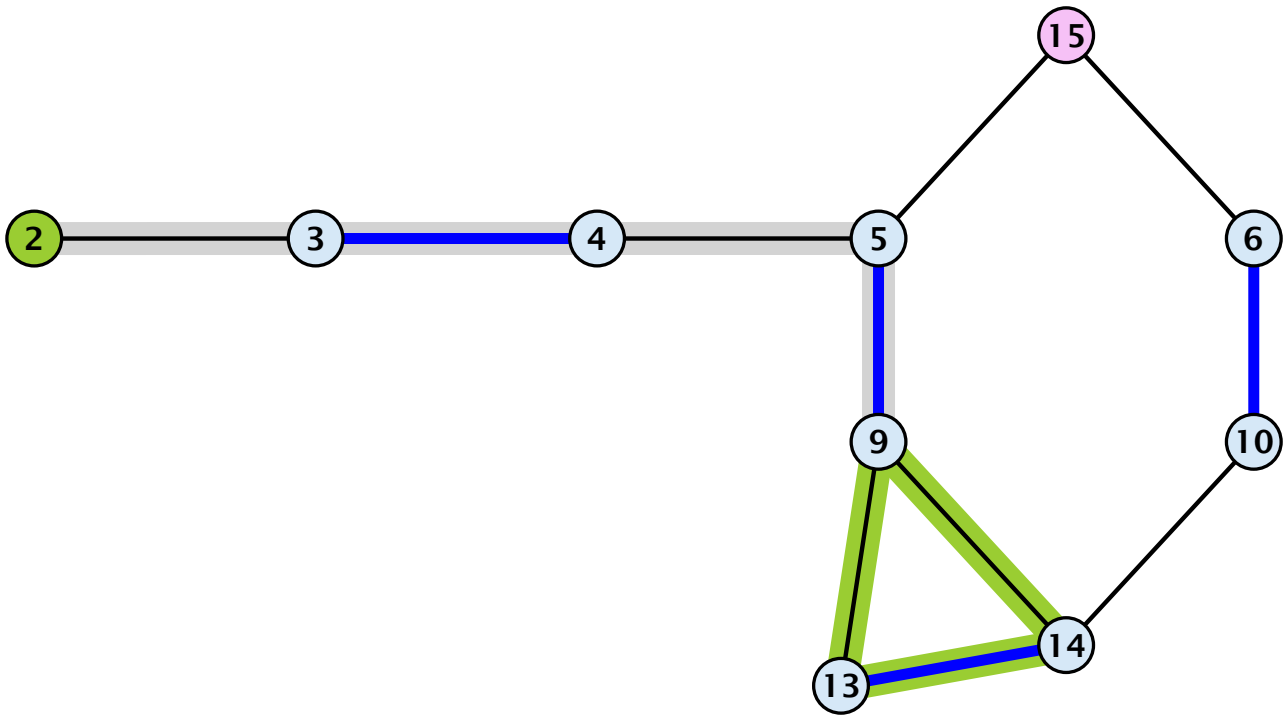
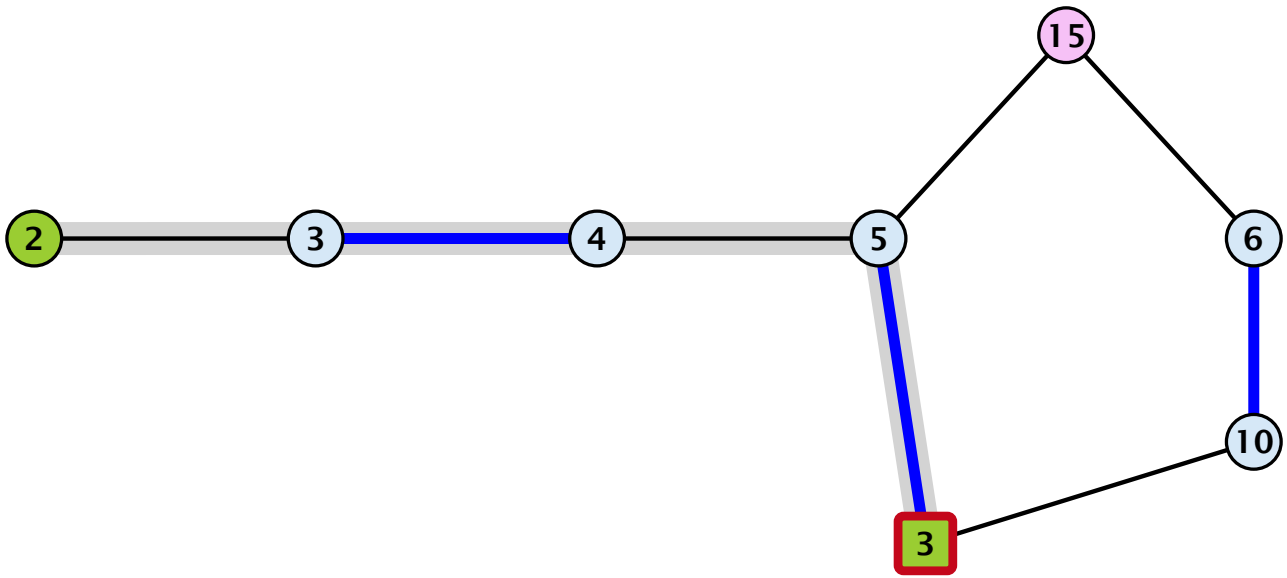


