

Part II

Linear Programming

Brewery Problem

Brewery brews ale and beer.

- ▶ Production limited by supply of corn, hops and barley malt
- ▶ Recipes for ale and beer require different amounts of resources

	<i>Corn (kg)</i>	<i>Hops (kg)</i>	<i>Malt (kg)</i>	<i>Profit (€)</i>
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beer (barrel)	15	4	20	23
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How can brewer maximize profits?

- ▶ only brew ale: 34 barrels of ale $\Rightarrow 442 \text{ €}$
- ▶ only brew beer: 32 barrels of beer $\Rightarrow 736 \text{ €}$
- ▶ 7.5 barrels ale, 29.5 barrels beer $\Rightarrow 775 \text{ €}$
- ▶ 12 barrels ale, 28 barrels beer $\Rightarrow 800 \text{ €}$

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Brewery Problem

Linear Program

- Introduce variables a and b that define how much ale and stout to produce.
- Choose the variables in such a way that the profit is maximized.
- Make sure that no constraints (due to limited supply) are violated.

$$\begin{array}{ll}\max & 13a + 23b \\ \text{s.t.} & 5a + 15b \leq 480 \\ & 4a + 4b \leq 160 \\ & 35a + 20b \leq 1190 \\ & a, b \geq 0\end{array}$$

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Standard Form LPs

LP in standard form:

maximize $c^T x$
subject to
linear constraints, i.e. constraints
maximize linear objective function subject to linear
inequalities

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & Ax = b, \quad 1 \leq i \leq m \\ & x \geq 0, \quad 1 \leq j \leq n \end{aligned}$$

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LP in standard form:

- ▶ input: numbers a_{ij} , c_j , b_i
- ▶ output: numbers x_j
- ▶ n = #decision variables, m = #constraints
- ▶ maximize linear objective function subject to linear (in)equalities

$$\begin{aligned} \max \quad & \sum_{j=1}^n c_j x_j \\ \text{s.t.} \quad & \sum_{j=1}^n a_{ij} x_j = b_i \quad 1 \leq i \leq m \\ & x_j \geq 0 \quad 1 \leq j \leq n \end{aligned}$$

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$$\begin{aligned} \max \quad & c_1 x_1 + c_2 x_2 + \dots + c_n x_n \\ \text{s.t.} \quad & a_{11} x_1 + a_{12} x_2 + \dots + a_{1n} x_n = b_1 \\ & a_{21} x_1 + a_{22} x_2 + \dots + a_{2n} x_n = b_2 \\ & \vdots \\ & a_{m1} x_1 + a_{m2} x_2 + \dots + a_{mn} x_n = b_m \\ & x_1, x_2, \dots, x_n \geq 0 \end{aligned}$$

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Original LP

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Standard Form

Add a **slack variable** to every constraint.

$$\begin{array}{llllllll} \max & 13a & + & 23b & & & & \\ \text{s.t.} & 5a & + & 15b & + & s_c & & = 480 \\ & 4a & + & 4b & & & + & s_h = 160 \\ & 35a & + & 20b & & & + & s_m = 1190 \\ & a & , & b & , & s_c & , & s_h , s_m \geq 0 \end{array}$$

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Standard Form LPs

There are different standard forms:

standard form

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9. Maximize $c^T x$

10. Minimize $c^T x$

Standard Form LPs

It is easy to transform variants of LPs into (any) standard form:

- ▶ **less or equal to equality:**

$$a - 3b + 5c \leq 12 \Rightarrow \begin{aligned} a - 3b + 5c + s &= 12 \\ s &\geq 0 \end{aligned}$$

- ▶ greater or equal to equality:

$$\begin{aligned} a - 3b + 5c - s &= 12 \\ s &\geq 0 \end{aligned}$$

- ▶ min to max:

$$\min z \Rightarrow \max -z$$

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$$x \text{ unrestricted} \Rightarrow x = x^+ - x^-, x^+ \geq 0, x^- \geq 0$$

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Observations:

- ▶ a linear program does not contain x^2 , $\cos(x)$, etc.
- ▶ transformations between standard forms can be done efficiently and only change the size of the LP by a small constant factor
- ▶ for the standard minimization or maximization LPs we could include the nonnegativity constraints into the set of ordinary constraints; this is of course not possible for the standard form

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Fundamental Questions

Definition 1 (Linear Programming Problem (LP))

Let $A \in \mathbb{Q}^{m \times n}$, $b \in \mathbb{Q}^m$, $c \in \mathbb{Q}^n$, $\alpha \in \mathbb{Q}$. Does there exist $x \in \mathbb{Q}^n$
s.t. $Ax = b$, $x \geq 0$, $c^T x \geq \alpha$?

Questions:

1. Is LP in NP?

2. Is LP in co-NP?

3. Is LP in P?

Input size:

- ▶ n number of variables, m constraints, L number of bits to encode the input

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Let $A \in \mathbb{Q}^{m \times n}$, $b \in \mathbb{Q}^m$, $c \in \mathbb{Q}^n$, $\alpha \in \mathbb{Q}$. Does there exist $x \in \mathbb{Q}^n$
s.t. $Ax = b$, $x \geq 0$, $c^T x \geq \alpha$?

Questions:

- ▶ Is LP in NP?
- ▶ Is LP in co-NP?
- ▶ Is LP in P?

Input size:

- ▶ n number of variables, m constraints, L number of bits to encode the input

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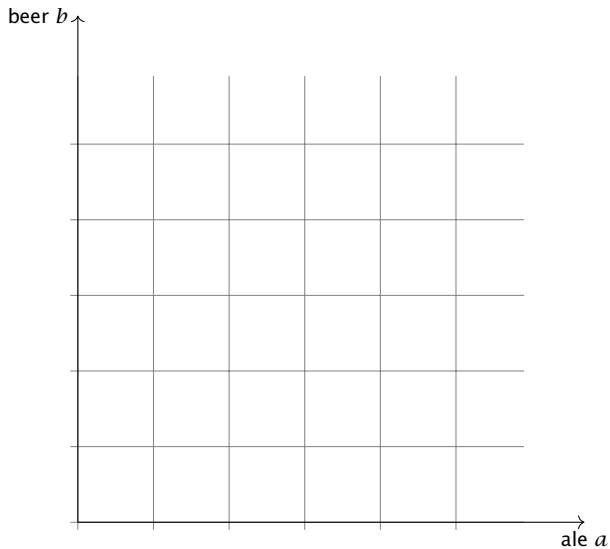
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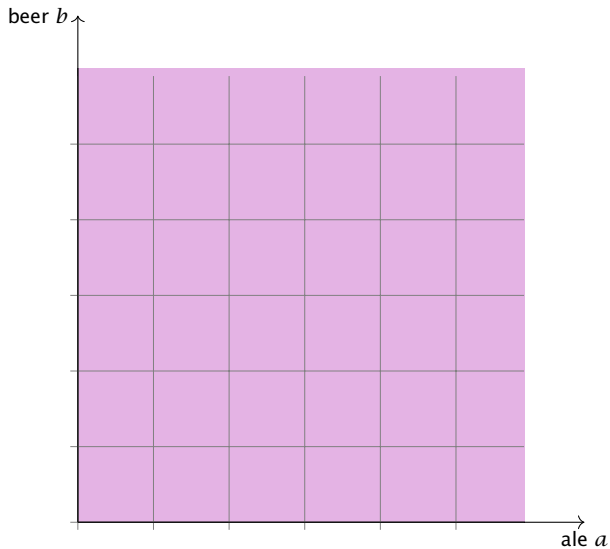
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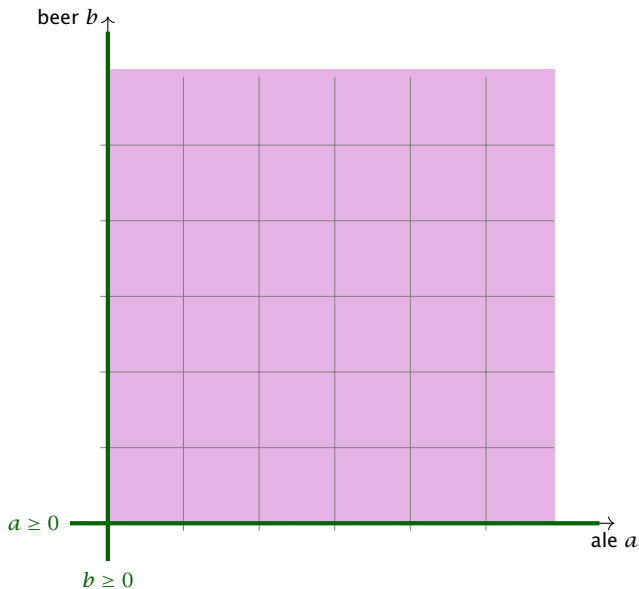
Geometry of Linear Programming



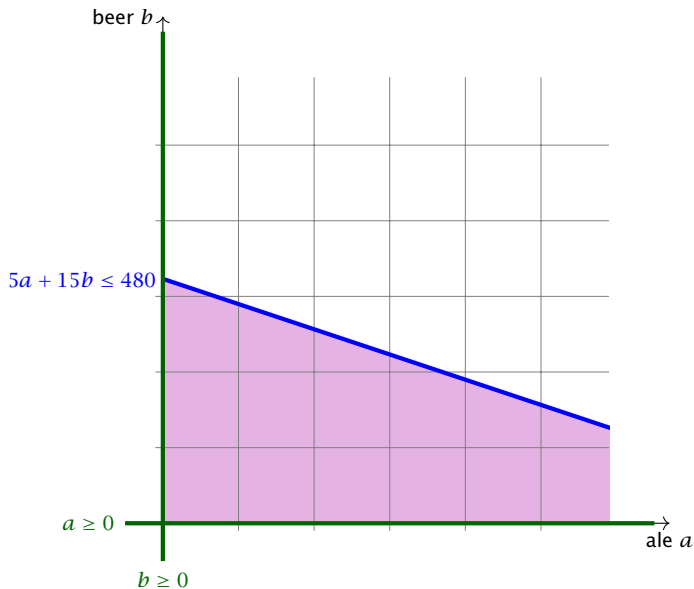
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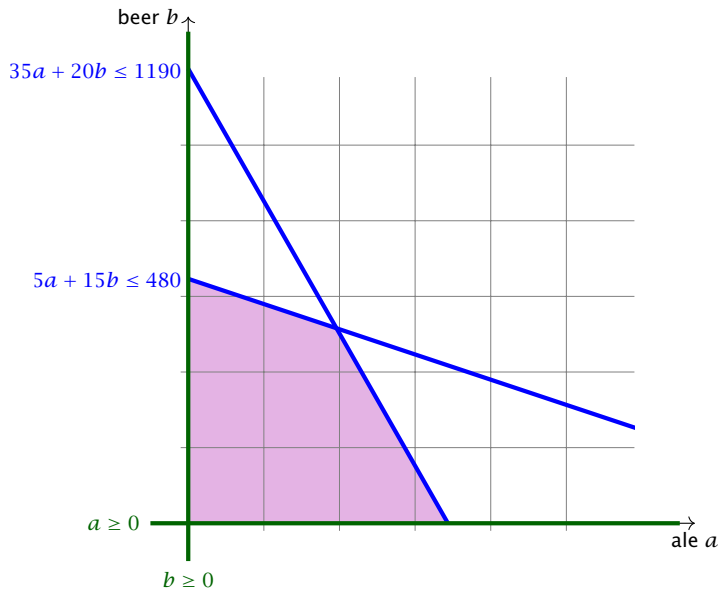
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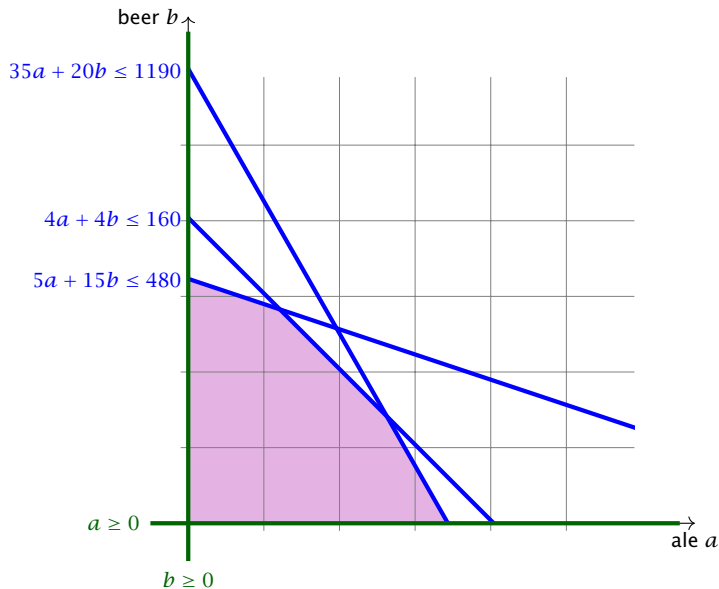
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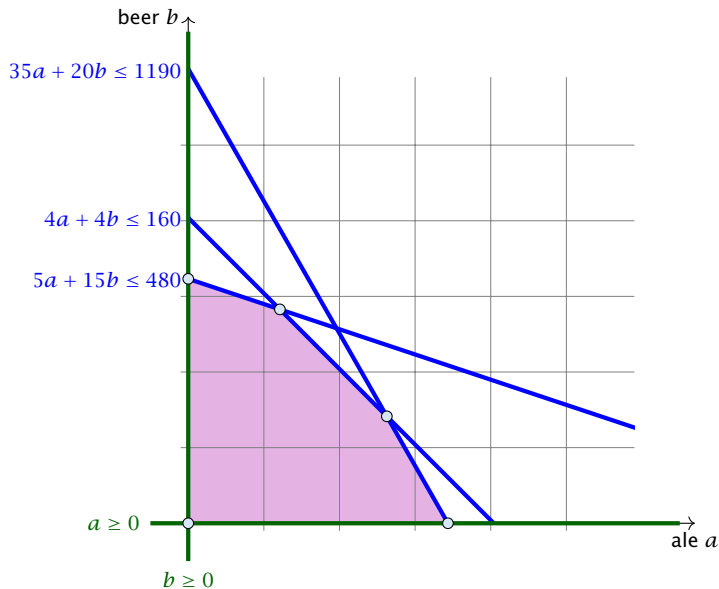
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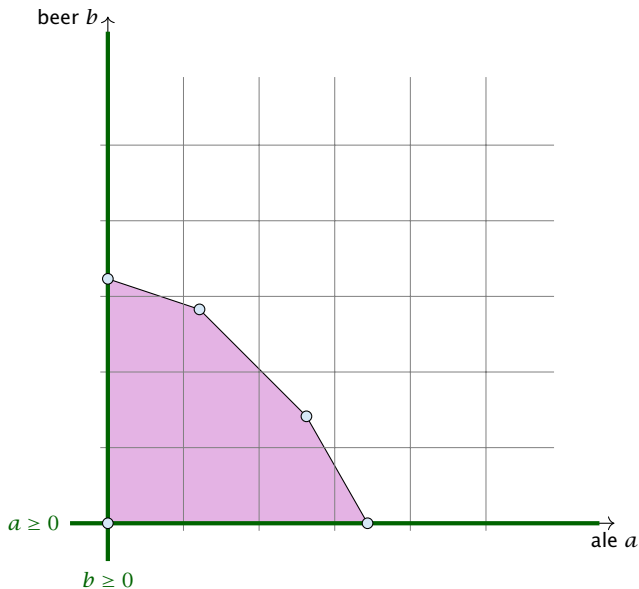
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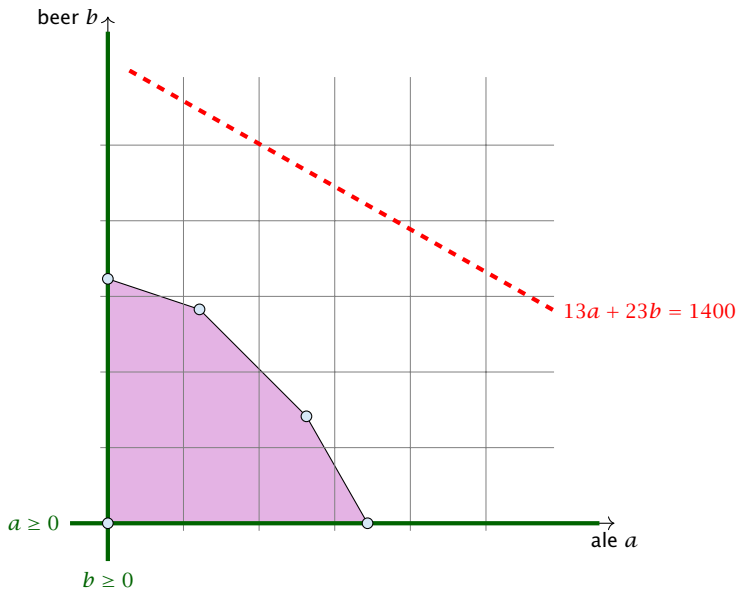
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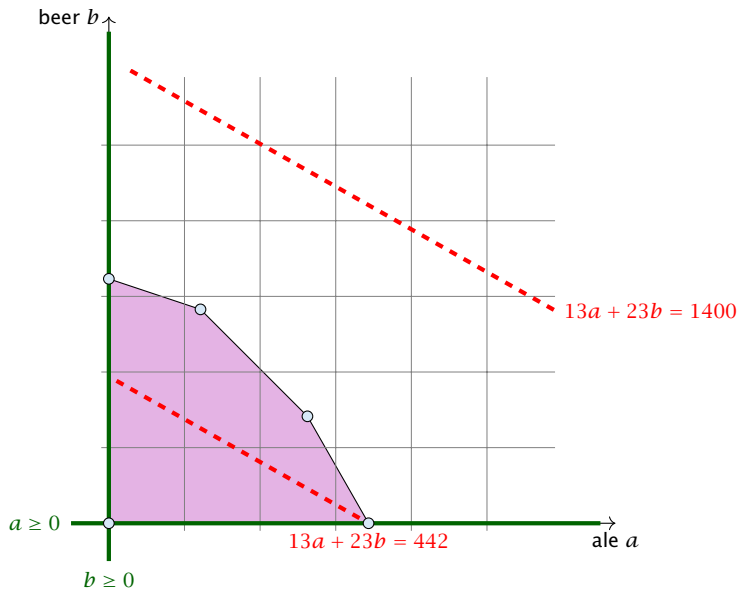
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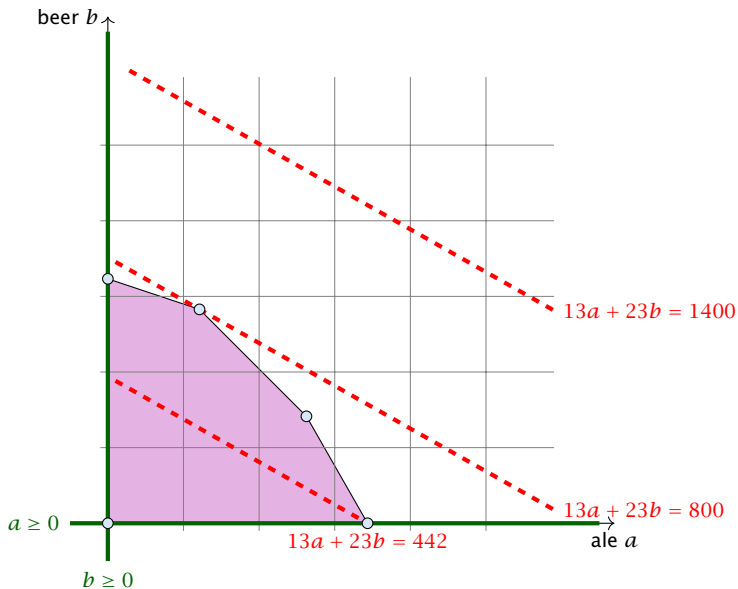
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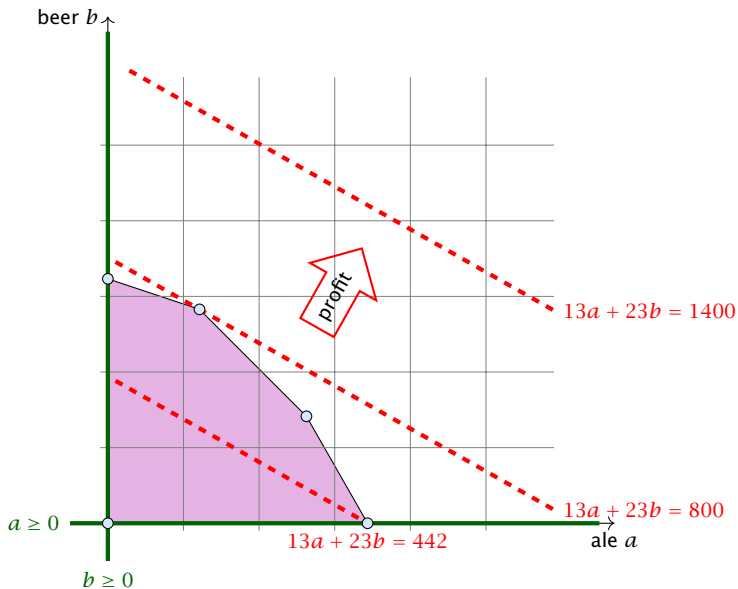
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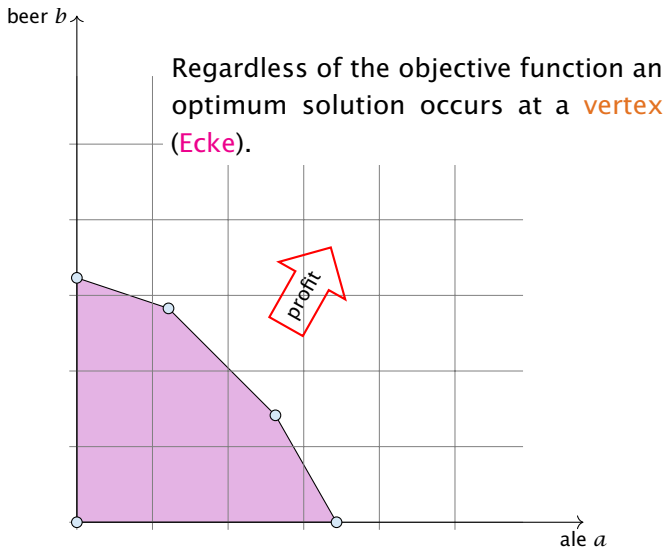
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- ▶ An LP is **bounded** (**beschränkt**) if it is feasible and
 $\exists M \in \mathbb{R}$ s.t. $x_i \leq M$ for all $i \in \{1, \dots, n\}$ (for maximization problems)
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Definition 2

Given vectors/points $x_1, \dots, x_k \in \mathbb{R}^n$, $\sum \lambda_i x_i$ is called

- ▶ **linear combination** if $\lambda_i \in \mathbb{R}$.
- ▶ **affine combination** if $\lambda_i \in \mathbb{R}$ and $\sum_i \lambda_i = 1$.
- ▶ **convex combination** if $\lambda_i \in \mathbb{R}$ and $\sum_i \lambda_i = 1$ and $\lambda_i \geq 0$.
- ▶ **conic combination** if $\lambda_i \in \mathbb{R}$ and $\lambda_i \geq 0$.

Note that a combination involves only finitely many vectors.

Definition 3

A set $X \subseteq \mathbb{R}^n$ is called

- ▶ a **linear subspace** if it is closed under linear combinations.
- ▶ an **affine subspace** if it is closed under affine combinations.
- ▶ **convex** if it is closed under convex combinations.
- ▶ a **convex cone** if it is closed under conic combinations.

Note that an affine subspace is **not** a vector space

Definition 4

Given a set $X \subseteq \mathbb{R}^n$.

- ▶ $\text{span}(X)$ is the set of all linear combinations of X
(linear hull, span)
- ▶ $\text{aff}(X)$ is the set of all affine combinations of X
(affine hull)
- ▶ $\text{conv}(X)$ is the set of all convex combinations of X
(convex hull)
- ▶ $\text{cone}(X)$ is the set of all conic combinations of X
(conic hull)

Definition 5

A function $f: \mathbb{R}^n \rightarrow \mathbb{R}$ is **convex** if for $x, y \in \mathbb{R}^n$ and $\lambda \in [0, 1]$ we have

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

Lemma 6

If $P \subseteq \mathbb{R}^n$, and $f: \mathbb{R}^n \rightarrow \mathbb{R}$ convex then also

$$Q = \{x \in P \mid f(x) \leq t\}$$

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Dimensions

Definition 7

The **dimension** $\dim(A)$ of an affine subspace $A \subseteq \mathbb{R}^n$ is the dimension of the vector space $\{x - a \mid x \in A\}$, where $a \in A$.

Definition 8

The **dimension** $\dim(X)$ of a convex set $X \subseteq \mathbb{R}^n$ is the dimension of its affine hull $\text{aff}(X)$.

Definition 9

A set $H \subseteq \mathbb{R}^n$ is a **hyperplane** if $H = \{x \mid a^T x = b\}$, for $a \neq 0$.

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A set $H' \subseteq \mathbb{R}^n$ is a (closed) **halfspace** if $H = \{x \mid a^T x \leq b\}$, for $a \neq 0$.

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Definition 11

A **polytop** is a set $P \subseteq \mathbb{R}^n$ that is the convex hull of a **finite** set of points, i.e., $P = \text{conv}(X)$ where $|X| = c$.

Definitions

Definition 12

A **polyhedron** is a set $P \subseteq \mathbb{R}^n$ that can be represented as the intersection of **finitely** many half-spaces $\{H(a_1, b_1), \dots, H(a_m, b_m)\}$, where

$$H(a_i, b_i) = \{x \in \mathbb{R}^n \mid a_i x \leq b_i\} \quad .$$

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A polyhedron P is **bounded** if there exists B s.t. $\|x\|_2 \leq B$ for all $x \in P$.

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Theorem 14

P is a bounded polyhedron iff P is a polytop.

Definition 15

Let $P \subseteq \mathbb{R}^n$, $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$. The hyperplane

$$H(a, b) = \{x \in \mathbb{R}^n \mid a^T x = b\}$$

is a **supporting hyperplane** of P if $\max\{a^T x \mid x \in P\} = b$.

Definition 16

Let $P \subseteq \mathbb{R}^n$. F is a **face** of P if $F = P$ or $F = P \cap H$ for some supporting hyperplane H .

Definition 17

Let $P \subseteq \mathbb{R}^n$.

- ▶ a face v is a **vertex** of P if $\{v\}$ is a face of P .
- ▶ a face e is an **edge** of P if e is a face and $\dim(e) = 1$.
- ▶ a face F is a **facet** of P if F is a face and $\dim(F) = \dim(P) - 1$.

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Equivalent definition for vertex:

Definition 18

Given polyhedron P . A point $x \in P$ is a **vertex** if $\exists c \in \mathbb{R}^n$ such that $c^T y < c^T x$, for all $y \in P$, $y \neq x$.

Definition 19

Given polyhedron P . A point $x \in P$ is an **extreme point** if $\nexists a, b \neq x$, $a, b \in P$, with $\lambda a + (1 - \lambda)b = x$ for $\lambda \in [0, 1]$.

Lemma 20

A vertex is also an extreme point.

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Observation

The feasible region of an LP is a Polyhedron.

Theorem 21

If there exists an optimal solution to an LP (in standard form) then there exists an optimum solution that is an extreme point.

Proof

Let x^* be an optimal solution to an LP (in standard form). If x^* is an extreme point, we are done. If not, then x^* can be written as a convex combination of two distinct feasible points x^1 and x^2 . Since x^* is optimal, the objective value at x^* is at least as large as the objective value at x^1 and x^2 . But since x^* is a convex combination of x^1 and x^2 , the objective value at x^* is a weighted average of the objective values at x^1 and x^2 . This implies that the objective value at x^1 and x^2 must both be equal to the objective value at x^* . Therefore, x^1 and x^2 are also optimal solutions. If x^1 and x^2 are not extreme points, we can repeat the argument. This process must terminate at an extreme point, which is an optimum solution.

Theorem 21

If there exists an optimal solution to an LP (in standard form) then there exists an optimum solution that is an extreme point.

Proof

- ▶ suppose x is optimal solution that is not extreme point
- ▶ there exists direction $d \neq 0$ such that $x \pm d \in P$
- ▶ $Ad = 0$ because $A(x \pm d) = b$
- ▶ Wlog. assume $c^T d \geq 0$ (by taking either d or $-d$)
- ▶ Consider $x + \lambda d, \lambda > 0$

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Convex Sets

Case 1. $[\exists j \text{ s.t. } d_j < 0]$

• Increase λ to λ^* until first component of $d + \lambda c$ hits 0.
• $d + \lambda^* c$ is feasible. Since $-c^T d = \lambda^* c^T d$ and $c^T d > 0$, $\lambda^* > 0$.
• $d + \lambda^* c$ has the more zero-component $\lambda^* c > 0$ for $c_j > 0$ (if $d_j > 0$).
• (Another way: $(d + \lambda^* c)_j = 0$ for $c_j > 0$ and $d_j > 0$.)

Case 2. $[d_j \geq 0 \text{ for all } j \text{ and } c^T d > 0]$

• $d + \lambda c$ is feasible for all $\lambda \geq 0$ since $d_j \geq 0$ and $c_j \geq 0$.
• $c^T d + \lambda c^T c > 0$ for all $\lambda \geq 0$.
• As $\lambda \rightarrow \infty$, $c^T d + \lambda c^T c \rightarrow \infty$ as $c^T c > 0$.

Case 1. $[\exists j \text{ s.t. } d_j < 0]$

Choose $\lambda = 0$ until first component of d is < 0 . This λ is feasible. Since $\lambda = 0$ and $d_j < 0$, λ has the more zero-component $\lambda_j = 0$ for case 1. (i.e. $d_j < 0$)

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Case 2. $[d_j \geq 0 \text{ for all } j \text{ and } c^T d > 0]$

$\lambda = 0$ is feasible for all $\lambda \geq 0$ since $d_j \geq 0$ and $c^T d > 0$. Choose $\lambda = 0$ until first component of d is < 0 . This λ is feasible. Since $\lambda = 0$ and $d_j \geq 0$, λ has the more zero-component $\lambda_j = 0$ for case 2. (i.e. $d_j \geq 0$)

Case 1. [$\exists j$ s.t. $d_j < 0$]

- ▶ increase λ to λ' until first component of $x + \lambda d$ hits 0
- ▶ $x + \lambda' d$ is feasible. Since $A(x + \lambda' d) = b$ and $x + \lambda' d \geq 0$
- ▶ $x + \lambda' d$ has one more zero-component ($d_k = 0$ for $x_k = 0$ as $x \pm d \in P$)
- ▶ $c^T x' = c^T (x + \lambda' d) = c^T x + \lambda' c^T d \geq c^T x$

Case 2. [$d_j \geq 0$ for all j and $c^T d > 0$]

- ▶ $x + \lambda d$ is feasible for all $\lambda \geq 0$ since $A(x + \lambda d) = b$ and $x + \lambda d \geq 0$
- ▶ $c^T x + \lambda c^T d > c^T x$ for all $\lambda > 0$ since $c^T d > 0$
- ▶ x is not optimal

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Case 2. [$d_j \geq 0$ for all j and $c^T d > 0$]

- ▶ $x + \lambda d$ is feasible for all $\lambda \geq 0$ since $x + \lambda d \geq 0$ and $A(x + \lambda d) = b$
- ▶ $c^T x + \lambda c^T d > c^T x$ for all $\lambda > 0$
- ▶ P is unbounded

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- ▶ $x + \lambda d$ is feasible for all $\lambda \geq 0$ since $x + \lambda d \geq 0$ and $A(x + \lambda d) = b$
- ▶ $c^T x + \lambda c^T d > c^T x$ for all $\lambda > 0$
- ▶ P is unbounded

Case 1. $[\exists j \text{ s.t. } d_j < 0]$

- ▶ increase λ to λ' until first component of $x + \lambda d$ hits 0
- ▶ $x + \lambda' d$ is feasible. Since $A(x + \lambda' d) = b$ and $x + \lambda' d \geq 0$
- ▶ $x + \lambda' d$ has one more zero-component ($d_k = 0$ for $x_k = 0$ as $x \pm d \in P$)
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Convex Sets

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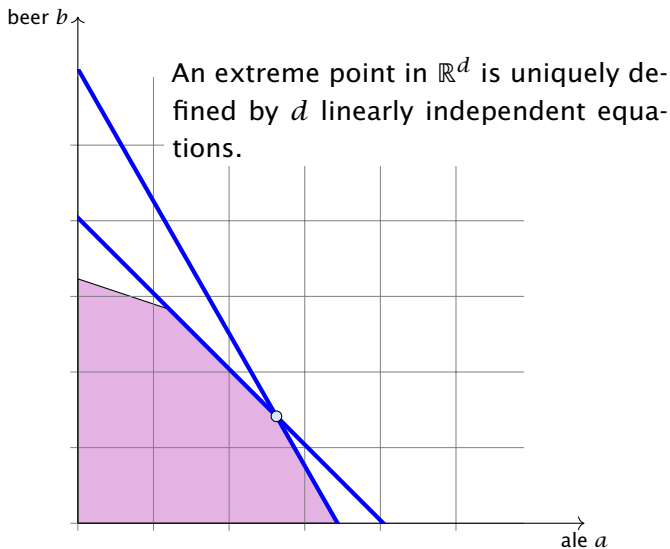
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Algebraic View



Notation

Suppose $B \subseteq \{1 \dots n\}$ is a set of column-indices. Define A_B as the subset of columns of A indexed by B .

Theorem 22

*Let $P = \{x \mid Ax = b, x \geq 0\}$. For $x \in P$, define $B = \{j \mid x_j > 0\}$.
Then x is extreme point iff A_B has linearly independent columns.*

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- ▶ $A_{B'}$ has linearly dependent columns as $Ad = 0$
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- ▶ Hence, $B' \subseteq B$, $A_{B'}$ is sub-matrix of A_B

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Let $P = \{x \mid Ax = b, x \geq 0\}$. For $x \in P$, define $B = \{j \mid x_j > 0\}$. If A_B has linearly independent columns then x is a vertex of P .

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- ▶ then $c^T x = 0$ and $c^T y \leq 0$ for $y \in P$
- ▶ assume $c^T y = 0$; then $y_j = 0$ for all $j \notin B$
- ▶ $b = Ay = A_B y_B = Ax = A_B x_B$ gives that $A_B(x_B - y_B) = 0$;
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C1 if now $b_1 = \sum_{i=2}^m \lambda_i \cdot b_i$ then for all x with $A_2 x \leq b_2, \dots, A_m x \leq b_m$ we also have $A_1 x \leq b_1$, hence the first constraint is superfluous

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- C2** if $b_1 \neq \sum_{i=2}^m \lambda_i \cdot b_i$ then the LP is infeasible, since for all x that fulfill constraints $A_2, \dots, A_m$ we have

$$A_1 x = \sum_{i=2}^m \lambda_i \cdot A_i x = \sum_{i=2}^m \lambda_i \cdot b_i \neq b_1$$

Observation

For an LP we can assume wlog. that the matrix A has full row-rank. This means $\text{rank}(A) = m$.

- ▶ assume that $\text{rank}(A) < m$
- ▶ assume wlog. that the first row A_1 lies in the span of the other rows $A_2, \dots, A_m$; this means

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From now on we will always assume that the constraint matrix of a standard form LP has full row rank.

Theorem 24

Given $P = \{x \mid Ax = b, x \geq 0\}$. x is extreme point iff there exists $B \subseteq \{1, \dots, n\}$ with $|B| = m$ and

- ▶ A_B is non-singular
- ▶ $x_B = A_B^{-1}b \geq 0$
- ▶ $x_N = 0$

where $N = \{1, \dots, n\} \setminus B$.

Proof

Take $B = \{j \mid x_j > 0\}$ and augment with linearly independent columns until $|B| = m$; always possible since $\text{rank}(A) = m$.

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Basic Feasible Solutions

$x \in \mathbb{R}^n$ is called **basic solution** (Basislösung) if $Ax = b$ and $\text{rank}(A_J) = |J|$ where $J = \{j \mid x_j \neq 0\}$;

x is a **basic feasible solution** (gültige Basislösung) if in addition $x \geq 0$.

A **basis** (Basis) is an index set $B \subseteq \{1, \dots, n\}$ with $\text{rank}(A_B) = m$ and $|B| = m$.

$x \in \mathbb{R}^n$ with $A_B x_B = b$ and $x_j = 0$ for all $j \notin B$ is **the basic solution associated to basis B** (die zu B assoziierte Basislösung)

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Basic Feasible Solutions

A BFS fulfills the m equality constraints.

In addition, at least $n - m$ of the x_i 's are zero. The corresponding non-negativity constraint is fulfilled with equality.

Fact:

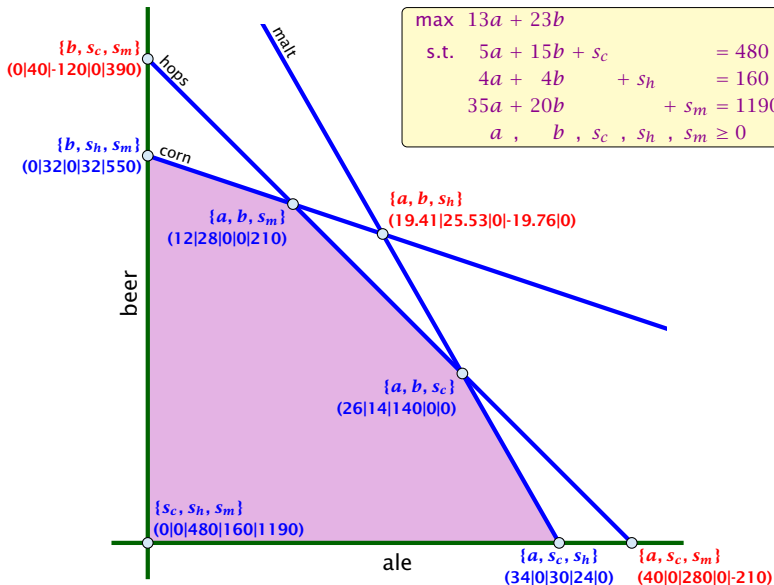
In a BFS at least n constraints are fulfilled with equality.

Basic Feasible Solutions

Definition 25

For a general LP ($\max\{c^T x \mid Ax \leq b\}$) with n variables a point x is a **basic feasible solution** if x is feasible and there exist n (linearly independent) constraints that are tight.

Algebraic View



Fundamental Questions

Linear Programming Problem (LP)

Let $A \in \mathbb{Q}^{m \times n}$, $b \in \mathbb{Q}^m$, $c \in \mathbb{Q}^n$, $\alpha \in \mathbb{Q}$. Does there exist $x \in \mathbb{Q}^n$
s.t. $Ax = b$, $x \geq 0$, $c^T x \geq \alpha$?

Questions:

- ▶ Is LP in NP? yes!
- ▶ Is LP in co-NP?
- ▶ Is LP in P?

Proof:

- ▶ Given a basis B we can compute the associated basis solution by calculating $A_B^{-1}b$ in polynomial time; then we can also compute the profit.

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Observation

We can compute an optimal solution to a linear program in time $\mathcal{O}\left(\binom{n}{m} \cdot \text{poly}(n, m)\right)$.

- ▶ there are only $\binom{n}{m}$ different bases.
- ▶ compute the profit of each of them and take the maximum

What happens if LP is unbounded?

4 Simplex Algorithm

Enumerating all basic feasible solutions (BFS), in order to find the optimum is slow.

Simplex Algorithm [George Dantzig 1947]

Move from BFS to adjacent BFS, without decreasing objective function.

Two BFSs are called adjacent if the bases just differ in one variable.

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4 Simplex Algorithm

$$\max 13a + 23b$$

$$\text{s.t. } 5a + 15b + s_c = 480$$

$$4a + 4b + s_h = 160$$

$$35a + 20b + s_m = 1190$$

$$a, b, s_c, s_h, s_m \geq 0$$

$$\max Z$$

$$13a + 23b - Z = 0$$

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Pivoting Step

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- ▶ choose variable to bring into the basis
 - ▶ chosen variable should have positive coefficient in objective function
 - ▶ apply min-ratio test to find out by how much the variable can be increased
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 - ▶ the existing basis variable in this row leaves the basis

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- Choose variable with coefficient > 0 as entering variable.

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- ▶ If we keep $a = 0$ and increase b from 0 to $\theta > 0$ s.t. all constraints ($Ax = b, x \geq 0$) are still fulfilled the objective value Z will strictly increase.

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- ▶ The basic variable in the row that gives $\min\{480/15, 160/4, 1190/20\}$ becomes the **leaving variable**.

max Z

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Substitute $b = \frac{1}{15}(480 - 5a - s_c)$.

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max Z

$$\frac{16}{3}a - \frac{23}{15}s_c - Z = -736$$

$$\frac{1}{3}a + b + \frac{1}{15}s_c = 32$$

$$\frac{8}{3}a - \frac{4}{15}s_c + s_h = 32$$

$$\frac{85}{3}a - \frac{4}{3}s_c + s_m = 550$$

$$a, b, s_c, s_h, s_m \geq 0$$

basis = $\{b, s_h, s_m\}$

$$a = s_c = 0$$

$$Z = 736$$

$$b = 32$$

$$s_h = 32$$

$$s_m = 550$$

max Z

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Choose variable a to bring into basis.

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Choose variable a to bring into basis.

Computing $\min\{3 \cdot 32, 3 \cdot 32/8, 3 \cdot 550/85\}$ means pivot on line 2.

max Z

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Substitute $a = \frac{3}{8}(32 + \frac{4}{15}s_c - s_h)$.

max Z

$$\frac{16}{3}a - \frac{23}{15}s_c - Z = -736$$

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Choose variable a to bring into basis.

Computing $\min\{3 \cdot 32, 3 \cdot 32/8, 3 \cdot 550/85\}$ means pivot on line 2.

Substitute $a = \frac{3}{8}(32 + \frac{4}{15}s_c - s_h)$.

max Z

$$-s_c - 2s_h - Z = -800$$

$$b + \frac{1}{10}s_c - \frac{1}{8}s_h = 28$$

$$a - \frac{1}{10}s_c + \frac{3}{8}s_h = 12$$

$$\frac{3}{2}s_c - \frac{85}{8}s_h + s_m = 210$$

$$a, b, s_c, s_h, s_m \geq 0$$

basis = $\{a, b, s_m\}$

$$s_c = s_h = 0$$

$$Z = 800$$

$$b = 28$$

$$a = 12$$

$$s_m = 210$$

4 Simplex Algorithm

Pivoting stops when all coefficients in the objective function are non-positive.

Solution is optimal:

- If any feasible solution satisfies all equations in the tableau, the current solution is optimal.
- If no feasible solution exists, the problem is infeasible.
- If no optimum solution value z , at most one feasible solution has a better value than the current solution has value.

4 Simplex Algorithm

Pivoting stops when all coefficients in the objective function are non-positive.

Solution is optimal:

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- ▶ any feasible solution satisfies all equations in the tableaux
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Matrix View

Let our linear program be

$$\begin{aligned}c_B^T x_B + c_N^T x_N &= Z \\ A_B x_B + A_N x_N &= b \\ x_B, x_N &\geq 0\end{aligned}$$

The simplex tableaux for basis B is

$$\begin{aligned}I x_B + (c_N^T - c_B^T A_B^{-1} A_N) x_N &= Z - c_B^T A_B^{-1} b \\ A_B^{-1} A_N x_N &= A_B^{-1} b \\ x_B, x_N &\geq 0\end{aligned}$$

The BFS is given by $x_N = 0, x_B = A_B^{-1} b$.

If $(c_N^T - c_B^T A_B^{-1} A_N) \leq 0$ we know that we have an optimum solution.