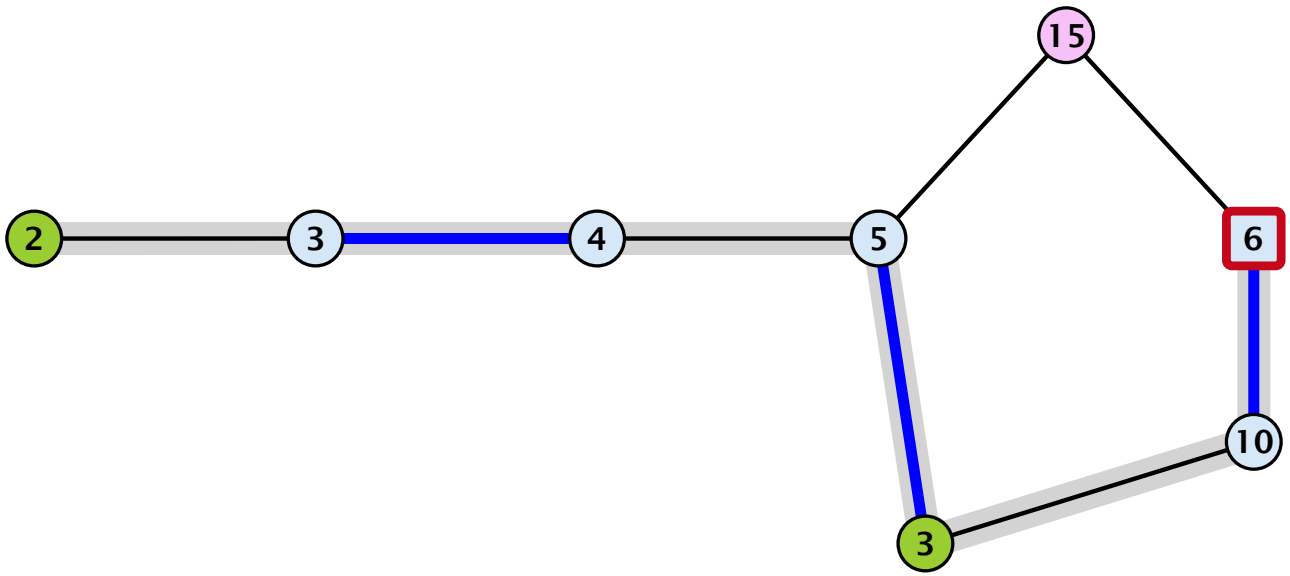
Example: Blossom Algorithm



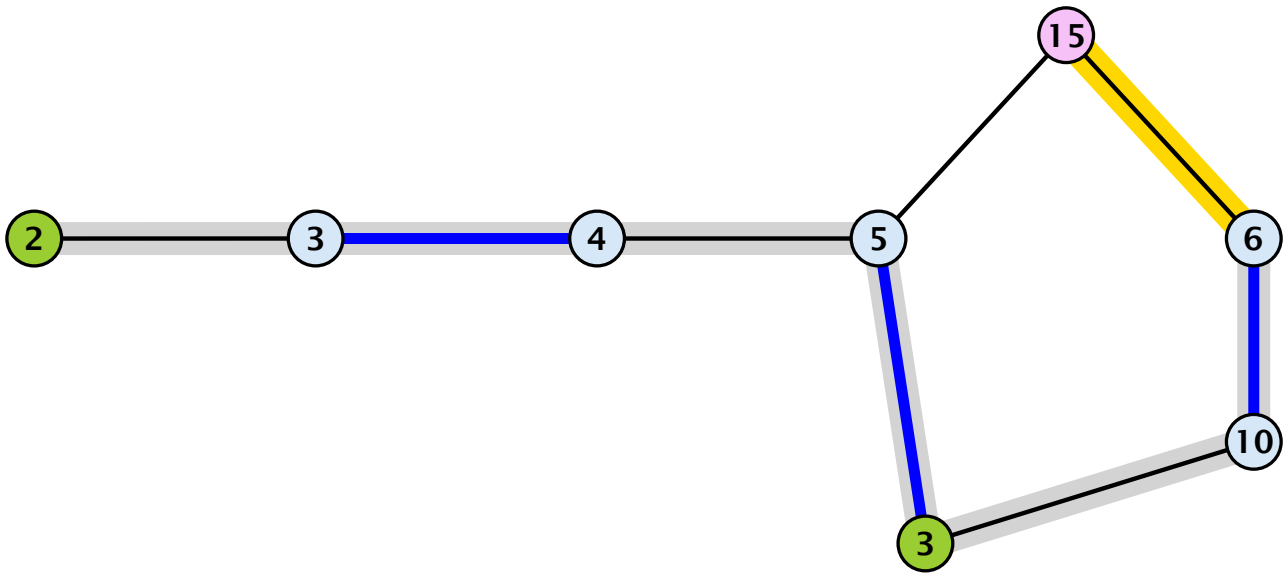