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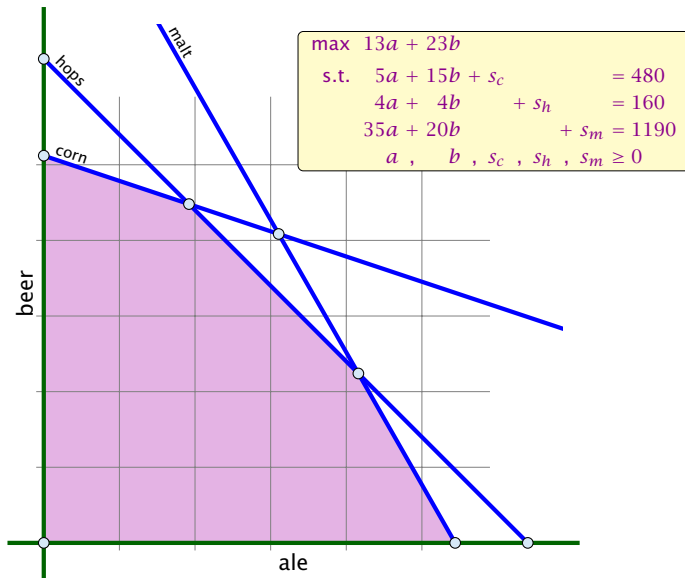
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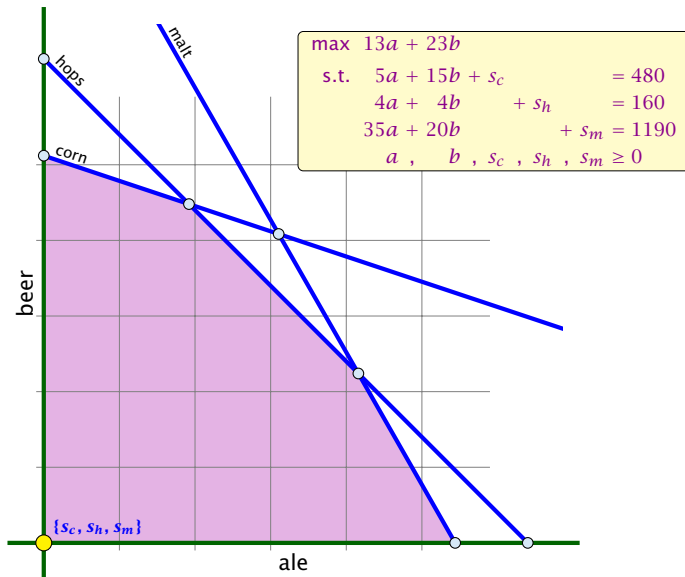
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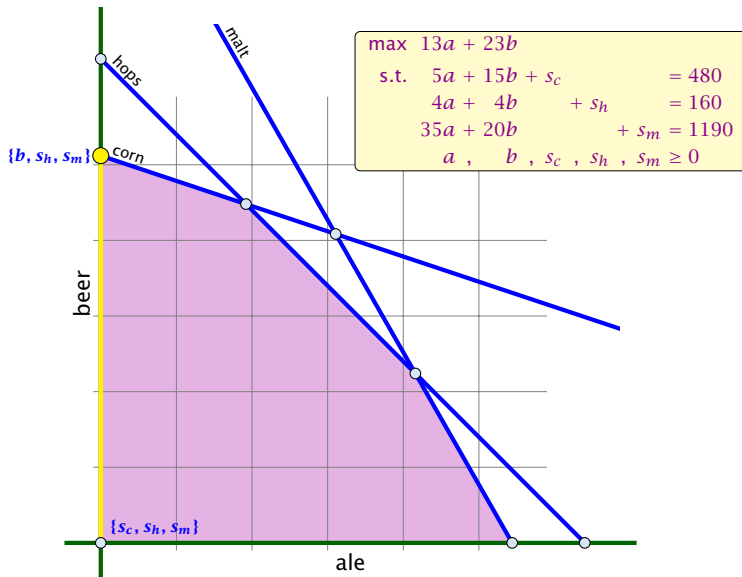
Geometric View of Pivoting



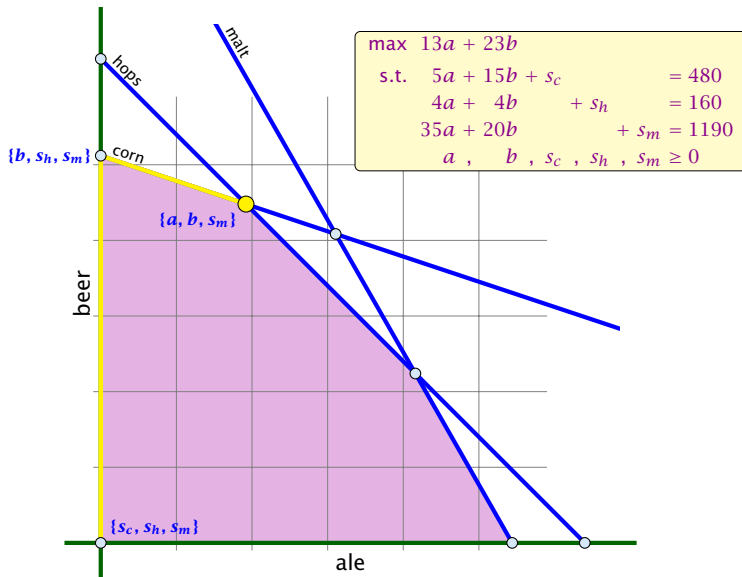
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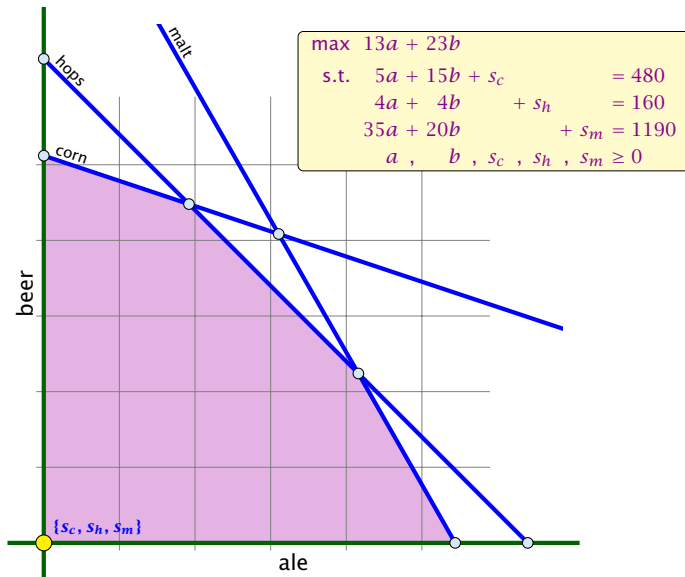
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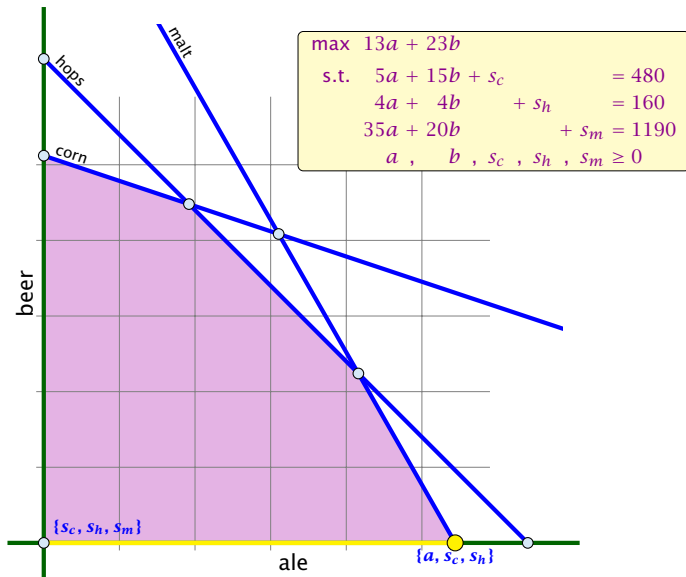
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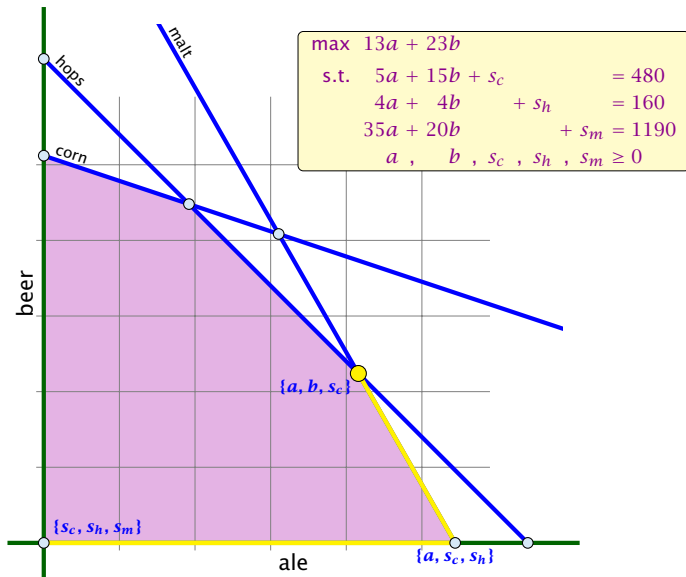
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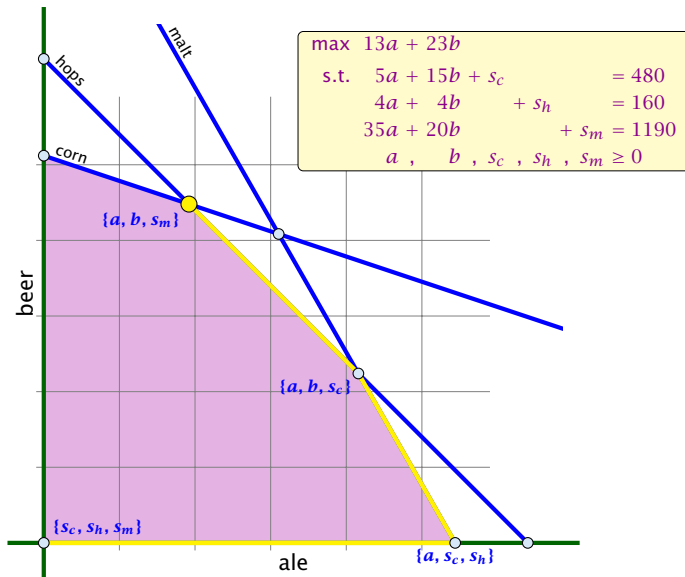
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Geometric View of Pivoting



Geometric View of Pivoting



Algebraic Definition of Pivoting

- ▶ Given basis B with BFS x^* .
- ▶ Choose index $j \notin B$ in order to increase x_j^* from 0 to $\theta > 0$.
 - Other non-basis variables should stay at 0.
 - Basis variables change to maintain feasibility.
- ▶ Go from x^* to $x^* + \theta \cdot d$.

Requirements for d :

1. $d_j = 1$ (normalization)

2. $d_B = -A_B^{-1} A_j$ (feasibility, d is a direction)

3. d must hold. Hence $d_B = -A_B^{-1} A_j$ is the only choice.

4. d is a direction. $d_j = 1$, which gives $d_B = -A_B^{-1} A_j$.

5. d is a direction.

Algebraic Definition of Pivoting

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Requirements for d :

- ▶ $d_j = 1$ (normalization)
 - ▶ $d_i = 0$ for all $i \in B$ (non-basis variables stay at 0)
 - ▶ $d_i \geq 0$ for all $i \in B$ (feasibility must hold, hence $\theta \geq 0$)
 - ▶ $d_i \leq 0$ for all $i \in B$ (feasibility must hold, hence $\theta \leq 0$)
- Altogether, d must satisfy $d_i = 0$ for all $i \in B$ and $d_j = 1$.

Algebraic Definition of Pivoting

- ▶ Given basis B with BFS x^* .
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Requirements for d :

- ▶ $d_j = 1$ (normalization)
 - ▶ $d_i = 0$ for all $i \notin B$
 - ▶ $d_i \leq 0$ for all $i \in B$ (feasibility must hold, hence $\theta \geq 0$)
- Remark: Algorithm 4.1 (p. 104) of [1] is a special case of this.

Algebraic Definition of Pivoting

- ▶ Given basis B with BFS x^* .
- ▶ Choose index $j \notin B$ in order to increase x_j^* from 0 to $\theta > 0$.
 - ▶ Other non-basis variables should stay at 0.
 - ▶ Basis variables change to maintain feasibility.
- ▶ Go from x^* to $x^* + \theta \cdot d$.

Requirements for d :

1. $d_j = 1$ (normalization)

2. $d_B = 0$ (non-basis variables)

3. d must be feasible, hence $d \geq 0$

4. d must be a direction, hence $d \neq 0$

Algebraic Definition of Pivoting

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- ▶ Choose index $j \notin B$ in order to increase x_j^* from 0 to $\theta > 0$.
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Requirements for d :

• $d_j = 1$ (normalization)

• $d_i = 0$ for all $i \notin B$

• $d_B = -a_{B,j}$ (maintain feasibility)

• $d_B = 0$ if $a_{B,j} = 0$ (no change in basis)

Algebraic Definition of Pivoting

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Requirements for d :

- ▶ $d_j = 1$ (normalization)
- ▶ $d_\ell = 0, \ell \notin B, \ell \neq j$
- ▶ $A(x^* + \theta d) = b$ must hold. Hence $Ad = 0$.
- ▶ Altogether: $A_B d_B + A_{*j} = Ad = 0$, which gives $d_B = -A_B^{-1} A_{*j}$.

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Algebraic Definition of Pivoting

Definition 26 (j -th basis direction)

Let B be a basis, and let $j \notin B$. The vector d with $d_j = 1$ and $d_\ell = 0, \ell \notin B, \ell \neq j$ and $d_B = -A_B^{-1}A_{*j}$ is called the j -th basis direction for B .

Going from x^* to $x^* + \theta \cdot d$ the objective function changes by

$$\theta \cdot c^T d = \theta(c_j - c_B^T A_B^{-1} A_{*j})$$

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Algebraic Definition of Pivoting

Definition 27 (Reduced Cost)

For a basis B the value

$$\tilde{c}_j = c_j - c_B^T A_B^{-1} A_{*j}$$

is called the **reduced cost** for variable x_j .

Note that this is defined for every j . If $j \in B$ then the above term is 0.

Algebraic Definition of Pivoting

Let our linear program be

$$\begin{aligned}c_B^T x_B + c_N^T x_N &= Z \\ A_B x_B + A_N x_N &= b \\ x_B, x_N &\geq 0\end{aligned}$$

The simplex tableaux for basis B is

$$\begin{aligned}(c_N^T - c_B^T A_B^{-1} A_N) x_N &= Z - c_B^T A_B^{-1} b \\ I x_B + A_B^{-1} A_N x_N &= A_B^{-1} b \\ x_B, x_N &\geq 0\end{aligned}$$

The BFS is given by $x_N = 0, x_B = A_B^{-1} b$.

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4 Simplex Algorithm

Questions:

What happens if the min ratio test fails to give us a value by which we can safely increase the entering variable?

How do we find the initial basic feasible solution?

Is there always a basis? Why/How?

Then how can terminate? How can we know that the solution is optimal?

Answer: If yes how do we make sure that we reach such a basis?

4 Simplex Algorithm

Questions:

- ▶ What happens if the min ratio test fails to give us a value θ by which we can safely increase the entering variable?
- ▶ How do we find the initial basic feasible solution?
- ▶ Is there always a basis B such that

$$(c_N^T - c_B^T A_B^{-1} A_N) \leq 0 \quad ?$$

Then we can terminate because we know that the solution is optimal.

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Min Ratio Test

The min ratio test computes a value $\theta \geq 0$ such that after setting the entering variable to θ the leaving variable becomes 0 and all other variables stay non-negative.

For this, one computes b_i/A_{ie} for all constraints i and calculates the minimum positive value.

What does it mean that the ratio b_i/A_{ie} (and hence A_{ie}) is negative for a constraint?

This means that the corresponding basic variable will increase if we increase b . Hence, there is no danger of this basic variable becoming negative

What happens if all b_i/A_{ie} are negative? Then we do not have a leaving variable. Then the LP is unbounded!

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Because a variable x_ℓ with $\ell \in B$ is already 0.

The set of inequalities is **degenerate** (also the basis is degenerate).

Definition 28 (Degeneracy)

A BFS x^* is called **degenerate** if the set $J = \{j \mid x_j^* > 0\}$ fulfills $|J| < m$.

It is possible that the algorithm **cycles**, i.e., it cycles through a sequence of different bases without ever terminating. Happens, very rarely in practise.

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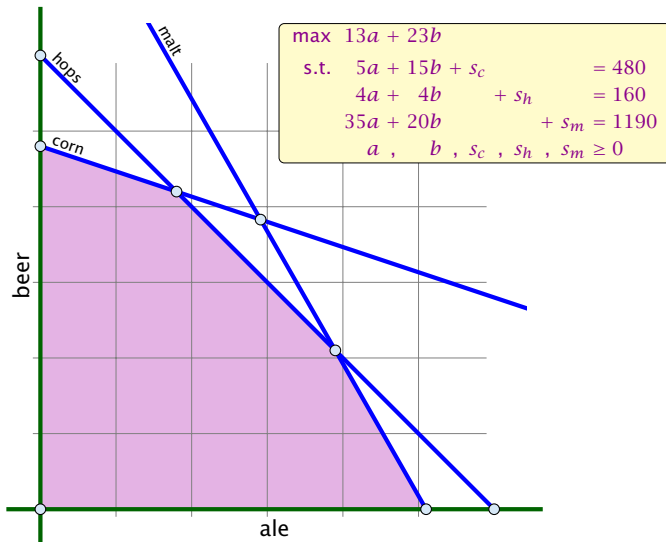
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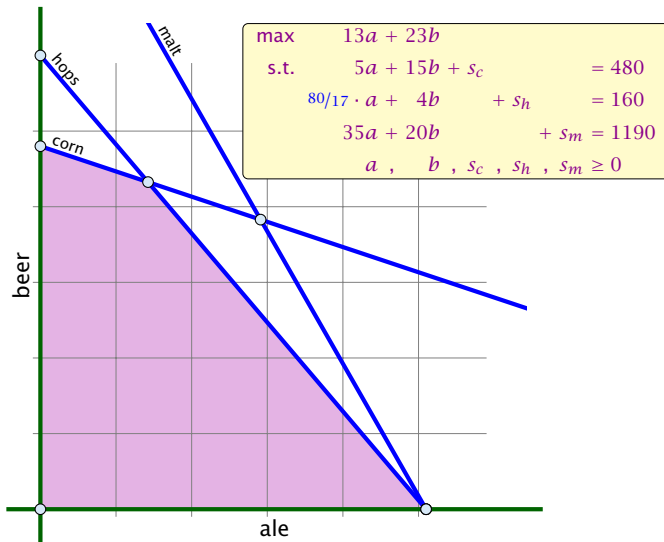
A BFS x^* is called **degenerate** if the set $J = \{j \mid x_j^* > 0\}$ fulfills $|J| < m$.

It is possible that the algorithm **cycles**, i.e., it cycles through a sequence of different bases without ever terminating. Happens, very rarely in practise.

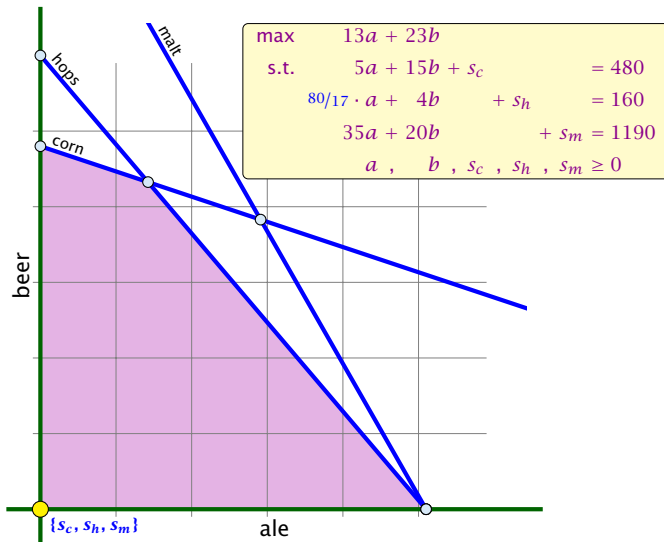
Non Degenerate Example



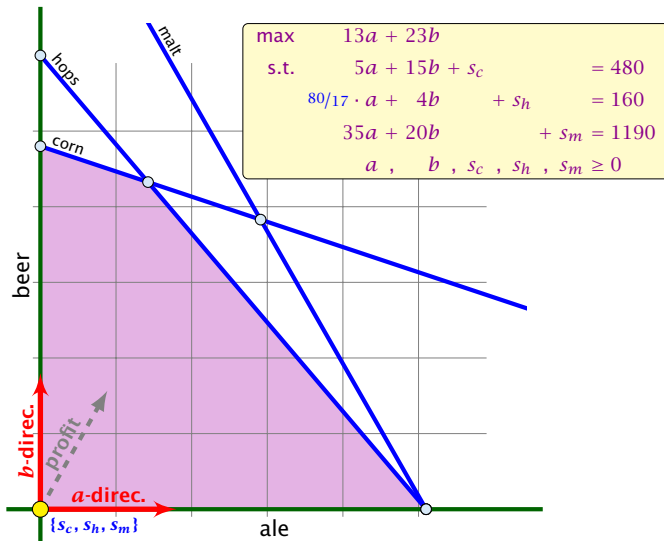
Degenerate Example



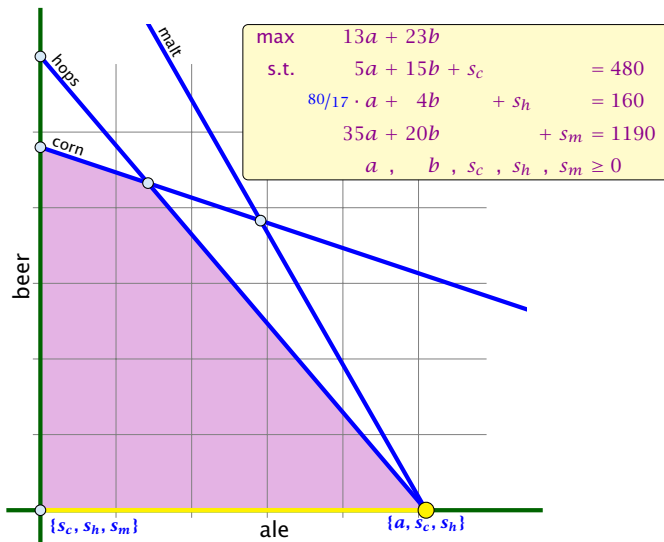
Degenerate Example



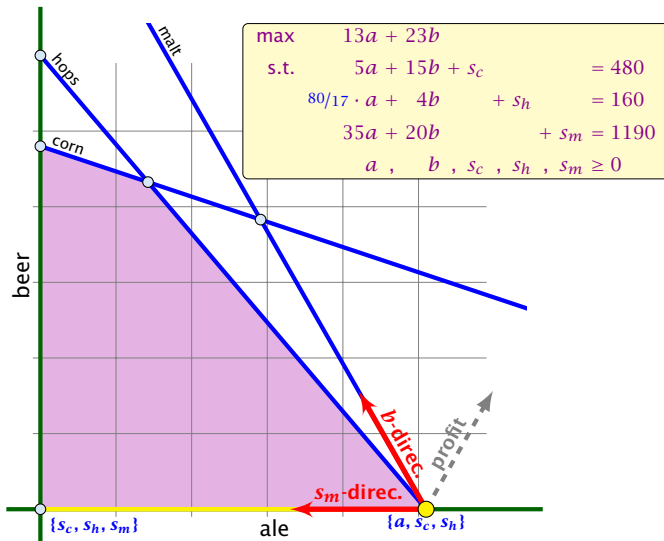
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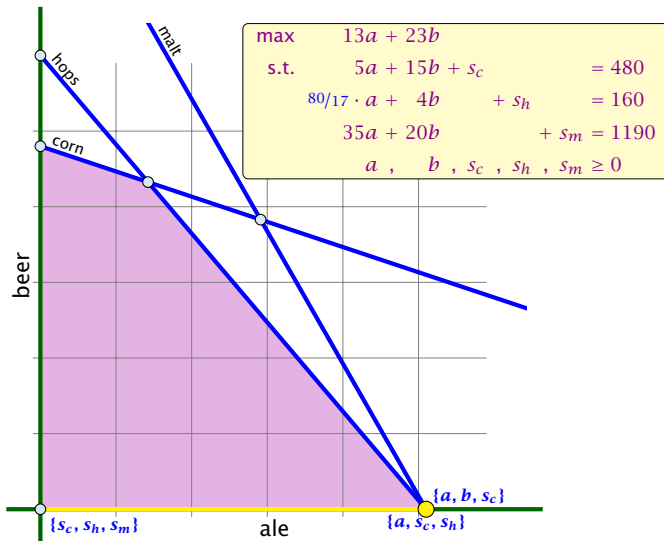
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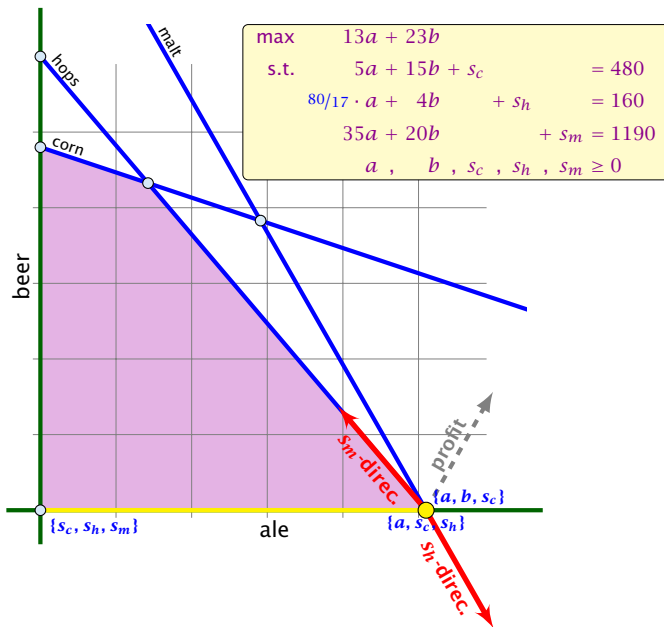
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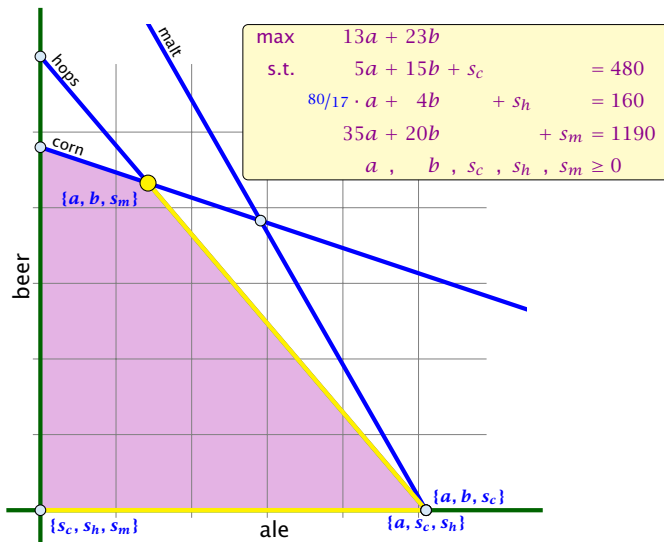
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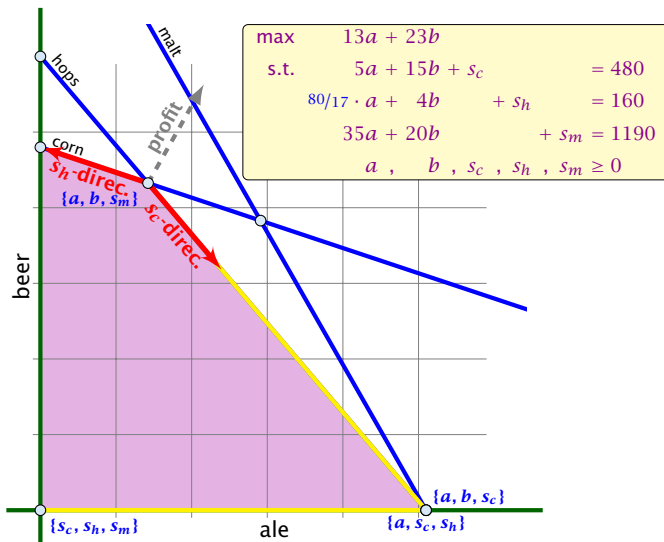
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Summary: How to choose pivot-elements

- ▶ We can choose a column e as an entering variable if $\tilde{c}_e > 0$ ($\tilde{c}_e$ is reduced cost for x_e).
- ▶ The standard choice is the column that maximizes $\tilde{c}_e$.
- ▶ If $A_{ie} \leq 0$ for all $i \in \{1, \dots, m\}$ then the maximum is not bounded.
- ▶ Otw. choose a leaving variable ℓ such that $b_\ell / A_{\ell e}$ is minimal among all variables i with $A_{ie} > 0$.
- ▶ If several variables have minimum $b_\ell / A_{\ell e}$ you reach a **degenerate** basis.
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What do we have so far?

Suppose we are given an initial feasible solution to an LP. If the LP is non-degenerate then Simplex will terminate.

Note that we either terminate because the min-ratio test fails and we can conclude that the LP is **unbounded**, or we terminate because the vector of reduced cost is non-positive. In the latter case we have an **optimum solution**.

How do we come up with an initial solution?

- ▶ $Ax \leq b, x \geq 0$, and $b \geq 0$.
- ▶ The standard slack form for this problem is $Ax + Is = b, x \geq 0, s \geq 0$, where s denotes the vector of slack variables.
- ▶ Then $s = b, x = 0$ is a basic feasible solution (how?).
- ▶ We directly can start the simplex algorithm.

How do we find an initial basic feasible solution for an arbitrary problem?

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How do we find an initial basic feasible solution for an arbitrary problem?

Two phase algorithm

Suppose we want to maximize $c^T x$ s.t. $Ax = b, x \geq 0$.

Multiply all rows with $b_i < 0$ by -1 .

maximize $-b^T$ s.t. $Ax = b, x \geq 0$, $b \geq 0$ using Simplex. If $b_i = 0$ is initial feasible.

If $b_i > 0$ then the original problem is infeasible.

Otherwise you have $x \geq 0$ with $Ax = b$.

From this you can get basic feasible solution.

Now you can start the Simplex for the original problem.

Two phase algorithm

Suppose we want to maximize $c^T x$ s.t. $Ax = b, x \geq 0$.

1. Multiply all rows with $b_i < 0$ by -1 .
2. maximize $-\sum_i v_i$ s.t. $Ax + Iv = b, x \geq 0, v \geq 0$ using Simplex. $x = 0, v = b$ is initial feasible.
3. If $\sum_i v_i > 0$ then the original problem is **infeasible**.
4. Otw. you have $x \geq 0$ with $Ax = b$.
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Lemma 29

Let B be a basis and x^ a BFS corresponding to basis B . $\tilde{c} \leq 0$ implies that x^* is an optimum solution to the LP.*

Duality

How do we get an upper bound to a maximization LP?

$$\begin{array}{llll} \max & 13a & + & 23b \\ \text{s.t.} & 5a & + & 15b \leq 480 \\ & 4a & + & 4b \leq 160 \\ & 35a & + & 20b \leq 1190 \\ & & & a, b \geq 0 \end{array}$$

Note that a lower bound is easy to derive. Every choice of $a, b \geq 0$ gives us a lower bound (e.g. $a = 12, b = 28$ gives us a lower bound of 800).

If you take a conic combination of the rows (multiply the i -th row with $y_i \geq 0$) such that $\sum_i y_i a_{ij} \geq c_j$ then $\sum_i y_i b_i$ will be an upper bound.

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Definition 30

Let $z = \max\{c^T x \mid Ax \leq b, x \geq 0\}$ be a linear program P (called the primal linear program).

The linear program D defined by

$$w = \min\{b^T y \mid A^T y \geq c, y \geq 0\}$$

is called the **dual problem**.

Lemma 31

The dual of the dual problem is the primal problem.

Proof:

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$$\blacktriangleright w = \min\{b^T y \mid A^T y \geq c, y \geq 0\}$$

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- ▶ $z = -\min\{-c^T x \mid -Ax \geq -b, x \geq 0\}$
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Weak Duality

Let $z = \max\{c^T x \mid Ax \leq b, x \geq 0\}$ and
 $w = \min\{b^T y \mid A^T y \geq c, y \geq 0\}$ be a primal dual pair.

x is primal feasible iff $x \in \{x \mid Ax \leq b, x \geq 0\}$

y is dual feasible, iff $y \in \{y \mid A^T y \geq c, y \geq 0\}$.

Theorem 32 (Weak Duality)

Let $\hat{x}$ be primal feasible and let $\hat{y}$ be dual feasible. Then

$$c^T \hat{x} \leq z \leq w \leq b^T \hat{y} .$$

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Weak Duality

$$A^T \hat{y} \geq c \Rightarrow \hat{x}^T A^T \hat{y} \geq \hat{x}^T c \quad (\hat{x} \geq 0)$$

$$A \hat{x} \leq b \Rightarrow y^T A \hat{x} \leq y^T b \quad (y \geq 0)$$

This gives

$$c^T \hat{x} \leq \hat{y}^T A \hat{x} \leq b^T \hat{y} .$$

Since, there exists primal feasible $\hat{x}$ with $c^T \hat{x} = z$, and dual feasible $\hat{y}$ with $b^T \hat{y} = w$ we get $z \leq w$.

If P is unbounded then D is infeasible.

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5.2 Simplex and Duality

The following linear programs form a primal dual pair:

$$z = \max\{c^T x \mid Ax = b, x \geq 0\}$$
$$w = \min\{b^T y \mid A^T y \geq c\}$$

This means for computing the dual of a standard form LP, we do not have non-negativity constraints for the dual variables.

Proof

Primal:

$$\max\{c^T x \mid Ax = b, x \geq 0\}$$

Proof

Primal:

$$\begin{aligned}\max\{c^T x \mid Ax = b, x \geq 0\} \\ = \max\{c^T x \mid Ax \leq b, -Ax \leq -b, x \geq 0\}\end{aligned}$$

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Primal:

$$\begin{aligned} & \max\{c^T x \mid Ax = b, x \geq 0\} \\ &= \max\{c^T x \mid Ax \leq b, -Ax \leq -b, x \geq 0\} \\ &= \max\{c^T x \mid \begin{bmatrix} A \\ -A \end{bmatrix} x \leq \begin{bmatrix} b \\ -b \end{bmatrix}, x \geq 0\} \end{aligned}$$

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Dual:

$$\min\{[b^T \ -b^T]y \mid [A^T \ -A^T]y \geq c, y \geq 0\}$$

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Dual:

$$\begin{aligned} & \min\{[b^T \ -b^T]y \mid [A^T \ -A^T]y \geq c, y \geq 0\} \\ &= \min\left\{[b^T \ -b^T] \cdot \begin{bmatrix} y^+ \\ y^- \end{bmatrix} \mid [A^T \ -A^T] \cdot \begin{bmatrix} y^+ \\ y^- \end{bmatrix} \geq c, y^- \geq 0, y^+ \geq 0\right\} \end{aligned}$$

Proof

Primal:

$$\begin{aligned} \max\{c^T x \mid Ax = b, x \geq 0\} \\ = \max\{c^T x \mid Ax \leq b, -Ax \leq -b, x \geq 0\} \\ = \max\{c^T x \mid \begin{bmatrix} A \\ -A \end{bmatrix} x \leq \begin{bmatrix} b \\ -b \end{bmatrix}, x \geq 0\} \end{aligned}$$

Dual:

$$\begin{aligned} \min\{[b^T \ -b^T]y \mid [A^T \ -A^T]y \geq c, y \geq 0\} \\ = \min\left\{[b^T \ -b^T] \cdot \begin{bmatrix} y^+ \\ y^- \end{bmatrix} \mid [A^T \ -A^T] \cdot \begin{bmatrix} y^+ \\ y^- \end{bmatrix} \geq c, y^- \geq 0, y^+ \geq 0\right\} \\ = \min\{b^T \cdot (y^+ - y^-) \mid A^T \cdot (y^+ - y^-) \geq c, y^- \geq 0, y^+ \geq 0\} \end{aligned}$$

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Proof of Optimality Criterion for Simplex

Suppose that we have a basic feasible solution with **reduced cost**

$$\tilde{c} = c^T - c_B^T A_B^{-1} A \leq 0$$

This is equivalent to $A^T (A_B^{-1})^T c_B \geq c$

$y^* = (A_B^{-1})^T c_B$ is solution to the **dual** $\min\{b^T y \mid A^T y \geq c\}$.

Since y^* is feasible and $y^* b = c^T x^*$, x^* is optimal.

Hence, the solution is optimal.

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Hence, the solution is optimal.

5.3 Strong Duality

$$P = \max\{c^T x \mid Ax \leq b, x \geq 0\}$$

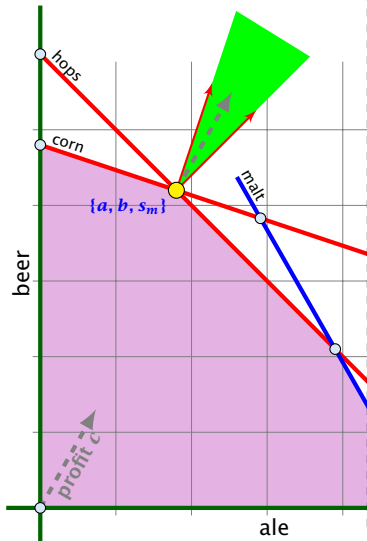
n_A : number of variables, m_A : number of constraints

We can put the non-negativity constraints into A (which gives us unrestricted variables): $\bar{P} = \max\{c^T x \mid \bar{A}x \leq \bar{b}\}$

$$n_{\bar{A}} = n_A, m_{\bar{A}} = m_A + n_A$$

$$\text{Dual } D = \min\{\bar{b}^T y \mid \bar{A}^T y = c, y \geq 0\}.$$

5.3 Strong Duality



If we have a conic combination y of c then $b^T y$ is an upper bound of the profit we can obtain (**weak duality**):

$$c^T x = (\bar{A}^T y)^T x = y^T \bar{A} x \leq y^T \bar{b}$$

If x and y are optimal then the **duality gap** is 0 (**strong duality**). This means

$$\begin{aligned} 0 &= c^T x - y^T \bar{b} \\ &= (\bar{A}^T y)^T x - y^T \bar{b} \\ &= y^T (\bar{A} x - \bar{b}) \end{aligned}$$

The last term can only be 0 if y_i is 0 whenever the i -th constraint is not tight. This means we have a conic combination of c by normals (columns of $\bar{A}^T$) of **tight** constraints.

Conversely, if we have x such that the normals of tight constraint (at x) give rise to a conic combination of c , we know that x is optimal.

The profit vector c lies in the cone generated by the normals for the hops and the corn constraint (the tight constraints).

Strong Duality

Theorem 33 (Strong Duality)

Let P and D be a primal dual pair of linear programs, and let z^* and w^* denote the optimal solution to P and D , respectively.
Then

$$z^* = w^*$$

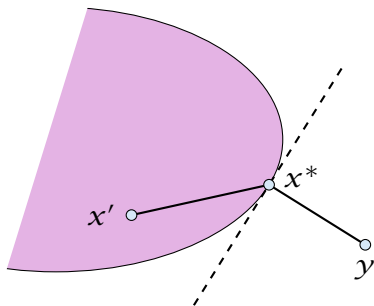
Lemma 34 (Weierstrass)

Let X be a compact set and let $f(x)$ be a continuous function on X . Then $\min\{f(x) : x \in X\}$ exists.

(without proof)

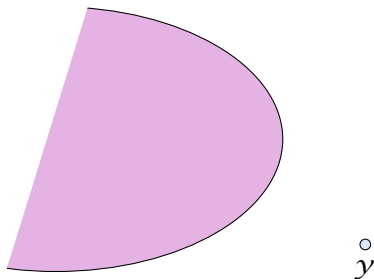
Lemma 35 (Projection Lemma)

Let $X \subseteq \mathbb{R}^m$ be a non-empty convex set, and let $y \notin X$. Then there exist $x^* \in X$ with minimum distance from y . Moreover for all $x \in X$ we have $(y - x^*)^T(x - x^*) \leq 0$.



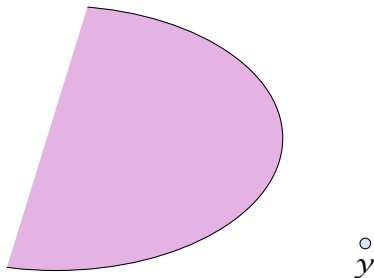
Proof of the Projection Lemma

- ▶ Define $f(x) = \|y - x\|$.
- ▶ We want to apply Weierstrass but X may not be bounded.
- ▶ $X \neq \emptyset$. Hence, there exists $x' \in X$.
- ▶ Define $X' = \{x \in X \mid \|y - x\| \leq \|y - x'\|\}$. This set is closed and bounded.
- ▶ Applying Weierstrass gives the existence.



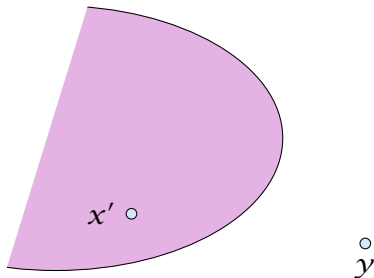
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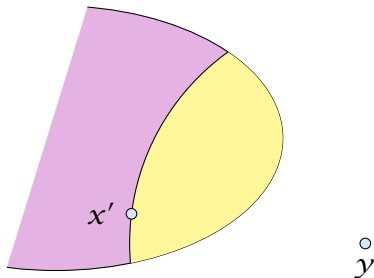
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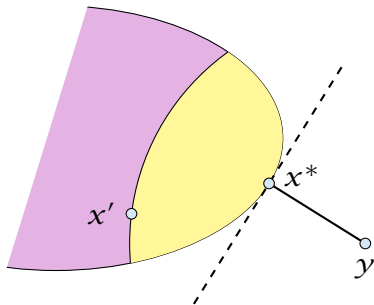
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Proof of the Projection Lemma (continued)

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x^* is minimum. Hence $\|y - x^*\|^2 \leq \|y - x\|^2$ for all $x \in X$.

Proof of the Projection Lemma (continued)

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By **convexity**: $x \in X$ then $x^* + \epsilon(x - x^*) \in X$ for all $0 \leq \epsilon \leq 1$.

Proof of the Projection Lemma (continued)

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$$\|y - x^*\|^2$$

Proof of the Projection Lemma (continued)

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By **convexity**: $x \in X$ then $x^* + \epsilon(x - x^*) \in X$ for all $0 \leq \epsilon \leq 1$.

$$\|y - x^*\|^2 \leq \|y - x^* - \epsilon(x - x^*)\|^2$$

Proof of the Projection Lemma (continued)

x^* is minimum. Hence $\|y - x^*\|^2 \leq \|y - x\|^2$ for all $x \in X$.

By **convexity**: $x \in X$ then $x^* + \epsilon(x - x^*) \in X$ for all $0 \leq \epsilon \leq 1$.

$$\begin{aligned}\|y - x^*\|^2 &\leq \|y - x^* - \epsilon(x - x^*)\|^2 \\ &= \|y - x^*\|^2 + \epsilon^2 \|x - x^*\|^2 - 2\epsilon(y - x^*)^T(x - x^*)\end{aligned}$$

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x^* is minimum. Hence $\|y - x^*\|^2 \leq \|y - x\|^2$ for all $x \in X$.

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Hence, $(y - x^*)^T(x - x^*) \leq \frac{1}{2}\epsilon \|x - x^*\|^2$.

Proof of the Projection Lemma (continued)

x^* is minimum. Hence $\|y - x^*\|^2 \leq \|y - x\|^2$ for all $x \in X$.

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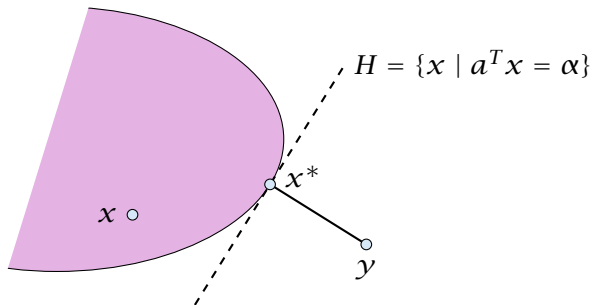
Letting $\epsilon \rightarrow 0$ gives the result.

Theorem 36 (Separating Hyperplane)

Let $X \subseteq \mathbb{R}^m$ be a non-empty closed convex set, and let $y \notin X$. Then there exists a separating hyperplane $\{x \in \mathbb{R}^m : a^T x = \alpha\}$ where $a \in \mathbb{R}^m$, $\alpha \in \mathbb{R}$ that separates y from X . ($a^T y < \alpha$; $a^T x \geq \alpha$ for all $x \in X$)

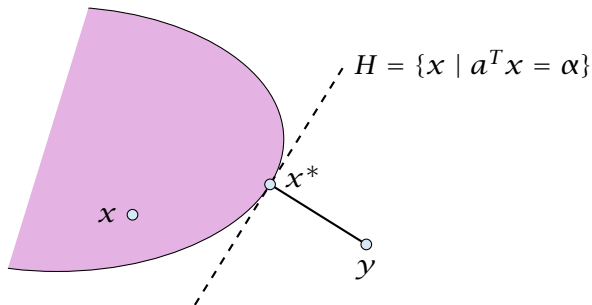
Proof of the Hyperplane Lemma

- ▶ Let $x^* \in X$ be closest point to y in X .
- ▶ By previous lemma $(y - x^*)^T(x - x^*) \leq 0$ for all $x \in X$.
- ▶ Choose $a = (x^* - y)$ and $\alpha = a^T x^*$.
- ▶ For $x \in X$: $a^T(x - x^*) \geq 0$, and, hence, $a^T x \geq \alpha$.
- ▶ Also, $a^T y = a^T(x^* - a) = \alpha - \|a\|^2 < \alpha$



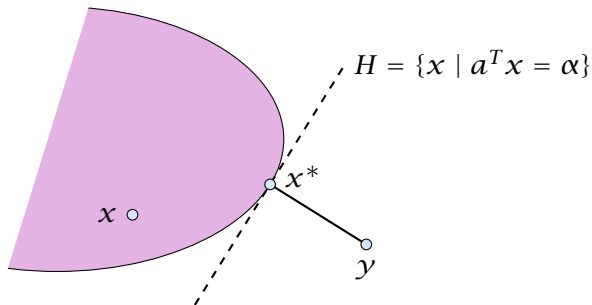
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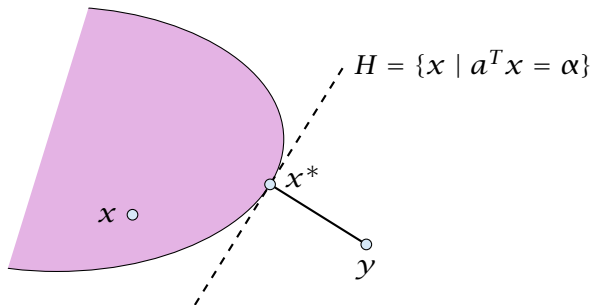
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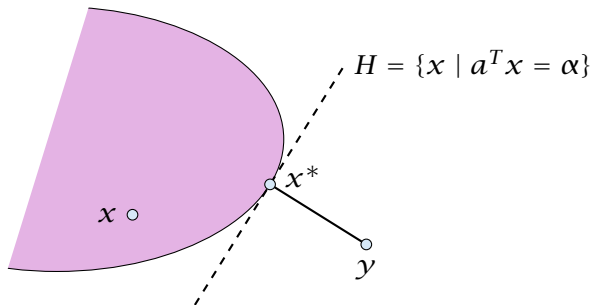
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Lemma 37 (Farkas Lemma)

Let A be an $m \times n$ matrix, $b \in \mathbb{R}^m$. Then *exactly one* of the following statements holds.

1. $\exists x \in \mathbb{R}^n$ with $Ax = b, x \geq 0$
2. $\exists y \in \mathbb{R}^m$ with $A^T y \geq 0, b^T y < 0$

Assume $\hat{x}$ satisfies 1. and $\hat{y}$ satisfies 2. Then

$$0 > \hat{y}^T b = \hat{y}^T A \hat{x} \geq 0$$

Hence, at most one of the statements can hold.

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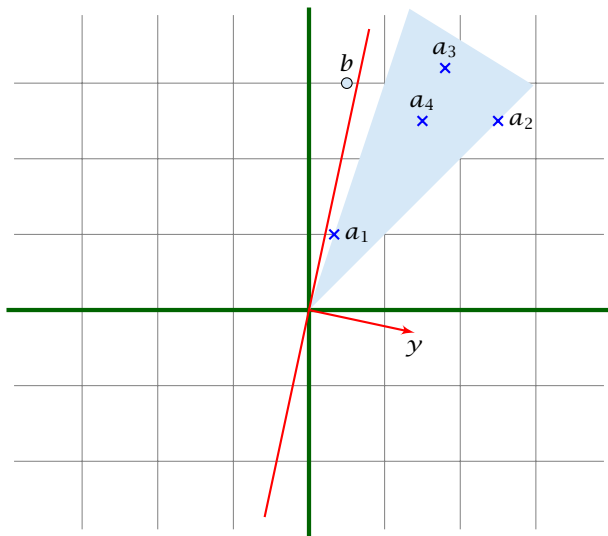
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Hence, at most one of the statements can hold.

Farkas Lemma



If b is not in the cone generated by the columns of A , there exists a hyperplane y that separates b from the cone.

Proof of Farkas Lemma

Now, assume that 1. does not hold.