Example: Blossom Algorithm



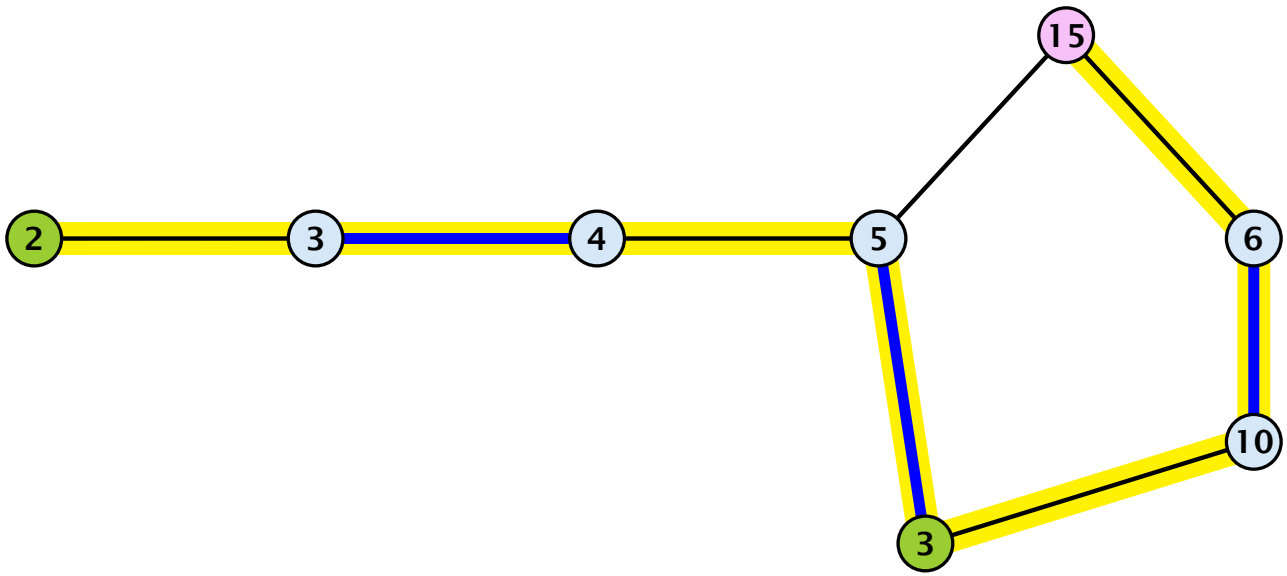
Example: Blossom Algorithm