Consider $S = \{Ax : x \geq 0\}$ so that S closed, convex, $b \notin S$.

We want to show that there is y with $A^T y \geq 0$, $b^T y < 0$.

Let y be a hyperplane that separates b from S . Hence, $y^T b < \alpha$ and $y^T s \geq \alpha$ for all $s \in S$.

$$0 \in S \Rightarrow \alpha \leq 0 \Rightarrow y^T b < 0$$

$y^T Ax \geq \alpha$ for all $x \geq 0$. Hence, $y^T A \geq 0$ as we can choose x arbitrarily large.

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Let y be a hyperplane that separates b from S . Hence, $y^T b < \alpha$ and $y^T s \geq \alpha$ for all $s \in S$.

$$0 \in S \Rightarrow \alpha \leq 0 \Rightarrow y^T b < 0$$

$y^T Ax \geq \alpha$ for all $x \geq 0$. Hence, $y^T A \geq 0$ as we can choose x arbitrarily large.

Proof of Farkas Lemma

Now, assume that 1. does not hold.

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Lemma 38 (Farkas Lemma; different version)

Let A be an $m \times n$ matrix, $b \in \mathbb{R}^m$. Then exactly one of the following statements holds.

1. $\exists x \in \mathbb{R}^n$ with $Ax \leq b, x \geq 0$
2. $\exists y \in \mathbb{R}^m$ with $A^T y \geq 0, b^T y < 0, y \geq 0$

Rewrite the conditions:

1. $\exists x \in \mathbb{R}^n$ with $\begin{bmatrix} A & I \end{bmatrix} \cdot \begin{bmatrix} x \\ s \end{bmatrix} = b, x \geq 0, s \geq 0$
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Proof of Strong Duality

$$P: z = \max\{c^T x \mid Ax \leq b, x \geq 0\}$$

$$D: w = \min\{b^T y \mid A^T y \geq c, y \geq 0\}$$

Theorem 39 (Strong Duality)

Let P and D be a primal dual pair of linear programs, and let z and w denote the optimal solution to P and D , respectively (i.e., P and D are non-empty). Then

$$z = w \text{ .}$$

Proof of Strong Duality

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$$\begin{array}{lll} \exists x \in \mathbb{R}^n & & \\ \text{s.t.} & Ax & \leq b \\ & -c^T x & \leq -\alpha \\ & x & \geq 0 \end{array}$$

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$$\begin{array}{lll} \text{s.t.} & A^T y - cv & \geq 0 \\ & b^T y - \alpha v & < 0 \\ & y, v & \geq 0 \end{array}$$

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From the definition of α we know that the first system is infeasible; hence the second must be feasible.

Proof of Strong Duality

$$\begin{array}{ll}\exists \mathbf{y} \in \mathbb{R}^m; v \in \mathbb{R} & \\ \text{s.t.} & A^T \mathbf{y} - c v \geq 0 \\ & b^T \mathbf{y} - \alpha v < 0 \\ & \mathbf{y}, v \geq 0\end{array}$$

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If the solution $\mathbf{y}, v$ has $v = 0$ we have that

$$\begin{array}{ll}\exists \mathbf{y} \in \mathbb{R}^m & \\ \text{s.t.} & A^T \mathbf{y} \geq 0 \\ & b^T \mathbf{y} < 0 \\ & \mathbf{y} \geq 0\end{array}$$

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is feasible. By Farkas lemma this gives that LP P is infeasible.
Contradiction to the assumption of the lemma.

Proof of Strong Duality

Hence, there exists a solution y, v with $v > 0$.

We can rescale this solution (scaling both y and v) s.t. $v = 1$.

Then y is feasible for the dual but $b^T y < \alpha$. This means that $w < \alpha$.

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Fundamental Questions

Definition 40 (Linear Programming Problem (LP))

Let $A \in \mathbb{Q}^{m \times n}$, $b \in \mathbb{Q}^m$, $c \in \mathbb{Q}^n$, $\alpha \in \mathbb{Q}$. Does there exist $x \in \mathbb{Q}^n$ s.t. $Ax = b$, $x \geq 0$, $c^T x \geq \alpha$?

Questions:

- ▶ Is LP in NP?
- ▶ Is LP in co-NP? yes!
- ▶ Is LP in P?

Proof:

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Proof:

- ▶ Given a primal maximization problem P and a parameter α . Suppose that $\alpha > \text{opt}(P)$.
 - ▶ We can prove this by providing an optimal basis for the dual.
 - ▶ A verifier can check that the associated dual solution fulfills all dual constraints and that it has dual cost $< \alpha$.

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Complementary Slackness

Lemma 41

Assume a linear program $P = \max\{c^T x \mid Ax \leq b; x \geq 0\}$ has solution x^* and its dual $D = \min\{b^T y \mid A^T y \geq c; y \geq 0\}$ has solution y^* .

1. If $x_j^* > 0$ then the j -th constraint in D is tight.
2. If the j -th constraint in D is not tight then $x_j^* = 0$.
3. If $y_i^* > 0$ then the i -th constraint in P is tight.
4. If the i -th constraint in P is not tight then $y_i^* = 0$.

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3. If $y_i^* > 0$ then the i -th constraint in P is tight.
4. If the i -th constraint in P is not tight then $y_i^* = 0$.

If we say that a variable x_j^* (y_i^*) has slack if $x_j^* > 0$ ($y_i^* > 0$), (i.e., the corresponding variable restriction is not tight) and a constraint has slack if it is not tight, then the above says that for a primal-dual solution pair it is not possible that a constraint **and** its corresponding (dual) variable has slack.

Proof: Complementary Slackness

Analogous to the proof of weak duality we obtain

$$c^T x^* \leq y^{*T} A x^* \leq b^T y^*$$

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From the constraint of the dual it follows that $y^T A \geq c^T$. Hence the left hand side is a sum over the product of non-negative numbers. Hence, if e.g. $(y^T A - c^T)_j > 0$ (the j -th constraint in the dual is not tight) then $x_j^* = 0$ (2.). The result for (1./3./4.) follows similarly.

Interpretation of Dual Variables

- Brewer: find mix of ale and beer that maximizes profits

$$\begin{array}{lll} \max & 13a & + \quad 23b \\ \text{s.t.} & 5a & + \quad 15b \leq 480 \\ & 4a & + \quad 4b \leq 160 \\ & 35a & + \quad 20b \leq 1190 \\ & & a, b \geq 0 \end{array}$$

- Entrepreneur: buy resources from brewer at minimum cost
 C, H, M : unit price for corn, hops and malt.

$$\begin{array}{lll} \min & 480C & + \quad 160H \quad + \quad 1190M \\ \text{s.t.} & 5C & + \quad 4H \quad + \quad 35M \geq 13 \\ & 15C & + \quad 4H \quad + \quad 20M \geq 23 \\ & & C, H, M \geq 0 \end{array}$$

Note that brewer won't sell (at least not all) if e.g.

$5C + 4H + 35M < 13$ as then brewing ale would be advantageous.

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Marginal Price:

- ▶ How much money is the brewer willing to pay for additional amount of Corn, Hops, or Malt?
- ▶ We are interested in the marginal price, i.e., what happens if we increase the amount of Corn, Hops, and Malt by ϵ_C , ϵ_H , and ϵ_M , respectively.

The profit increases to $\max\{c^T x \mid Ax \leq b + \epsilon; x \geq 0\}$. Because of strong duality this is equal to

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If ϵ is “small” enough then the optimum dual solution y^* might not change. Therefore the profit increases by $\sum_i \epsilon_i y_i^*$.

Therefore we can interpret the dual variables as **marginal prices**.

Note that with this interpretation, complementary slackness becomes obvious.

• If the browser has slack of some resource (e.g. cord) then he is not willing to pay anything for it (corresponding dual variable is zero).

• If the dual variable for some resource is non-zero, then an increase of this resource increases the profit of the browser. Hence, it makes no sense to have left-overs of this resource. Therefore no slack must be zero.

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If the primal has slack on some resource (e.g. only 1000 lbs of flour) then the primal is not willing to pay anything for it (corresponding dual variable is zero). If the primal has no slack on a resource (e.g. 1000 lbs of flour and 1000 lbs of flour are needed) then the primal is willing to pay the corresponding dual value for it.

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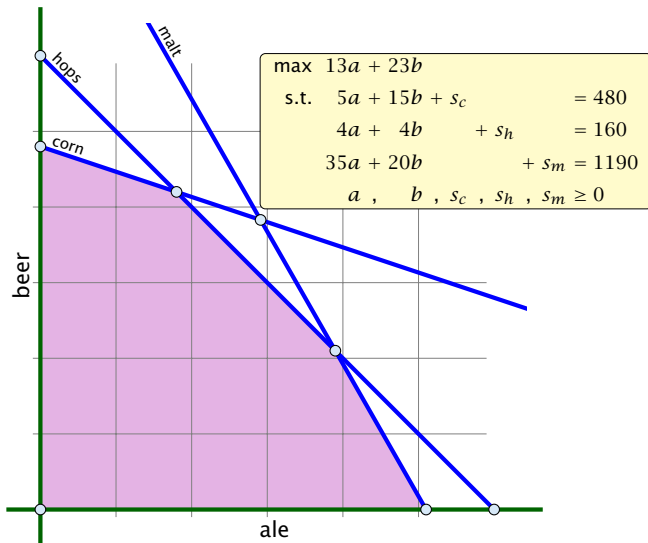
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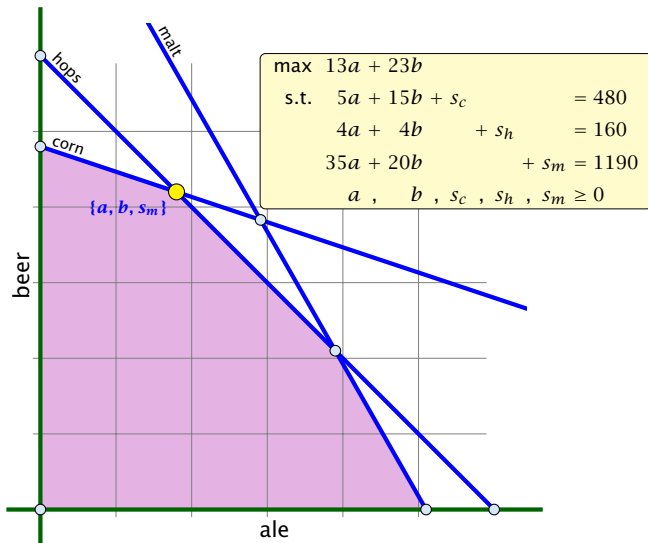
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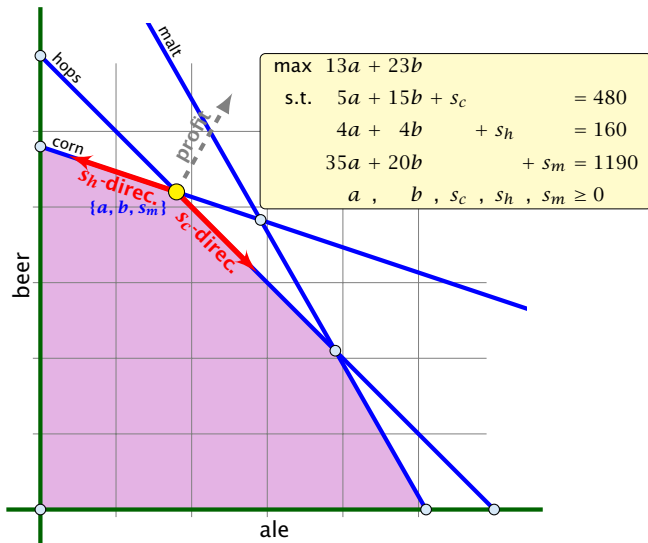
Example



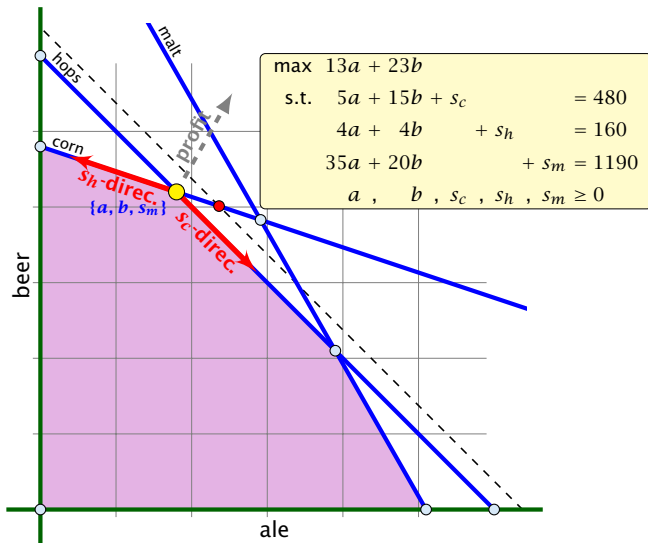
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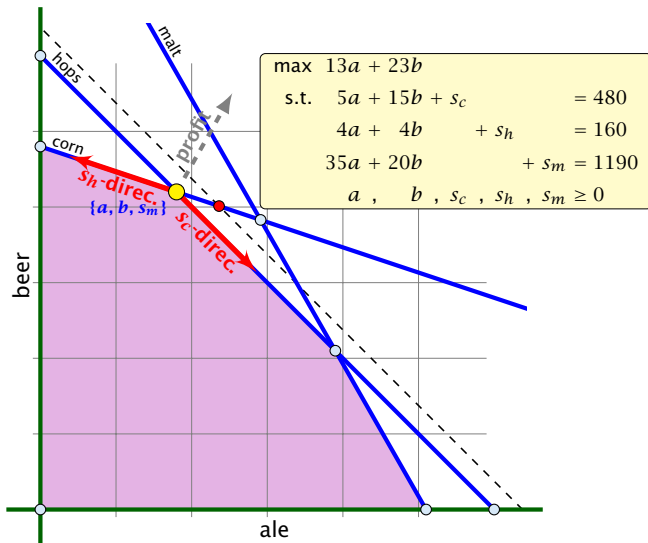
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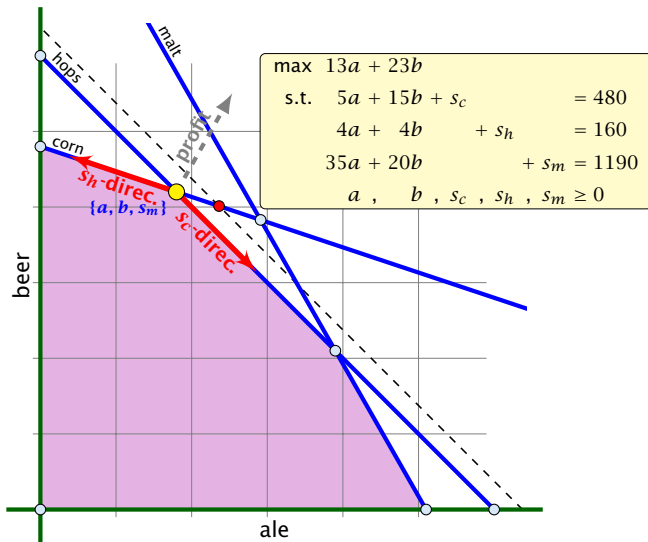
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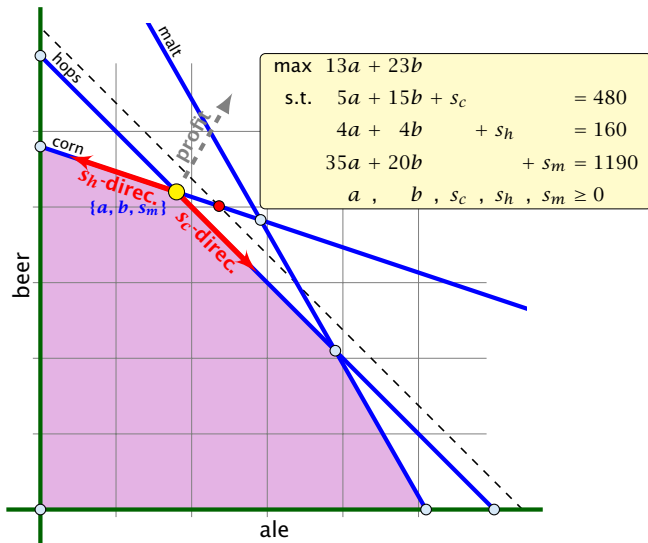
Example



The change in profit when increasing hops by one unit is

$$= c_B^T A_B^{-1} e_h.$$

Example



The change in profit when increasing hops by one unit is

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Of course, the previous argument about the increase in the primal objective only holds for the non-degenerate case.

If the optimum basis is degenerate then increasing the supply of one resource may not allow the objective value to increase.

Definition 42

An (s, t) -flow in a (complete) directed graph $G = (V, V \times V, c)$ is a function $f : V \times V \mapsto \mathbb{R}_0^+$ that satisfies

1. For each edge (x, y)

$$0 \leq f_{xy} \leq c_{xy} .$$

(capacity constraints)

2. For each $v \in V \setminus \{s, t\}$

$$\sum_x f_{vx} = \sum_x f_{xv} .$$

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Flows

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The **value of an (s, t) -flow f** is defined as

$$\text{val}(f) = \sum_x f_{sx} - \sum_x f_{xs} .$$

Maximum Flow Problem:

Find an (s, t) -flow with maximum value.

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LP-Formulation of Maxflow

$$\begin{array}{ll}\max & \sum_z f_{sz} - \sum_z f_{zs} \\ \text{s.t.} & \forall (z, w) \in V \times V \quad f_{zw} \leq c_{zw} \quad \ell_{zw} \\ & \forall w \neq s, t \quad \sum_z f_{zw} - \sum_z f_{wz} = 0 \quad p_w \\ & f_{zw} \geq 0\end{array}$$

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 \end{array}$$

$$\begin{array}{ll}
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 \text{s.t.} & f_{xy} \ (x, y \neq s, t) : \quad 1\ell_{xy} - 1p_x + 1p_y \geq 0 \\
 & f_{sy} \ (y \neq s, t) : \quad 1\ell_{sy} \quad \quad + 1p_y \geq 1 \\
 & f_{xs} \ (x \neq s, t) : \quad 1\ell_{xs} - 1p_x \quad \quad \geq -1 \\
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LP-Formulation of Maxflow

$$\begin{array}{ll} \min & \sum_{(x,y)} c_{xy} \ell_{xy} \\ \text{s.t.} & f_{xy} \ (x, y \neq s, t) : \quad 1\ell_{xy} - 1p_x + 1p_y \geq 0 \\ & f_{sy} \ (y \neq s, t) : \quad 1\ell_{sy} - \quad 1 + 1p_y \geq 0 \\ & f_{xs} \ (x \neq s, t) : \quad 1\ell_{xs} - 1p_x + \quad 1 \geq 0 \\ & f_{ty} \ (y \neq s, t) : \quad 1\ell_{ty} - \quad 0 + 1p_y \geq 0 \\ & f_{xt} \ (x \neq s, t) : \quad 1\ell_{xt} - 1p_x + \quad 0 \geq 0 \\ & f_{st} : \quad 1\ell_{st} - \quad 1 + \quad 0 \geq 0 \\ & f_{ts} : \quad 1\ell_{ts} - \quad 0 + \quad 1 \geq 0 \\ & \ell_{xy} \geq 0 \end{array}$$

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with $p_t = 0$ and $p_s = 1$.

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We can interpret the ℓ_{xy} value as assigning a length to every edge.

The value p_x for a variable, then can be seen as the distance of x to t (where the distance from s to t is required to be 1 since $p_s = 1$).

The constraint $p_x \leq \ell_{xy} + p_y$ then simply follows from triangle inequality ($d(x,t) \leq d(x,y) + d(y,t) \Rightarrow d(x,t) \leq \ell_{xy} + d(y,t)$).

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One can show that there is an optimum LP-solution for the dual problem that gives an integral assignment of variables.

This means $p_x = 1$ or $p_x = 0$ for our case. This gives rise to a cut in the graph with vertices having value 1 on one side and the other vertices on the other side. The objective function then evaluates the capacity of this cut.

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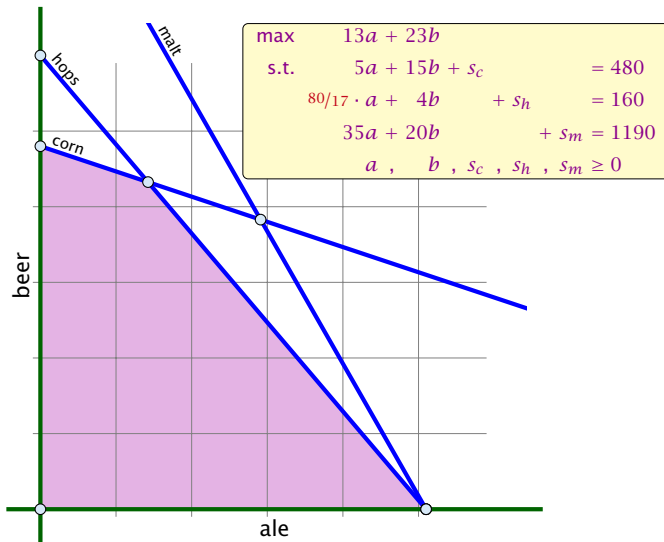
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Degeneracy Revisited

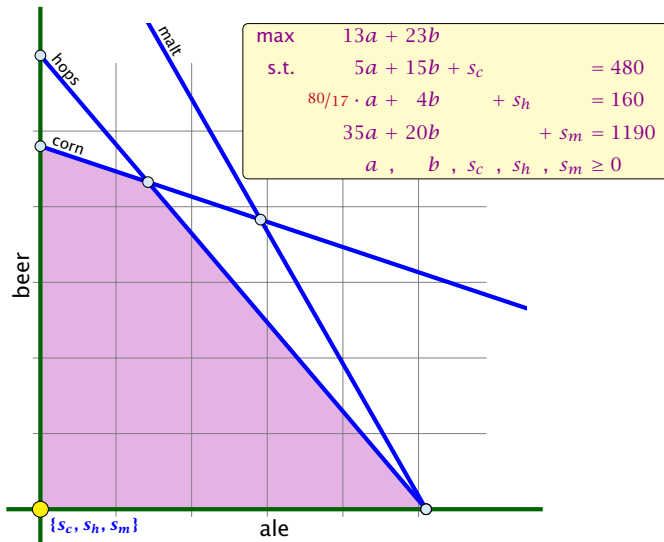
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If a basis variable is 0 in the basic feasible solution then we may not make progress during an iteration of simplex.

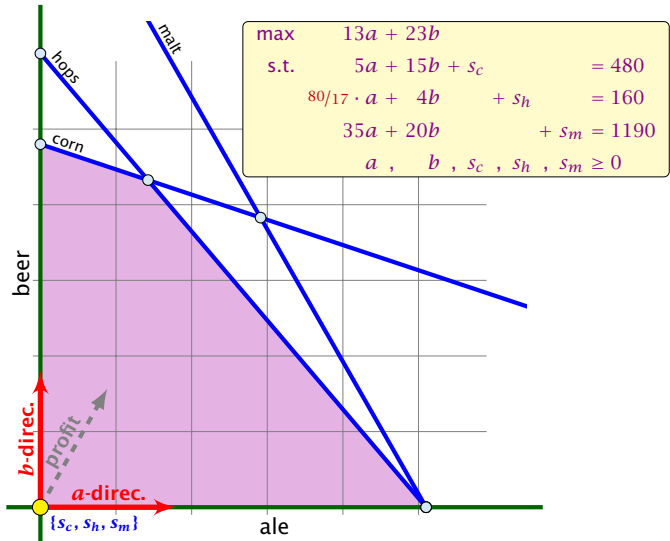
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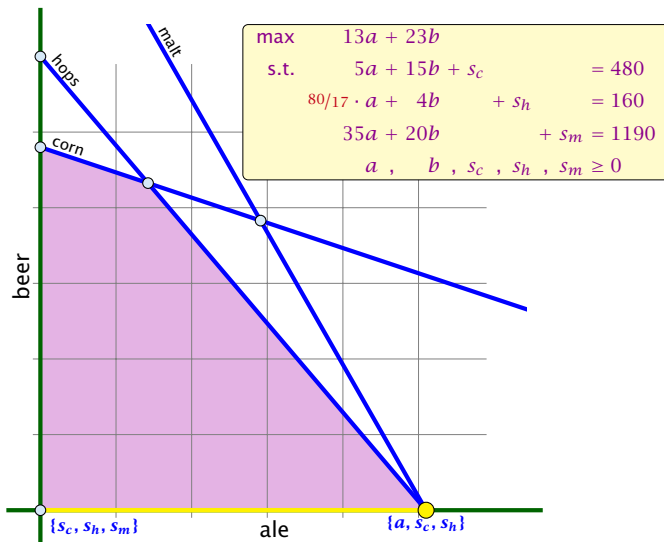
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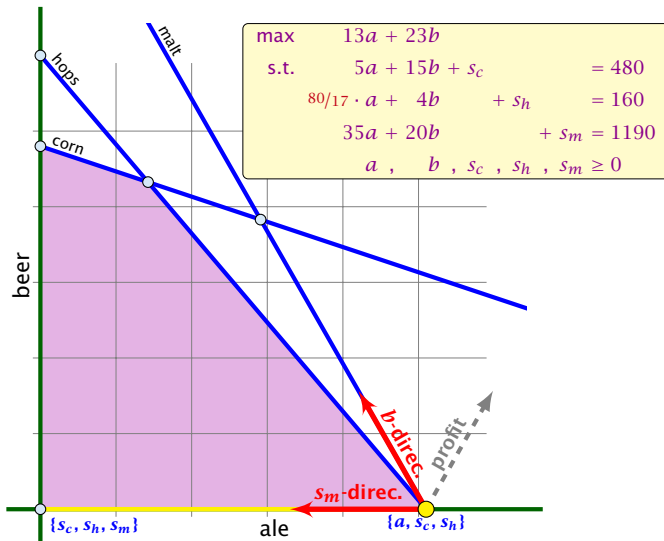
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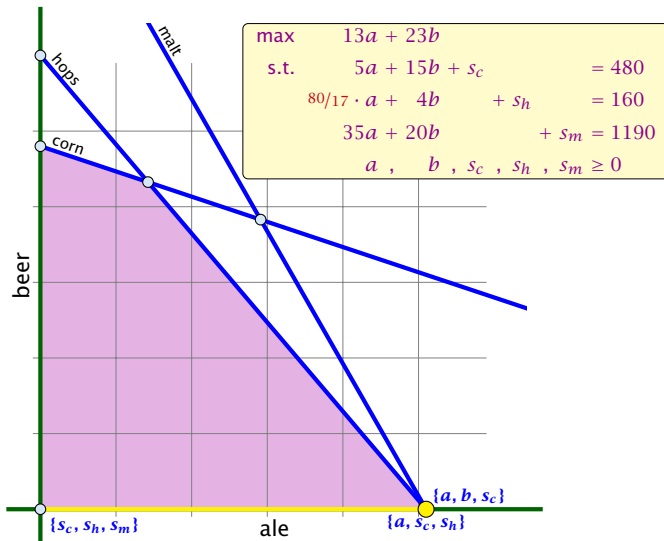
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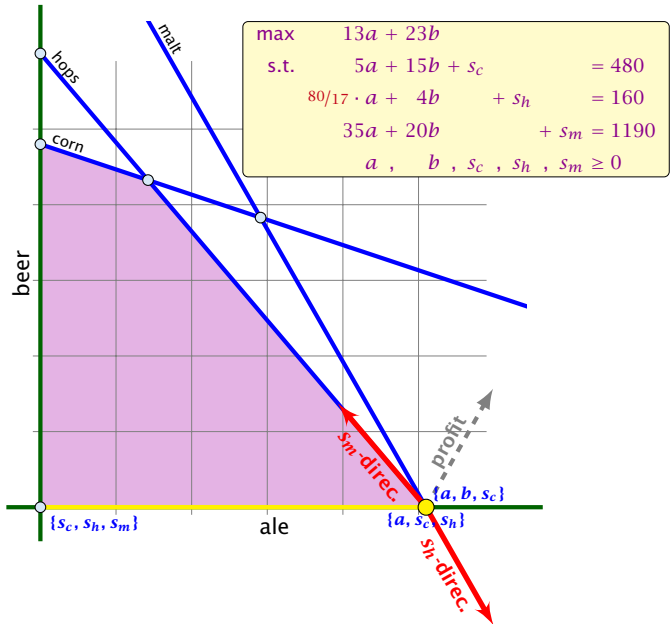
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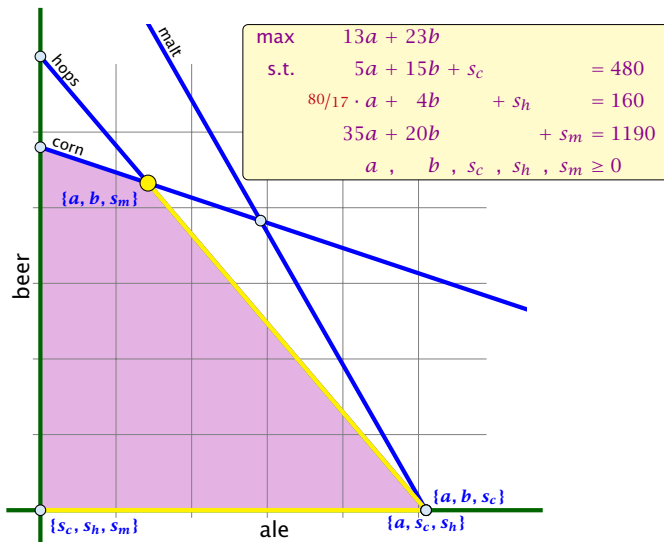
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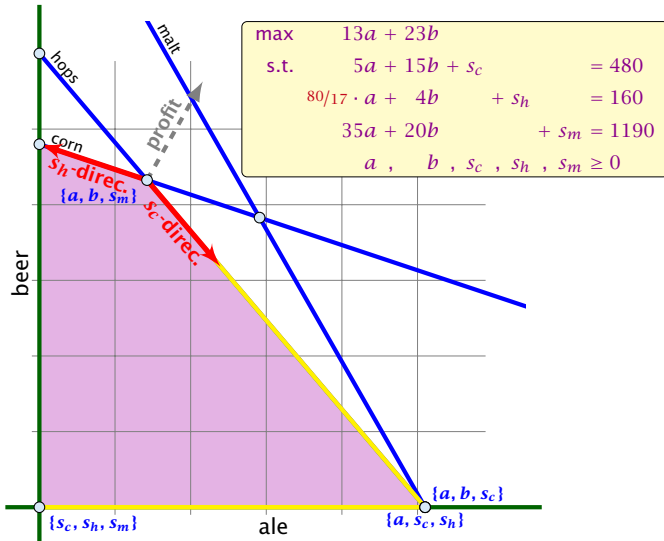
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Given feasible $LP := \max\{c^T x, Ax = b; x \geq 0\}$. Change it into $LP' := \max\{c^T x, Ax = b', x \geq 0\}$ such that

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if a set of basis variables corresponds to an infeasible basis (e.g., $b' \leq 0$) then it corresponds to an infeasible basis in LP . Hence that column must be linearly independent from the other columns having no degenerate basic solution.

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- II. If a set B of basis variables corresponds to an infeasible basis (i.e. $A_B^{-1}b \not\geq 0$) then B corresponds to an infeasible basis in LP' (note that columns in A_B are linearly independent).
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Perturbation

Let B be index set of **some** basis with basic solution

$$x_B^* = A_B^{-1}b \geq 0, x_N^* = 0 \quad (\text{i.e. } B \text{ is feasible})$$

Fix

$$b' := b + A_B \begin{pmatrix} \varepsilon \\ \vdots \\ \varepsilon^m \end{pmatrix} \quad \text{for } \varepsilon > 0 .$$

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The new LP is feasible because the set B of basis variables provides a feasible basis:

$$A_B^{-1} \left(b + A_B \begin{pmatrix} \varepsilon \\ \vdots \\ \varepsilon^m \end{pmatrix} \right) = x_B^* + \begin{pmatrix} \varepsilon \\ \vdots \\ \varepsilon^m \end{pmatrix} \geq 0 \ .$$

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Hence, $\tilde{B}$ is not feasible.

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Let $\tilde{B}$ be a basis. It has an associated solution

$$x_{\tilde{B}}^* = A_{\tilde{B}}^{-1}b + A_{\tilde{B}}^{-1}A_B \begin{pmatrix} \varepsilon \\ \vdots \\ \varepsilon^m \end{pmatrix}$$

in the perturbed instance.

We can view each component of the vector as a polynomial with variable ε of degree at most m .

$A_{\tilde{B}}^{-1}A_B$ has rank m . Therefore no polynomial is 0.

A polynomial of degree at most m has at most m roots (Nullstellen).

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- ▶ If it terminates because it finds a variable x_j with $\tilde{c}_j > 0$ for which the j -th basis direction d , fulfills $d \geq 0$ we know that LP' is unbounded. The basis direction **does not depend on b** . Hence, we also know that LP is unbounded.

Lexicographic Pivoting

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Simulate behaviour of LP' without explicitly doing a perturbation.

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We choose the entering variable arbitrarily as before ($\tilde{c}_e > 0$, of course).

If we do not have a choice for the leaving variable then LP' and LP do the same (i.e., choose the same variable).

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In the following we assume that $b \geq 0$. This can be obtained by replacing the initial system $(A \mid b)$ by $(A_B^{-1}A \mid A_B^{-1}b)$ where B is the index set of a feasible basis (found e.g. by the first phase of the Two-phase algorithm).

Then the perturbed instance is

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Matrix View

Let our linear program be

$$\begin{aligned}c_B^T x_B + c_N^T x_N &= Z \\ A_B x_B + A_N x_N &= b \\ x_B, x_N &\geq 0\end{aligned}$$

The simplex tableaux for basis B is

$$\begin{aligned}(c_N^T - c_B^T A_B^{-1} A_N) x_N &= Z - c_B^T A_B^{-1} b \\ I x_B + A_B^{-1} A_N x_N &= A_B^{-1} b \\ x_B, x_N &\geq 0\end{aligned}$$

The BFS is given by $x_N = 0, x_B = A_B^{-1} b$.

If $(c_N^T - c_B^T A_B^{-1} A_N) \leq 0$ we know that we have an optimum solution.

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LP chooses an arbitrary leaving variable that has $\hat{A}_{\ell e} > 0$ and minimizes

$$\theta_{\ell} = \frac{\hat{b}_{\ell}}{\hat{A}_{\ell e}} = \frac{(A_B^{-1}b)_{\ell}}{(A_B^{-1}A_{*e})_{\ell}} .$$

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Definition 44

$u \leq_{\text{lex}} v$ if and only if the first component in which u and v differ fulfills $u_i \leq v_i$.

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This means you can choose the variable/row ℓ for which the vector

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This means you can choose the variable/row ℓ for which the vector

$$\frac{\ell\text{-th row of } A_B^{-1}(b \mid I)}{(A_B^{-1}A_{*e})_\ell}$$

is lexicographically minimal.

Of course only including rows with $(A_B^{-1}A_{*e})_\ell > 0$.

This technique guarantees that your pivoting is the same as in the perturbed case. This guarantees that cycling does not occur.

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Can we obtain a better analysis?

Number of Simplex Iterations

Observation

Simplex visits every **feasible** basis at most once.

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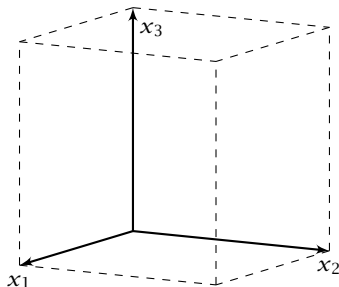
Observation

Simplex visits every **feasible** basis at most once.

However, also the number of feasible bases can be very large.

Example

$$\begin{array}{ll}\max & c^T x \\ \text{s.t.} & 0 \leq x_1 \leq 1 \\ & 0 \leq x_2 \leq 1 \\ & \vdots \\ & 0 \leq x_n \leq 1\end{array}$$

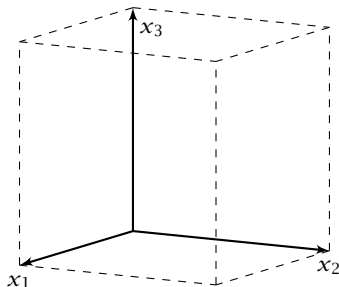


$2n$ constraint on n variables define an n -dimensional hypercube as feasible region.

The feasible region has 2^n vertices.

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However, Simplex may still run quickly as it usually does not visit all feasible bases.

In the following we give an example of a feasible region for which there is a bad **Pivoting Rule**.

Pivoting Rule

A Pivoting Rule defines how to choose the entering and leaving variable for an iteration of Simplex.

In the non-degenerate case after choosing the entering variable the leaving variable is unique.

Klee Minty Cube

$$\max x_n$$

$$\text{s.t.} \quad 0 \leq x_1 \leq 1$$

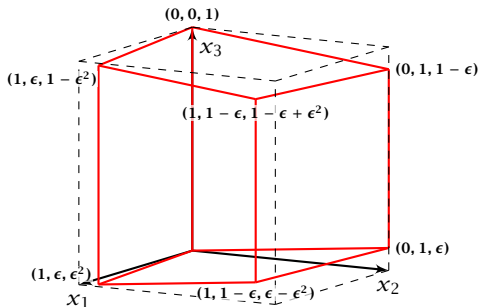
$$\epsilon x_1 \leq x_2 \leq 1 - \epsilon x_1$$

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$$\vdots$$

$$\epsilon x_{n-1} \leq x_n \leq 1 - \epsilon x_{n-1}$$

$$x_i \geq 0$$



Observations

- ▶ We have $2n$ constraints, and $3n$ variables (after adding slack variables to every constraint).
- ▶ Every basis is defined by $2n$ variables, and n non-basic variables.
- ▶ There exist degenerate vertices.
- ▶ The degeneracies come from the non-negativity constraints, which are superfluous.
- ▶ In the following all variables x_i stay in the basis at all times.
- ▶ Then, we can uniquely specify a basis by choosing for each variable whether it should be equal to its lower bound, or equal to its upper bound (the slack variable corresponding to the non-tight constraint is part of the basis).
- ▶ We can also simply identify each basis/vertex with the corresponding hypercube vertex obtained by letting $\epsilon \rightarrow 0$.

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- ▶ In the following we specify a sequence of bases (identified by the corresponding hypercube node) along which the objective function strictly increases.
- ▶ The basis $(0, \dots, 0, 1)$ is the unique optimal basis.
- ▶ Our sequence S_n starts at $(0, \dots, 0)$ ends with $(0, \dots, 0, 1)$ and visits every node of the hypercube.
- ▶ An unfortunate Pivoting Rule may choose this sequence, and, hence, require an exponential number of iterations.

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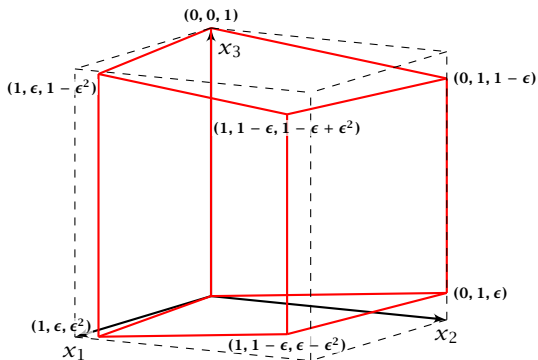
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$$\max x_n$$

$$\text{s.t.} \quad 0 \leq x_1 \leq 1$$

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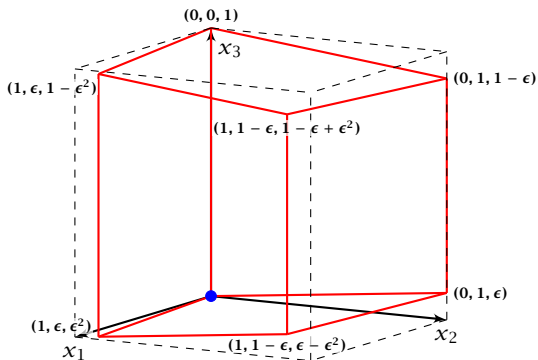
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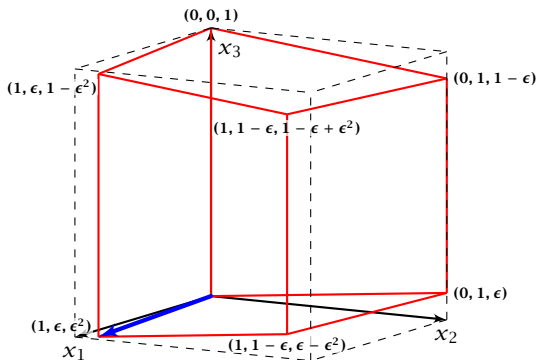
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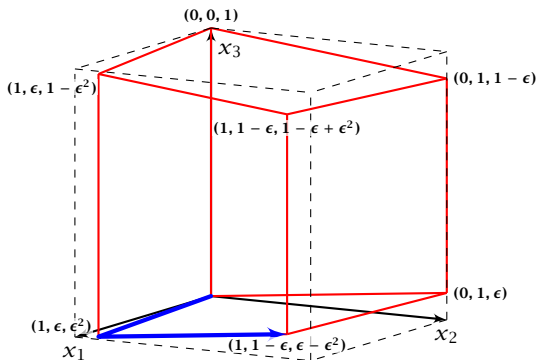
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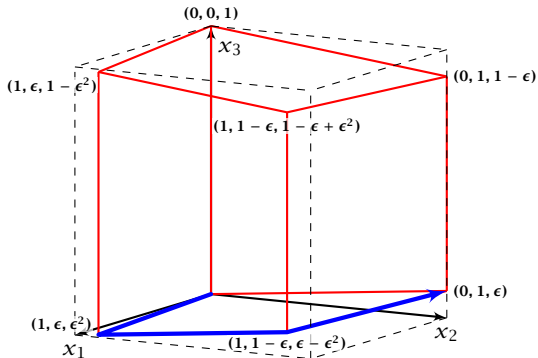
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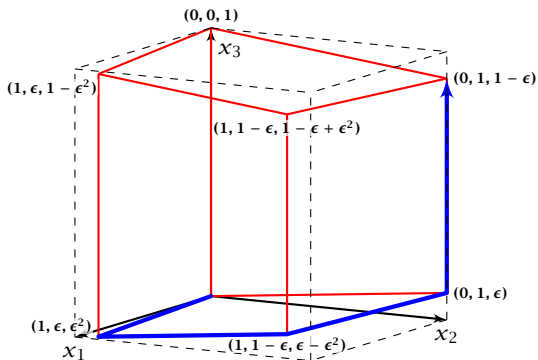
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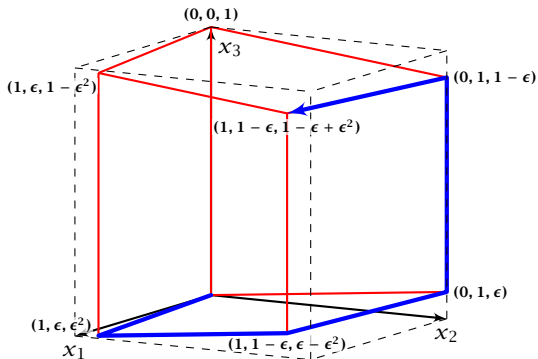
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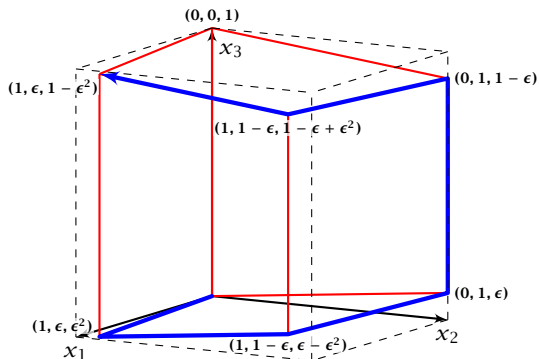
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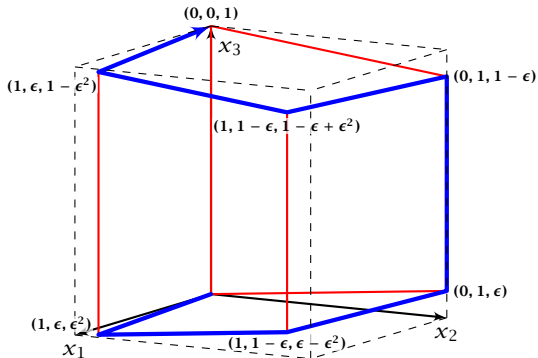
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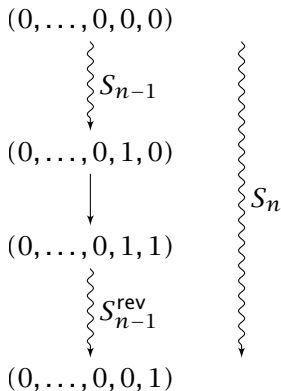
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Analysis

The sequence S_n that visits every node of the hypercube is defined recursively



The non-recursive case is $S_1 = 0 \rightarrow 1$

Analysis

Lemma 45

The objective value x_n is increasing along path S_n .

Proof by induction:

$n = 1$: obvious, since $S_1 = 0 \rightarrow 1$, and $1 > 0$.

$n - 1 \rightarrow n$

For the first part the value of x_n is increasing along S_n by definition.

By induction hypothesis x_{n-1} is increasing along S_{n-1} .
Hence, x_n is increasing twice, also.

Going from S_{n-1} to S_n , x_n is increasing. It decreases ϵ for ϵ small enough, but x_n is increasing again.

For the remaining path S_n we have x_n is increasing again.

By induction hypothesis x_{n-1} is increasing along S_{n-1} , hence x_n is increasing along S_n .

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For the first part the value of x_n is increasing along S_n if and only if x_n is increasing along S_{n-1} . By induction hypothesis x_{n-1} is increasing along S_{n-1} . If x_{n-1} is decreasing, also x_n is decreasing along S_n .
Going from $(x_{n-1}, 0)$ to $(x_n, 0)$ is a step of size ϵ . This means that for ϵ small enough $x_{n-1} > x_n$ implies $x_{n-1} > x_n + \epsilon$.
For the remaining path S_n we have $x_{n-1} > x_n + \epsilon$. By induction hypothesis x_{n-1} is increasing along S_{n-1} , hence x_n is increasing along S_n .
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At the first part the value of x_n is increasing, since x_n is increasing by induction hypothesis, and x_n is increasing along S_{n-1} as well, also. Then, x_n is decreasing along S_{n-1} for n large enough, since x_n is decreasing along S_{n-1} for n large enough. For the remaining part, we have x_n is increasing along S_{n-1} for n large enough, and x_n is increasing along S_{n-1} for n large enough. If increasing along S_{n-1} for n large enough, then x_n is increasing along S_n .