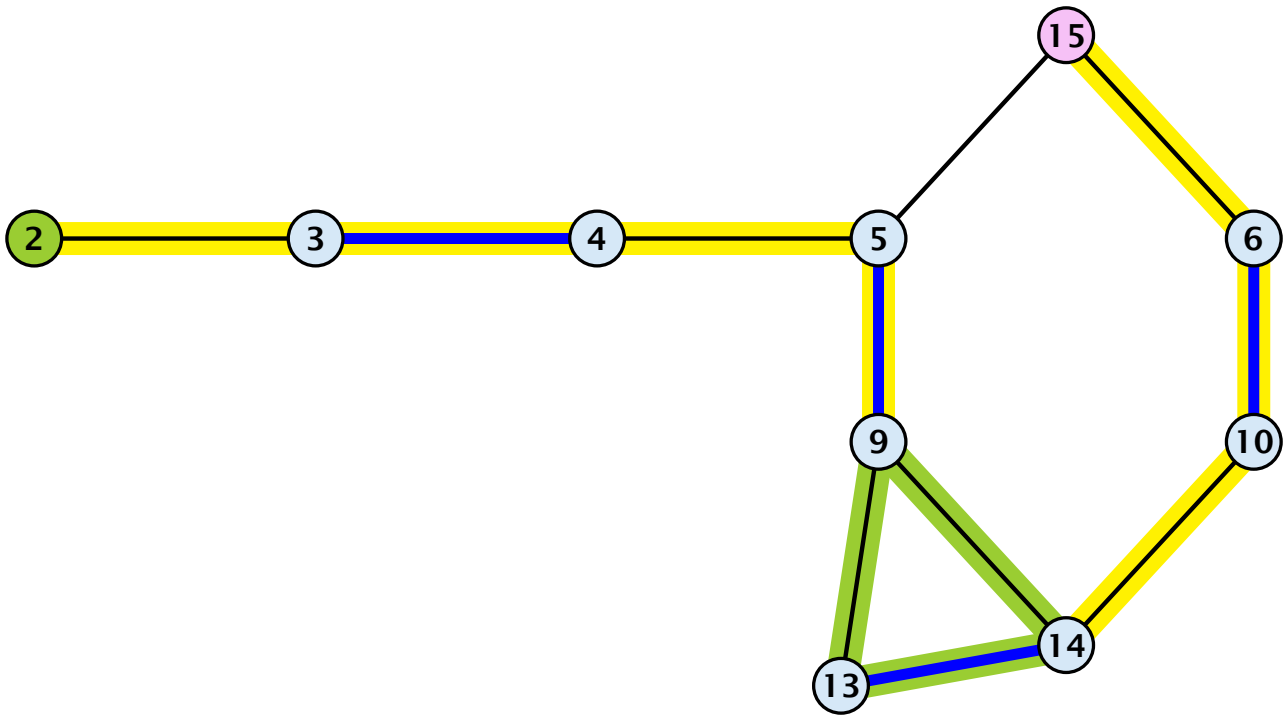
Example: Blossom Algorithm



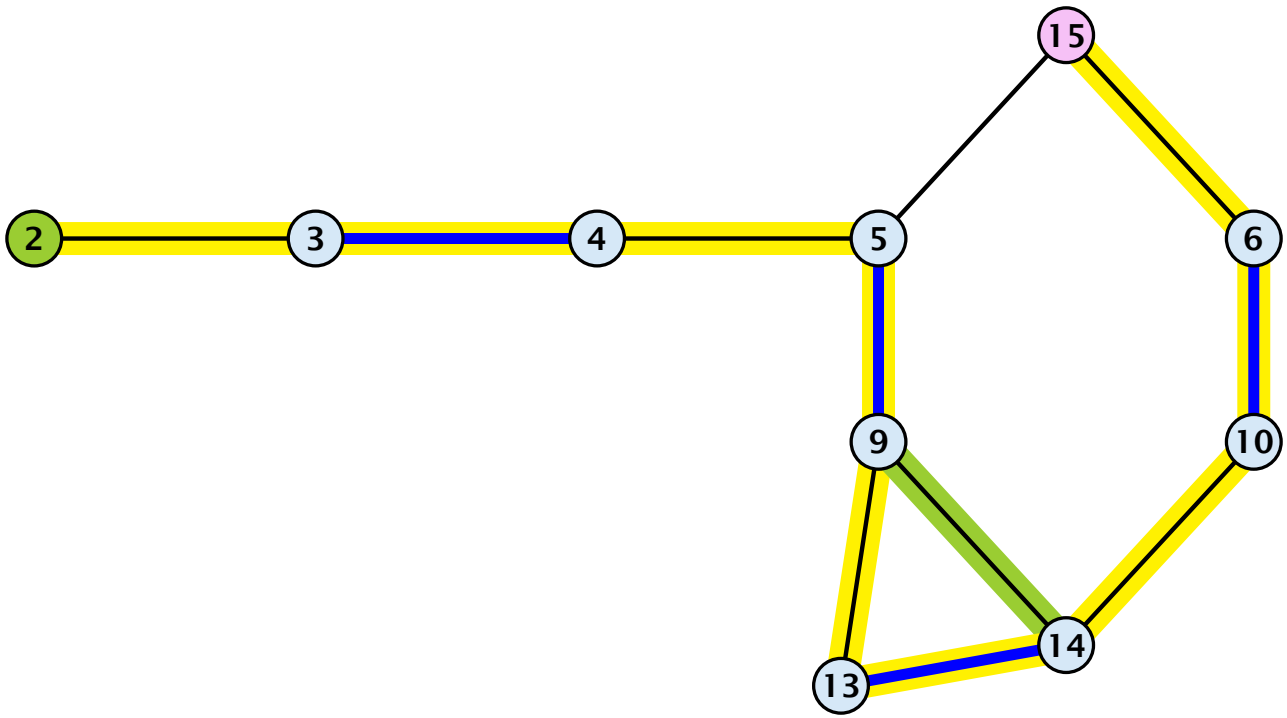