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- ▶ Going from $(0, \dots, 0, 1, 0)$ to $(0, \dots, 0, 1, 1)$ increases x_n for small enough ϵ .
- ▶ For the remaining path S_{n-1}^{rev} we have $x_n = 1 - \epsilon x_{n-1}$.
- ▶ By induction hypothesis x_{n-1} is increasing along S_{n-1} , hence $-\epsilon x_{n-1}$ is increasing along S_{n-1}^{rev} .

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Remarks about Simplex

Observation

The simplex algorithm takes at most $\binom{n}{m}$ iterations. Each iteration can be implemented in time $\mathcal{O}(mn)$.

In practise it usually takes a linear number of iterations.

Remarks about Simplex

Theorem

For almost all known **deterministic** pivoting rules (rules for choosing entering and leaving variables) there exist lower bounds that require the algorithm to have exponential running time ($\Omega(2^{\Omega(n)})$) (e.g. Klee Minty 1972).

Remarks about Simplex

Theorem

For some standard **randomized** pivoting rules there exist subexponential lower bounds ($\Omega(2^{\Omega(n^\alpha)})$ for $\alpha > 0$) (Friedmann, Hansen, Zwick 2011).

Remarks about Simplex

Conjecture (Hirsch 1957)

The edge-vertex graph of an m -facet polytope in d -dimensional Euclidean space has diameter no more than $m - d$.

The conjecture has been proven wrong in 2010.

But the question whether the diameter is perhaps of the form $\mathcal{O}(\text{poly}(m, d))$ is open.

8 Seidels LP-algorithm

- ▶ Suppose we want to solve $\min\{c^T x \mid Ax \geq b; x \geq 0\}$, where $x \in \mathbb{R}^d$ and we have m constraints.
- ▶ In the worst-case Simplex runs in time roughly $\mathcal{O}(m(m+d)\binom{m+d}{m}) \approx (m+d)^m$. (slightly better bounds on the running time exist, but will not be discussed here).
- ▶ If d is much smaller than m one can do a lot better.
- ▶ In the following we develop an algorithm with running time $\mathcal{O}(d! \cdot m)$, i.e., linear in m .

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8 Seidels LP-algorithm

Setting:

- ▶ We assume an LP of the form

$$\begin{array}{ll}\min & c^T x \\ \text{s.t.} & Ax \geq b \\ & x \geq 0\end{array}$$

- ▶ We assume that the LP is **bounded**.

Ensuring Conditions

Given a **standard minimization LP**

$$\begin{array}{ll}\min & c^T x \\ \text{s.t.} & Ax \geq b \\ & x \geq 0\end{array}$$

how can we obtain an LP of the required form?

- ▶ **Compute a lower bound on $c^T x$ for any basic feasible solution.**

Computing a Lower Bound

Let s denote the smallest common multiple of all denominators of entries in A, b .

Multiply entries in A, b by s to obtain integral entries. This does not change the feasible region.

Add slack variables to A ; denote the resulting matrix with $\tilde{A}$.

If B is an optimal basis then x_B with $\tilde{A}_B x_B = \tilde{b}$, gives an optimal assignment to the basis variables (non-basic variables are 0).

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Multiply entries in A, b by s to obtain integral entries. This does not change the feasible region.

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If B is an optimal basis then x_B with $\bar{A}_B x_B = \bar{b}$, gives an optimal assignment to the basis variables (non-basic variables are 0).

Theorem 46 (Cramers Rule)

Let M be a matrix with $\det(M) \neq 0$. Then the solution to the system $Mx = b$ is given by

$$x_i = \frac{\det(M_j)}{\det(M)} ,$$

where M_i is the matrix obtained from M by replacing the i -th column by the vector b .

Proof:

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- ▶ Define

$$X_i = \begin{pmatrix} | & & | & | & | & & | \\ e_1 & \cdots & e_{i-1} & x & e_{i+1} & \cdots & e_n \\ | & & | & | & | & & | \end{pmatrix}$$

Note that expanding along the i -th column gives that $\det(X_i) = x_i$.

- ▶ Further, we have

$$MX_i = \begin{pmatrix} | & & | & | & | & & | \\ Me_1 & \cdots & Me_{i-1} & Mx & Me_{i+1} & \cdots & Me_n \\ | & & | & | & | & & | \end{pmatrix} = M_i$$

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Bounding the Determinant

Let Z be the maximum absolute entry occurring in $\bar{A}$, $\bar{b}$ or c . Let C denote the matrix obtained from $\bar{A}_B$ by replacing the j -th column with vector $\bar{b}$ (for some j).

Observe that

$$|\det(C)|$$

Here $\text{sgn}(\pi)$ denotes the **sign** of the permutation, which is 1 if the permutation can be generated by an even number of transpositions (exchanging two elements), and -1 if the number of transpositions is odd. The first identity is known as Leibniz formula.

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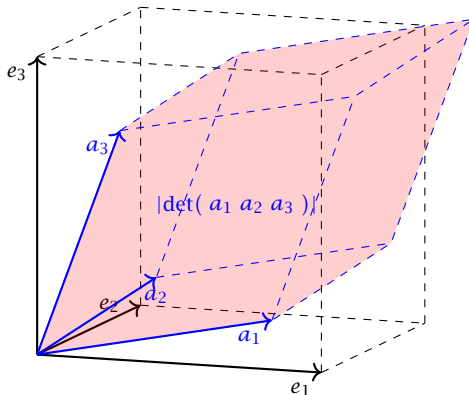
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$$\begin{aligned} |\det(C)| &\leq \prod_{i=1}^m \|C_{*i}\| \leq \prod_{i=1}^m (\sqrt{m}Z) \\ &\leq m^{m/2} Z^m . \end{aligned}$$

Hadamards Inequality



Hadamard's inequality says that the volume of the red parallelepiped (**Spat**) is smaller than the volume in the black cube (if $\|e_1\| = \|a_1\|$, $\|e_2\| = \|a_2\|$, $\|e_3\| = \|a_3\|$).

Ensuring Conditions

Given a **standard minimization LP**

$$\begin{array}{ll} \min & c^T x \\ \text{s.t.} & Ax \geq b \\ & x \geq 0 \end{array}$$

how can we obtain an LP of the required form?

- **Compute a lower bound on $c^T x$ for any basic feasible solution.** Add the constraint $c^T x \geq -dZ(m! \cdot Z^m) - 1$. **Note that this constraint is superfluous unless the LP is unbounded.**

Ensuring Conditions

Compute an optimum basis for the new LP.

- ▶ If the cost is $c^T x = -(dZ)(m! \cdot Z^m) - 1$ we know that the original LP is unbounded.
- ▶ Otw. we have an optimum basis.

In the following we use $\mathcal{H}$ to denote the set of all constraints apart from the constraint $c^T x \geq -dZ(m! \cdot Z^m) - 1$.

We give a routine $\text{SeidelLP}(\mathcal{H}, d)$ that is given a set $\mathcal{H}$ of **explicit, non-degenerate** constraints over d variables, and minimizes $c^T x$ over all feasible points.

In addition it obeys the implicit constraint $c^T x \geq -(dZ)(m! \cdot Z^m) - 1$.

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- 12: **if** $\hat{x}^* = \text{infeasible}$ **then**
- 13: **return** infeasible
- 14: **else**
- 15: add the value of x_ℓ to $\hat{x}^*$ and return the solution

8 Seidels LP-algorithm

Note that for the case $d = 1$, the asymptotic bound $\mathcal{O}(\max\{m, 1\})$ is valid also for the case $m = 0$.

- ▶ If $d = 1$ we can solve the 1-dimensional problem in time $\mathcal{O}(\max\{m, 1\})$.
- ▶ If $d > 1$ and $m = 0$ we take time $\mathcal{O}(d)$ to return d -dimensional vector x .
- ▶ The first recursive call takes time $T(m - 1, d)$ for the call plus $\mathcal{O}(d)$ for checking whether the solution fulfills h .
- ▶ If we are unlucky and $\hat{x}^*$ does not fulfill h we need time $\mathcal{O}(d(m + 1)) = \mathcal{O}(dm)$ to eliminate x_ℓ . Then we make a recursive call that takes time $T(m - 1, d - 1)$.
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This gives the recurrence

$$T(m, d) = \begin{cases} \mathcal{O}(\max\{1, m\}) & \text{if } d = 1 \\ \mathcal{O}(d) & \text{if } d > 1 \text{ and } m = 0 \\ \mathcal{O}(d) + T(m-1, d) + \frac{d}{m}(\mathcal{O}(dm) + T(m-1, d-1)) & \text{otw.} \end{cases}$$

Note that $T(m, d)$ denotes the **expected running time**.

8 Seidels LP-algorithm

Let C be the largest constant in the $\mathcal{O}$ -notations.

$$T(m, d) = \begin{cases} C \max\{1, m\} & \text{if } d = 1 \\ Cd & \text{if } d > 1 \text{ and } m = 0 \\ Cd + T(m - 1, d) + \frac{d}{m}(Cdm + T(m - 1, d - 1)) & \text{otw.} \end{cases}$$

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$$\begin{aligned} T(1, d) &= \mathcal{O}(d) + T(0, d) + d(\mathcal{O}(d) + T(0, d-1)) \\ &\leq Cd + Cd + Cd^2 + dCf(d-1) \end{aligned}$$

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$d > 1; m = 1$:

$$\begin{aligned} T(1, d) &= \mathcal{O}(d) + T(0, d) + d(\mathcal{O}(d) + T(0, d-1)) \\ &\leq Cd + Cd + Cd^2 + dCf(d-1) \\ &\leq Cf(d) \max\{1, m\} \end{aligned}$$

8 Seidels LP-algorithm

Let C be the largest constant in the $\mathcal{O}$ -notations.

We show $T(m, d) \leq Cf(d) \max\{1, m\}$.

$d = 1$:

$$T(m, 1) \leq C \max\{1, m\} \leq Cf(1) \max\{1, m\} \text{ for } f(1) \geq 1$$

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if $f(d) \geq df(d-1) + 2d^2$.

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since $\sum_{i \geq 1} \frac{i^2}{i!}$ is a constant.

$$\sum_{i \geq 1} \frac{i^2}{i!} = \sum_{i \geq 0} \frac{i+1}{i!} = e + \sum_{i \geq 1} \frac{i}{i!} = 2e$$

Complexity

LP Feasibility Problem (LP feasibility A)

Given $A \in \mathbb{Z}^{m \times n}$, $b \in \mathbb{Z}^m$. Does there exist $x \in \mathbb{R}^n$ with $Ax \leq b$, $x \geq 0$?

LP Feasibility Problem (LP feasibility B)

Given $A \in \mathbb{Z}^{m \times n}$, $b \in \mathbb{Z}^m$. Find $x \in \mathbb{R}^n$ with $Ax \leq b$, $x \geq 0$!

LP Optimization A

Given $A \in \mathbb{Z}^{m \times n}$, $b \in \mathbb{Z}^m$, $c \in \mathbb{Z}^n$. What is the maximum value of $c^T x$ for a feasible point $x \in \mathbb{R}^n$?

LP Optimization B

Given $A \in \mathbb{Z}^{m \times n}$, $b \in \mathbb{Z}^m$, $c \in \mathbb{Z}^n$. Return feasible point $x \in \mathbb{R}^n$ with maximum value of $c^T x$?

Note that allowing A, b to contain rational numbers does not make a difference, as we can multiply every number by a suitable large constant so that everything becomes integral but the **feasible region** does not change.

The Bit Model

Input size

- ▶ The number of bits to represent a number $a \in \mathbb{Z}$ is

$$\lceil \log_2(|a|) \rceil + 1$$

- ▶ Let for an $m \times n$ matrix M , $L(M)$ denote the number of bits required to encode all the numbers in M .

$$\langle M \rangle := \sum_{i,j} \lceil \log_2(|m_{ij}|) \rceil + 1$$

- ▶ In the following we assume that input matrices are encoded in a standard way, where each number is encoded in binary and then suitable separators are added in order to separate distinct number from each other.
- ▶ Then the input length is $L = \Theta(\langle A \rangle + \langle b \rangle)$.

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- ▶ In the following we sometimes refer to $L := \langle A \rangle + \langle b \rangle$ as the input size (even though the real input size is something in $\Theta(\langle A \rangle + \langle b \rangle)$).
- ▶ Sometimes we may also refer to $L := \langle A \rangle + \langle b \rangle + n \log_2 n$ as the input size. Note that $n \log_2 n = \Theta(\langle A \rangle + \langle b \rangle)$.
- ▶ In order to show that LP-decision is in NP we show that if there is a solution x then there exists a small solution for which feasibility can be verified in polynomial time (polynomial in L).

Note that $m \log_2 m$ may be much larger than $\langle A \rangle + \langle b \rangle$.

Suppose that $\bar{A}x = b$; $x \geq 0$ is feasible.

Then there exists a basic feasible solution. This means a set B of basic variables such that

$$x_B = \bar{A}_B^{-1}b$$

and all other entries in x are 0.

In the following we show that this x has small encoding length and we give an explicit bound on this length. So far we have only been handwaving and have said that we can compute x via Gaussian elimination and it will be short...

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Size of a Basic Feasible Solution

Note that n in the theorem denotes the number of columns in A which may be much smaller than m .

- ▶ A : original input matrix
- ▶ $\bar{A}$: transformation of A into standard form
- ▶ $\bar{A}_B$: submatrix of $\bar{A}$ corresponding to basis B

Lemma 47

Let $\bar{A}_B \in \mathbb{Z}^{m \times m}$ and $b \in \mathbb{Z}^m$. Define $L = \langle A \rangle + \langle b \rangle + n \log_2 n$. Then a solution to $\bar{A}_B x_B = b$ has rational components x_j of the form $\frac{D_j}{D}$, where $|D_j| \leq 2^L$ and $|D| \leq 2^L$.

Proof:

Cramers rules says that we can compute x_j as

$$x_j = \frac{\det(\bar{A}_B^j)}{\det(\bar{A}_B)}$$

where $\bar{A}_B^j$ is the matrix obtained from $\bar{A}_B$ by replacing the j -th column by the vector b .

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Bounding the Determinant

Let $X = \bar{A}_B$. Then

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Analogously for $\det(A_B^j)$.

When computing the determinant of $X = \bar{A}_B$ we first do expansions along columns that were introduced when transforming A into standard form, i.e., into $\bar{A}$.

Such a column contains a single 1 and the remaining entries of the column are 0. Therefore, these expansions do not increase the absolute value of the determinant. After we did expansions for all these columns we are left with a square sub-matrix of A of size at most $n \times n$.

Reducing LP-solving to LP decision.

Given an LP $\max\{c^T x \mid Ax \leq b; x \geq 0\}$ do a **binary search** for the optimum solution

(Add constraint $c^T x \geq M$). Then checking for feasibility shows whether optimum solution is larger or smaller than M).

If the LP is feasible then the binary search finishes in at most

$$\log_2 \left(\frac{2n2^{2L'}}{1/2^{L'}} \right) = \mathcal{O}(L') ,$$

as the range of the search is at most $-n2^{2L'}, \dots, n2^{2L'}$ and the distance between two adjacent values is at least $\frac{1}{\det(A)} \geq \frac{1}{2^{L'}}$.

Here we use $L' = \langle A \rangle + \langle b \rangle + \langle c \rangle + n \log_2 n$ (it also includes the encoding size of c).

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How do we detect whether the LP is unbounded?

Let $M_{\max} = n2^{2L'}$ be an upper bound on the objective value of a basic feasible solution.

We can add a constraint $c^T x \geq M_{\max} + 1$ and check for feasibility.

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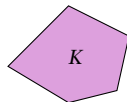
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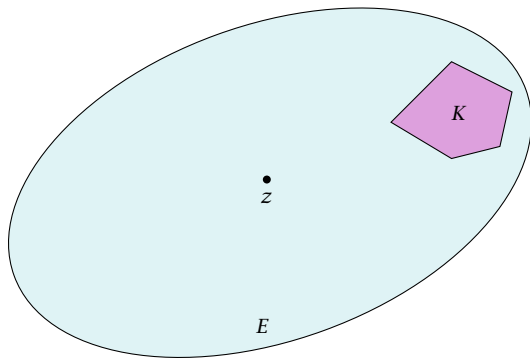
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- ▶ Let K be a convex set.



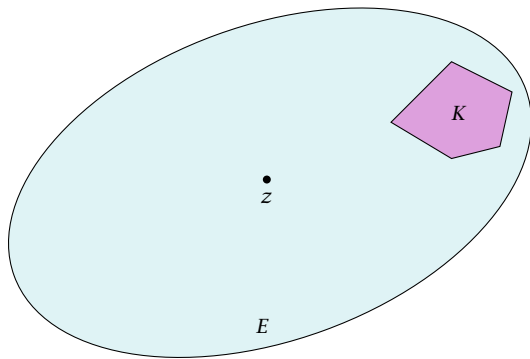
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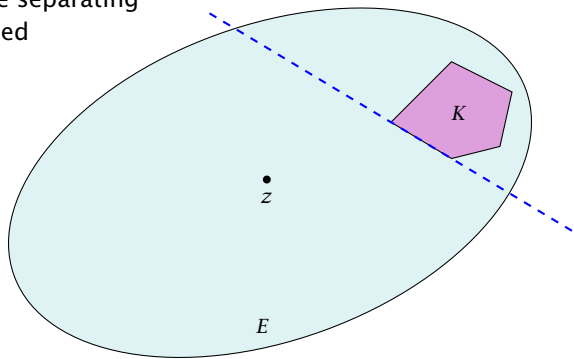
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- ▶ Let K be a convex set.
- ▶ Maintain ellipsoid E that is guaranteed to contain K provided that K is non-empty.
- ▶ If center $z \in K$ STOP.



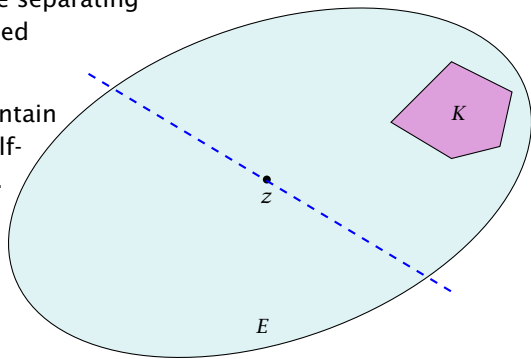
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- ▶ Maintain ellipsoid E that is guaranteed to contain K provided that K is non-empty.
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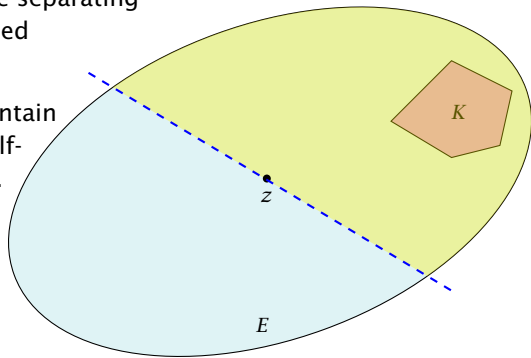
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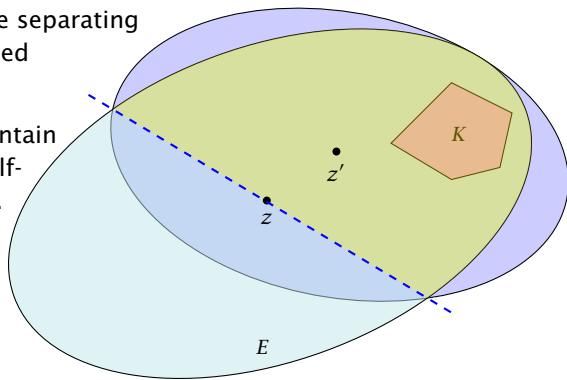
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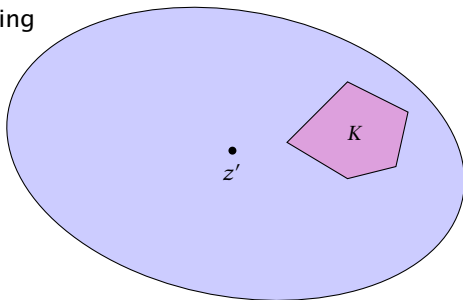
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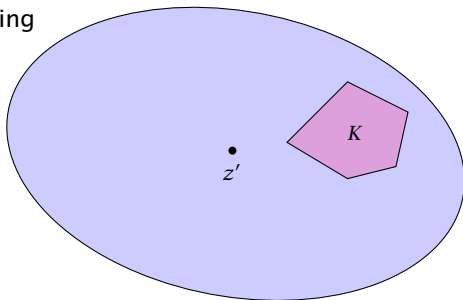
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- ▶ REPEAT



Issues/Questions:

- ▶ How do you choose the first Ellipsoid? What is its volume?
- ▶ How do you measure progress? By how much does the volume decrease in each iteration?
- ▶ When can you stop? What is the minimum volume of a non-empty polytop?

Definition 48

A mapping $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ with $f(x) = Lx + t$, where L is an invertible matrix is called an **affine transformation**.

Definition 49

A ball in $\mathbb{R}^n$ with center c and radius r is given by

$$\begin{aligned} B(c, r) &= \{x \mid (x - c)^T (x - c) \leq r^2\} \\ &= \{x \mid \sum_i (x - c)_i^2 / r^2 \leq 1\} \end{aligned}$$

$B(0, 1)$ is called the **unit ball**.

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An affine transformation of the unit ball is called an **ellipsoid**.

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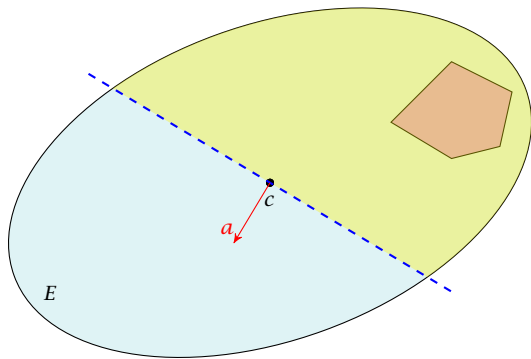
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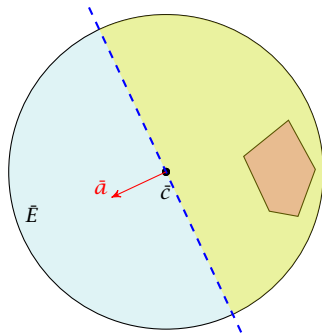
where $Q = LL^T$ is an invertible matrix.

How to Compute the New Ellipsoid



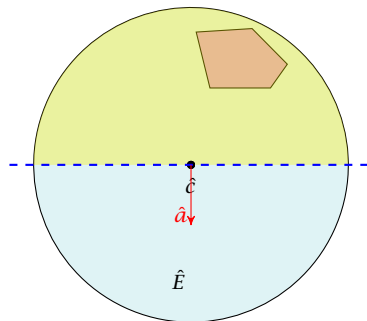
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- Use f^{-1} (recall that $f = Lx + t$ is the affine transformation of the unit ball) to rotate/distort the ellipsoid (back) into the unit ball.