Example: Blossom Algorithm



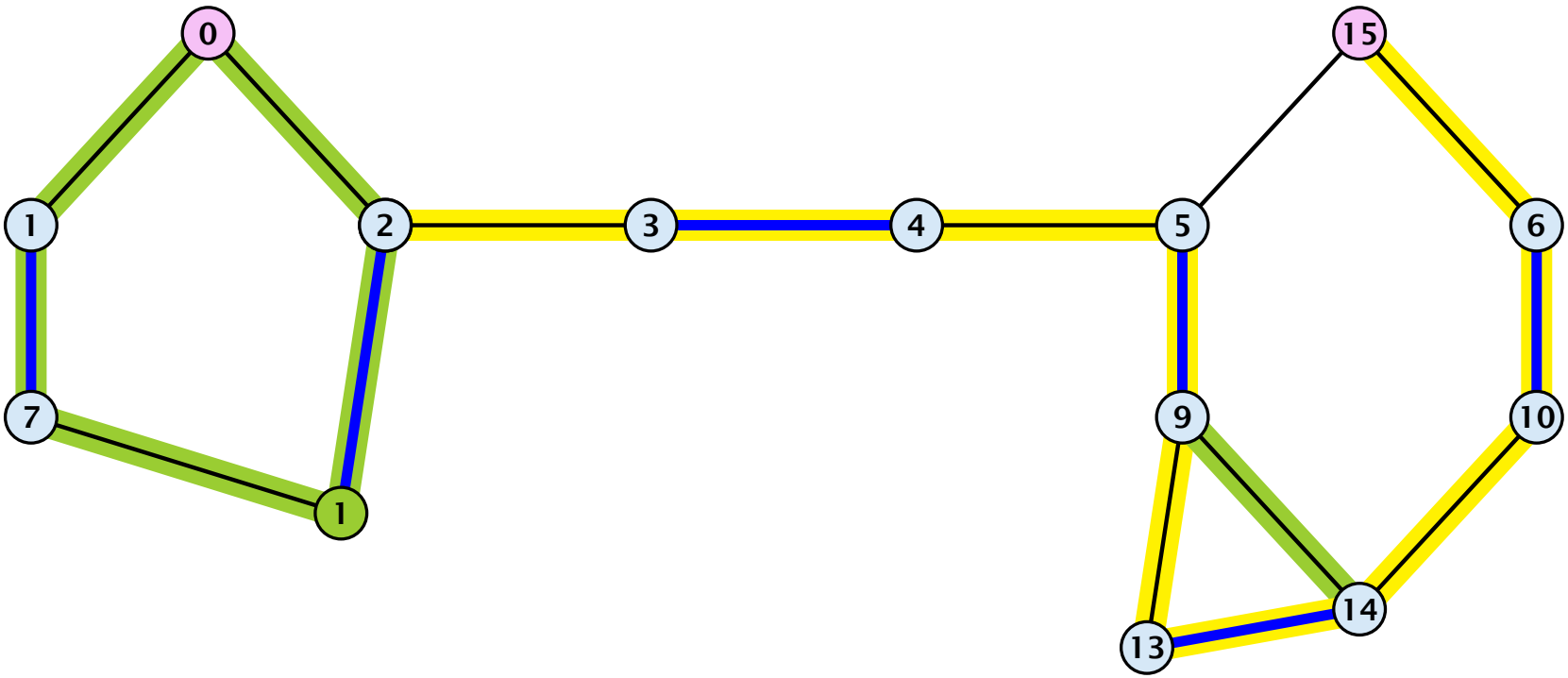
Example: Blossom Algorithm



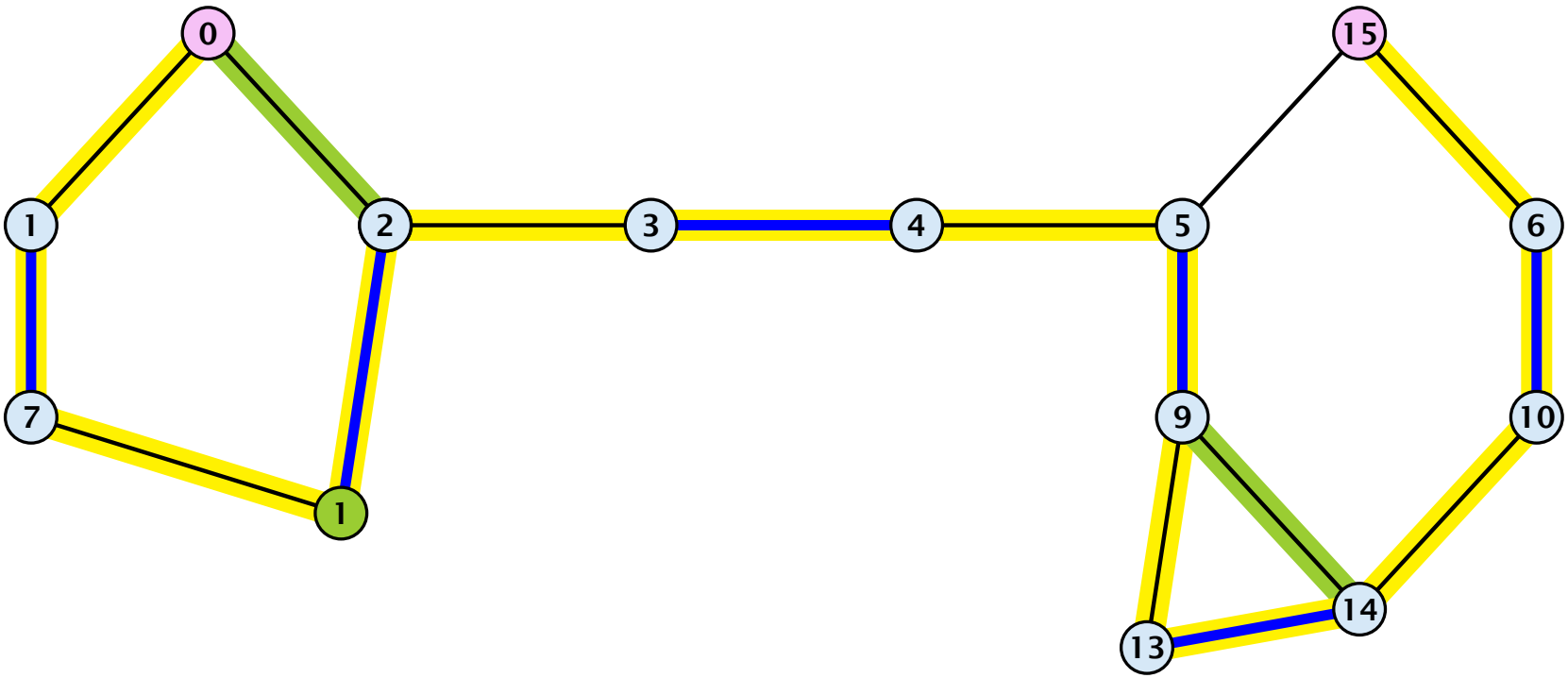
Example: Blossom Algorithm



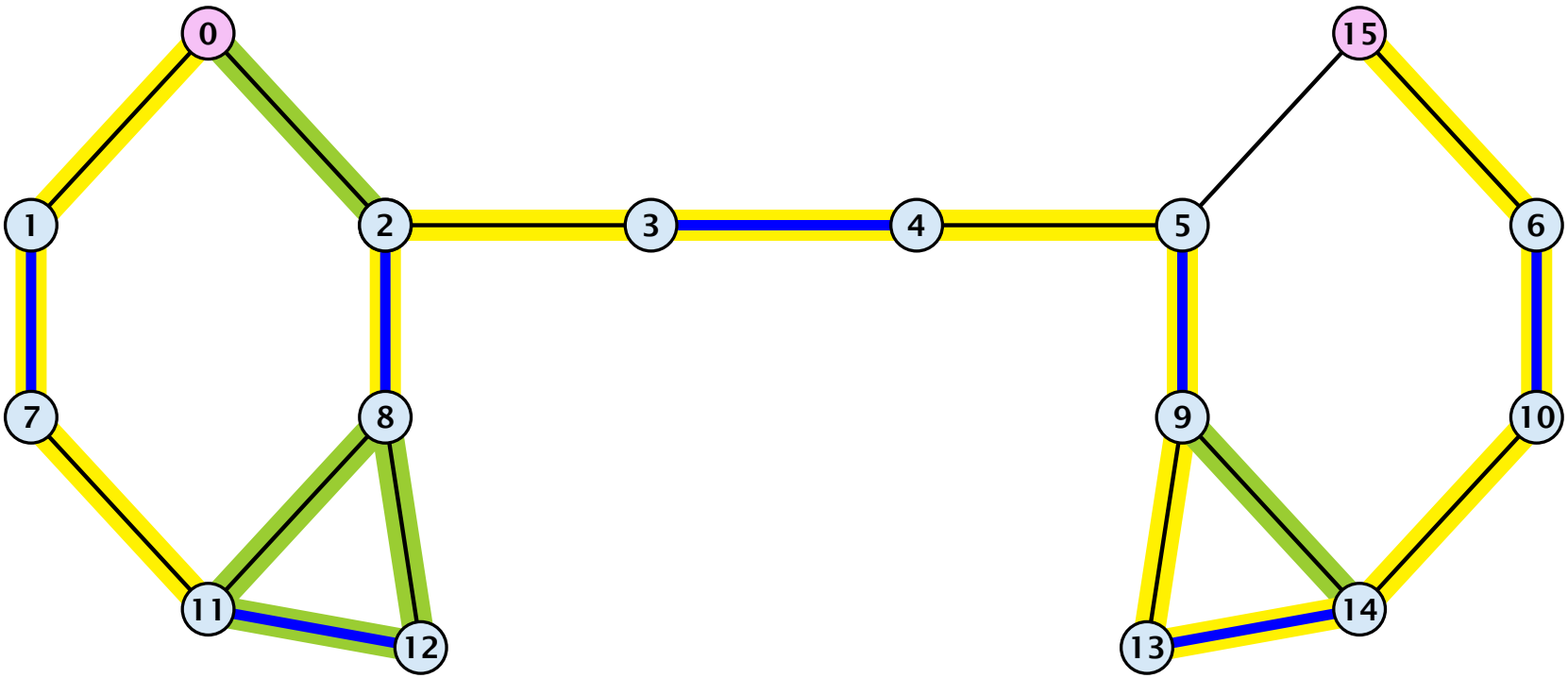
Example: Blossom Algorithm



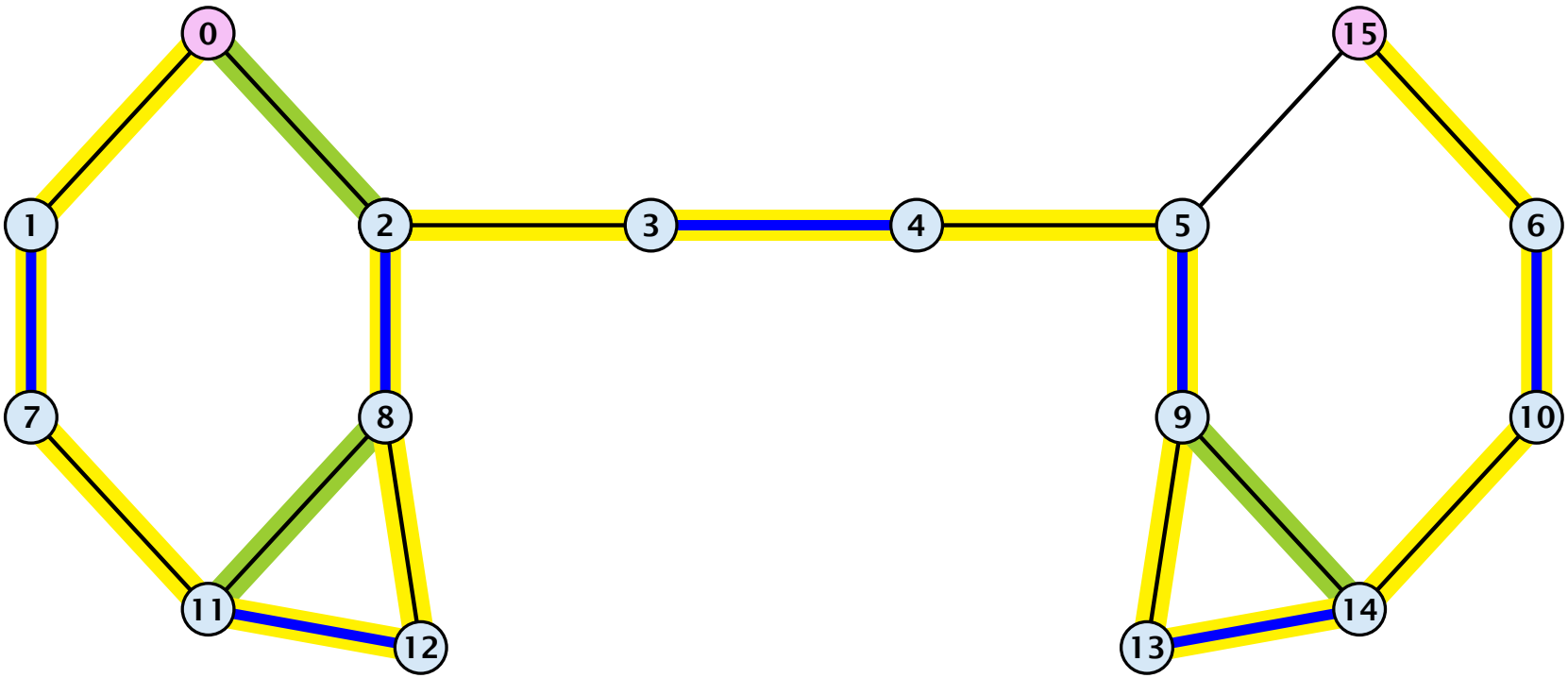
Example: Blossom Algorithm



Example: Blossom Algorithm



Example: Blossom Algorithm



Example: Blossom Algorithm