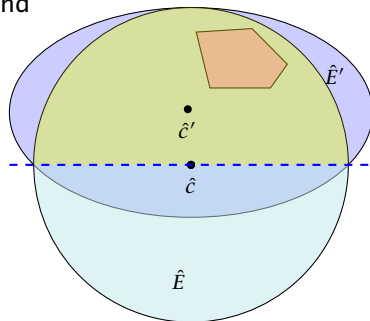
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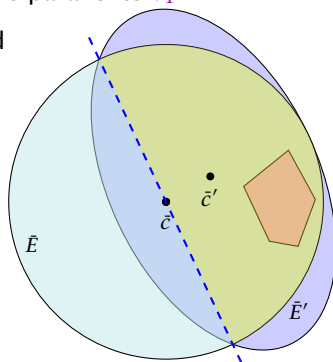
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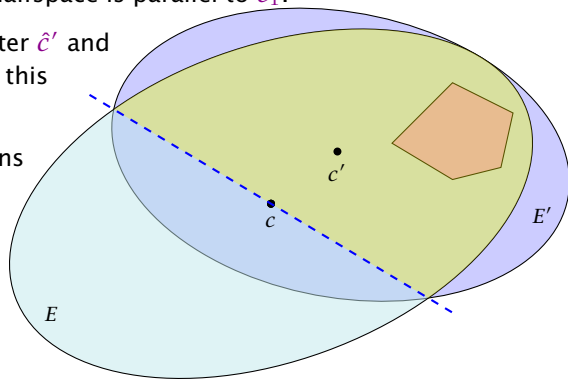
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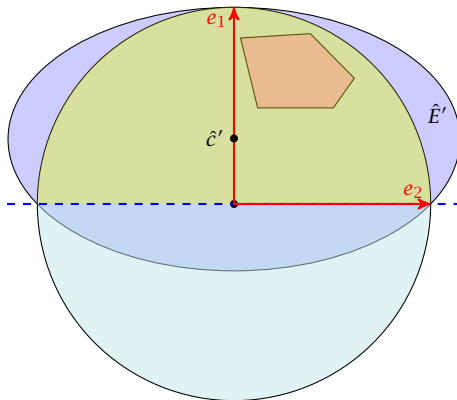


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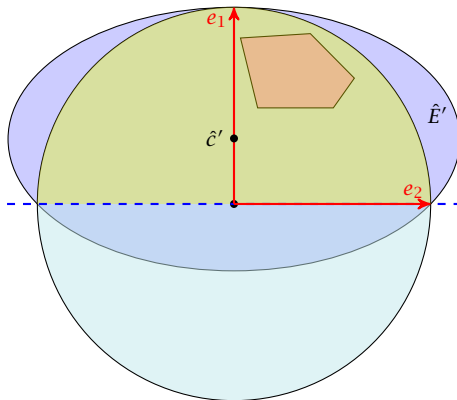


The Easy Case



- ▶ The new center lies on axis x_1 . Hence, $\hat{c}' = te_1$ for $t > 0$.
- ▶ The vectors $e_1, e_2, \dots$ have to fulfill the ellipsoid constraint with equality. Hence $(e_i - \hat{c}')^T \hat{Q}'^{-1} (e_i - \hat{c}') = 1$.

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- ▶ To obtain the matrix $\hat{Q}'^{-1}$ for our ellipsoid $\hat{E}'$ note that $\hat{E}'$ is **axis-parallel**.
- ▶ Let a denote the radius along the x_1 -axis and let b denote the (common) radius for the other axes.
- ▶ The matrix

$$\hat{L}' = \begin{pmatrix} a & 0 & \dots & 0 \\ 0 & b & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & b \end{pmatrix}$$

maps the unit ball (via function $\hat{f}'(x) = \hat{L}'x$) to an axis-parallel ellipsoid with radius a in direction x_1 and b in all other directions.

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The Easy Case

- As $\hat{Q}' = \hat{L}'\hat{L}'^t$ the matrix $\hat{Q}'^{-1}$ is of the form

$$\hat{Q}'^{-1} = \begin{pmatrix} \frac{1}{a^2} & 0 & \dots & 0 \\ 0 & \frac{1}{b^2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \frac{1}{b^2} \end{pmatrix}$$

The Easy Case

- ▶ $(e_1 - \hat{c}')^T \hat{Q}'^{-1} (e_1 - \hat{c}') = 1$ gives

$$\begin{pmatrix} 1-t \\ 0 \\ \vdots \\ 0 \end{pmatrix}^T \cdot \begin{pmatrix} \frac{1}{a^2} & 0 & \dots & 0 \\ 0 & \frac{1}{b^2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \frac{1}{b^2} \end{pmatrix} \cdot \begin{pmatrix} 1-t \\ 0 \\ \vdots \\ 0 \end{pmatrix} = 1$$

- ▶ This gives $(1-t)^2 = a^2$.

The Easy Case

- ▶ For $i \neq 1$ the equation $(e_i - \hat{c}')^T \hat{Q}'^{-1} (e_i - \hat{c}') = 1$ looks like (here $i = 2$)

$$\begin{pmatrix} -t \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}^T \cdot \begin{pmatrix} \frac{1}{a^2} & 0 & \dots & 0 \\ 0 & \frac{1}{b^2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \frac{1}{b^2} \end{pmatrix} \cdot \begin{pmatrix} -t \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = 1$$

- ▶ This gives $\frac{t^2}{a^2} + \frac{1}{b^2} = 1$, and hence

$$\frac{1}{b^2} = 1 - \frac{t^2}{a^2}$$

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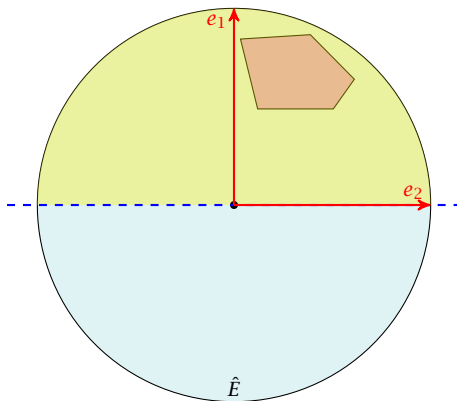
Summary

So far we have

$$a = 1 - t \quad \text{and} \quad b = \frac{1 - t}{\sqrt{1 - 2t}}$$

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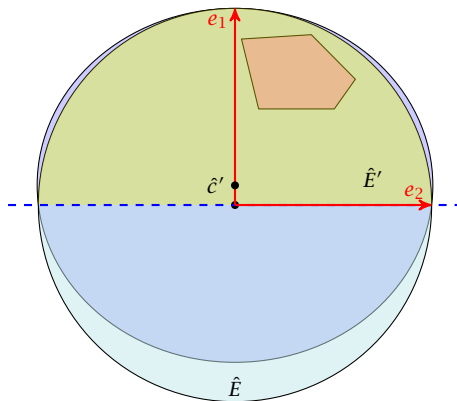
We still have many choices for t :



Choose t such that the volume of $\hat{E}'$ is minimal!!!

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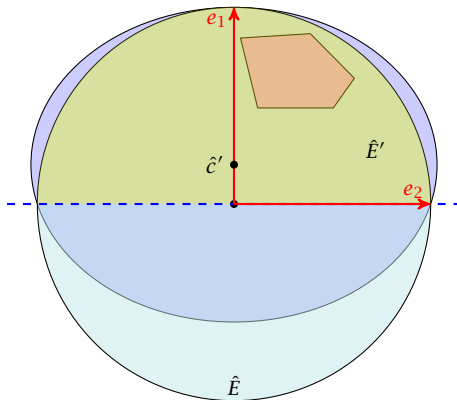
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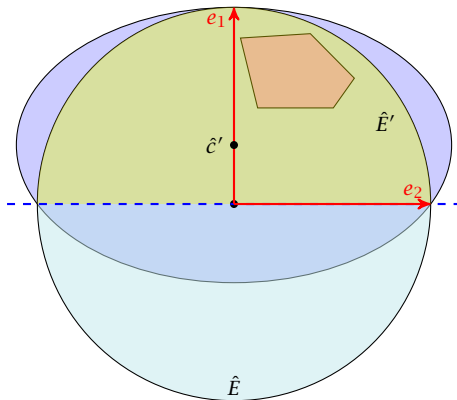
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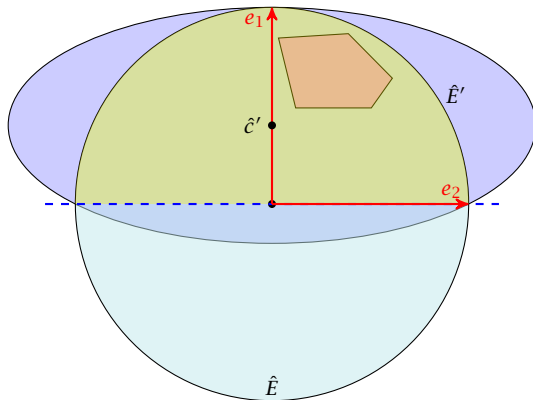
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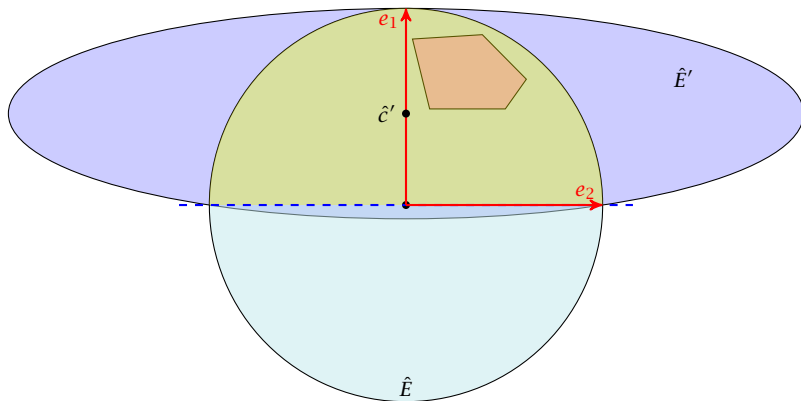
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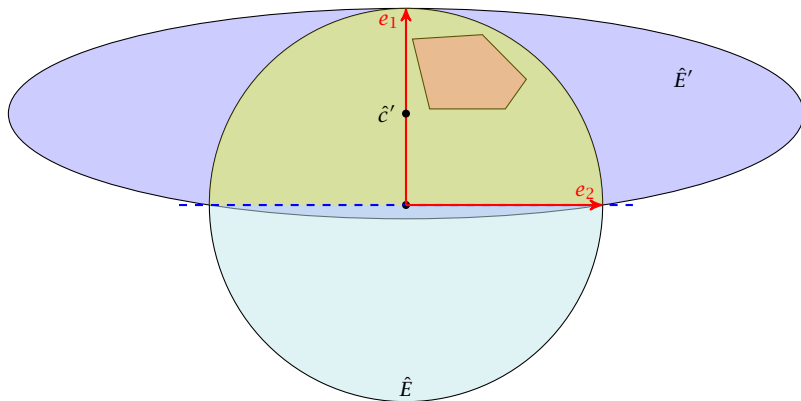
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We want to choose t such that the volume of $\hat{E}'$ is minimal.

Lemma 51

Let L be an affine transformation and $K \subseteq \mathbb{R}^n$. Then

$$\text{vol}(L(K)) = |\det(L)| \cdot \text{vol}(K) .$$

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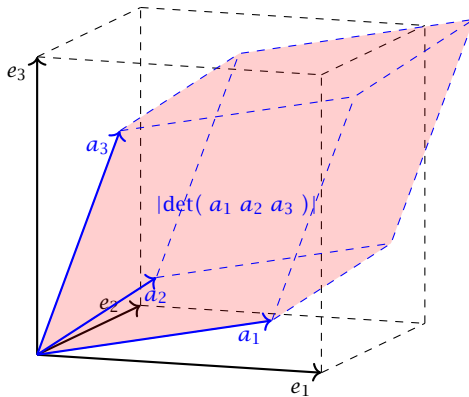
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n-dimensional volume



The Easy Case

- ▶ We want to choose t such that the volume of $\hat{E}'$ is minimal.

$$\text{vol}(\hat{E}') = \text{vol}(B(0, 1)) \cdot |\det(\hat{L}')| ,$$

- ▶ Recall that

$$\hat{L}' = \begin{pmatrix} a & 0 & \dots & 0 \\ 0 & b & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & b \end{pmatrix}$$

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We use the shortcut $\Phi := \text{vol}(B(0, 1))$.

The Easy Case

$$\frac{d \operatorname{vol}(\hat{E}')}{d t}$$

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$$\begin{aligned}\frac{d \operatorname{vol}(\hat{E}')}{d t} &= \frac{d}{d t} \left(\Phi \frac{(1-t)^n}{(\sqrt{1-2t})^{n-1}} \right) \\ &= \frac{\Phi}{N^2}\end{aligned}$$

$N = \text{denominator}$

The Easy Case

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denominator

The Easy Case

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The Easy Case

$$\begin{aligned}\frac{d \operatorname{vol}(\hat{E}')}{d t} &= \frac{d}{d t} \left(\Phi \frac{(1-t)^n}{(\sqrt{1-2t})^{n-1}} \right) \\ &= \frac{\Phi}{N^2} \cdot \left((-1) \cdot n(1-t)^{n-1} \cdot (\sqrt{1-2t})^{n-1} \right. \\ &\quad \left. - (n-1)(\sqrt{1-2t})^{n-2} \cdot \frac{1}{2\sqrt{1-2t}} \cdot (-2) \right)\end{aligned}$$

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 &= \frac{\Phi}{N^2} \cdot (\sqrt{1-2t})^{n-3} \cdot (1-t)^{n-1} \cdot \left((n+1)t - 1 \right)
 \end{aligned}$$

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Let $\gamma_n = \frac{\text{vol}(\hat{E}')}{\text{vol}(B(0,1))} = ab^{n-1}$ be the ratio by which the volume changes:

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where we used $(1+x)^a \leq e^{ax}$ for $x \in \mathbb{R}$ and $a > 0$.

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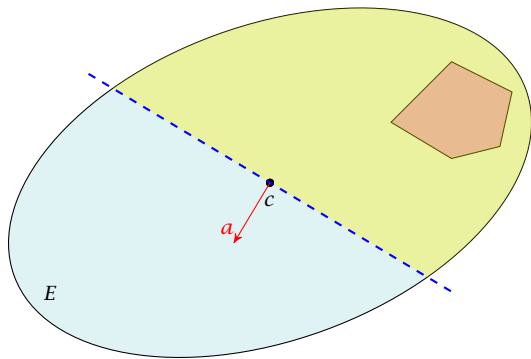
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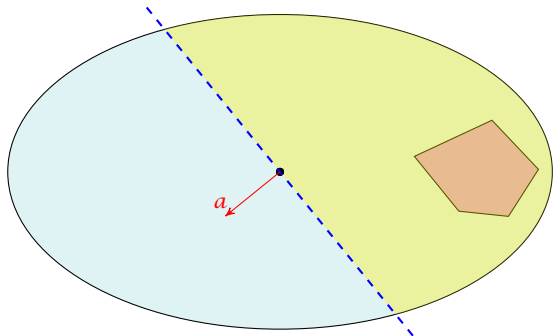
This gives $y_n \leq e^{-\frac{1}{2(n+1)}}$.

How to Compute the New Ellipsoid



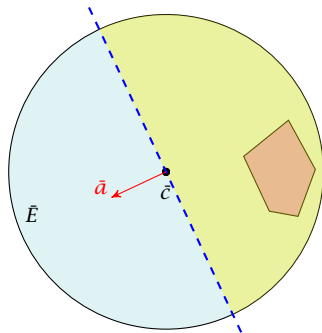
How to Compute the New Ellipsoid

- Use f^{-1} (recall that $f = Lx + t$ is the affine transformation of the unit ball) to translate/distort the ellipsoid (back) into the unit ball.



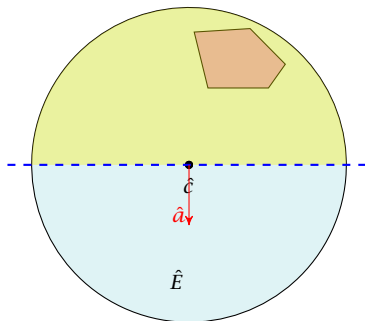
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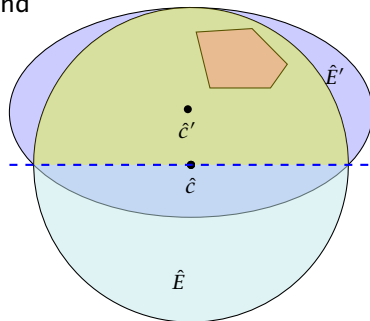
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- ▶ Use f^{-1} (recall that $f = Lx + t$ is the affine transformation of the unit ball) to translate/distort the ellipsoid (back) into the unit ball.
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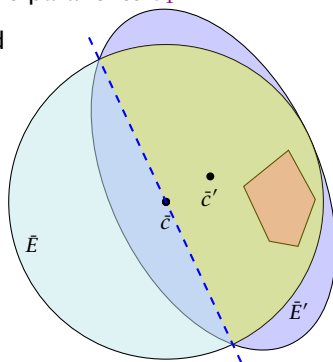
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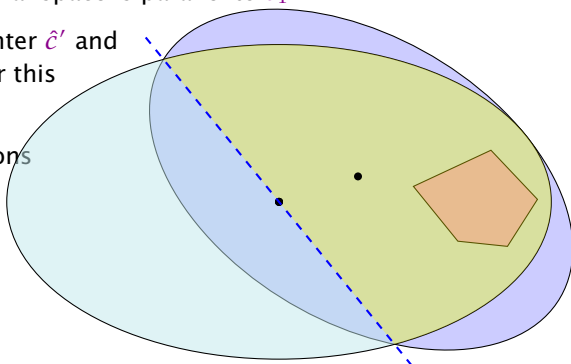
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- ▶ Use the transformations R and f to get the new center c' and the new matrix Q' for the original ellipsoid E .



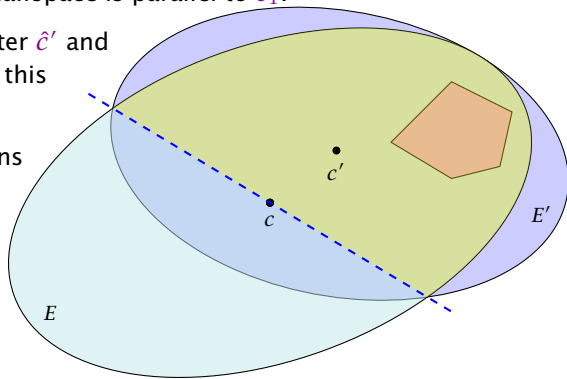
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Here it is important that mapping a set with affine function $f(x) = Lx + t$ changes the volume by factor $\det(L)$.

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$$\begin{aligned} f^{-1}(H) &= \{f^{-1}(x) \mid a^T(x - c) \leq 0\} \\ &= \{f^{-1}(f(y)) \mid a^T(f(y) - c) \leq 0\} \\ &= \{y \mid a^T(f(y) - c) \leq 0\} \\ &= \{y \mid a^T(Ly + c - c) \leq 0\} \\ &= \{y \mid (a^T L)y \leq 0\} \end{aligned}$$

This means $\bar{a} = L^T a$.

The center $\bar{c}$ is of course at the origin.

The Ellipsoid Algorithm

After rotating back (applying R^{-1}) the normal vector of the halfspace points in negative x_1 -direction. Hence,

$$R^{-1} \left(\frac{L^T a}{\|L^T a\|} \right) = -e_1 \quad \Rightarrow \quad -\frac{L^T a}{\|L^T a\|} = R \cdot e_1$$

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Hence,

$$\bar{c}' = R \cdot \hat{c}' = R \cdot \frac{1}{n+1} e_1$$

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The Ellipsoid Algorithm

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$$\begin{aligned} c' &= f(\tilde{c}') = L \cdot \tilde{c}' + c \\ &= -\frac{1}{n+1} L \frac{L^T a}{\|L^T a\|} + c \\ &= c - \frac{1}{n+1} \frac{Qa}{\sqrt{a^T Qa}} \end{aligned}$$

For computing the matrix Q' of the new ellipsoid we assume in the following that $\hat{E}', \bar{E}'$ and E' refer to the ellipsoids centered in the origin.

Recall that

$$\hat{Q}' = \begin{pmatrix} a^2 & 0 & \dots & 0 \\ 0 & b^2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & b^2 \end{pmatrix}$$

This gives

$$\hat{Q}' = \frac{n^2}{n^2 - 1} \left(I - \frac{2}{n + 1} e_1 e_1^T \right)$$

Note that $e_1 e_1^T$ is a matrix M that has $M_{11} = 1$ and all other entries equal to 0.

because for $a^2 = n^2/(n+1)^2$ and $b^2 = n^2/n^2-1$

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$$\begin{aligned} b^2 - b^2 \frac{2}{n + 1} &= \frac{n^2}{n^2 - 1} - \frac{2n^2}{(n - 1)(n + 1)^2} \\ &= \frac{n^2(n + 1) - 2n^2}{(n - 1)(n + 1)^2} = \frac{n^2(n - 1)}{(n - 1)(n + 1)^2} = a^2 \end{aligned}$$

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9 The Ellipsoid Algorithm

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$$\begin{aligned}\bar{E}' &= R(\hat{E}') \\ &= \{R(x) \mid x^T \hat{Q}'^{-1} x \leq 1\}\end{aligned}$$

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9 The Ellipsoid Algorithm

Hence,

$$\bar{Q}'$$

Here we used the equation for Re_1 proved before, and the fact that $RR^T = I$, which holds for any rotation matrix. To see this observe that the length of a rotated vector x should not change, i.e.,

$$x^T I x = (Rx)^T (Rx) = x^T (R^T R) x$$

which means $x^T (I - R^T R) x = 0$ for every vector x . It is easy to see that this can only be fulfilled if $I - R^T R = 0$.

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Incomplete Algorithm

Algorithm 1 ellipsoid-algorithm

```
1: input: point  $c \in \mathbb{R}^n$ , convex set  $K \subseteq \mathbb{R}^n$ 
2: output: point  $x \in K$  or “ $K$  is empty”
3:  $Q \leftarrow ???$ 
4: repeat
5:     if  $c \in K$  then return  $c$ 
6:     else
7:         choose a violated hyperplane  $a$ 
8:         
$$c \leftarrow c - \frac{1}{n+1} \frac{Qa}{\sqrt{a^T Q a}}$$

9:         
$$Q \leftarrow \frac{n^2}{n^2 - 1} \left( Q - \frac{2}{n+1} \frac{Qaa^T Q}{a^T Q a} \right)$$

10:    endif
11: until  $???$ 
12: return “ $K$  is empty”
```

Repeat: Size of basic solutions

Lemma 52

Let $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$ be a bounded polyhedron. Let $L := 2\langle A \rangle + \langle b \rangle + 2n(1 + \log_2 n)$. Then every entry x_j in a basic solution fulfills $|x_j| = \frac{D_j}{D}$ with $D_j, D \leq 2^L$.

In the following we use $\delta := 2^L$.

Proof:

We can replace P by $P' := \{x \mid A'x \leq b; x \geq 0\}$ where $A' = \begin{bmatrix} A \\ -A \end{bmatrix}$. The lemma follows by applying Lemma 47, and observing that $\langle A' \rangle = 2\langle A \rangle$ and $n' = 2n$.

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How do we find the first ellipsoid?

For feasibility checking we can assume that the polytop P is bounded; it is sufficient to consider basic solutions.

Every entry x_i in a basic solution fulfills $|x_i| \leq \delta$.

Hence, P is contained in the cube $-\delta \leq x_i \leq \delta$.

A vector in this cube has at most distance $R := \sqrt{n}\delta$ from the origin.

Starting with the ball $E_0 := B(0, R)$ ensures that P is completely contained in the initial ellipsoid. This ellipsoid has volume at most $R^n \text{vol}(B(0, 1)) \leq (n\delta)^n \text{vol}(B(0, 1))$.

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When can we terminate?

Let $P := \{x \mid Ax \leq b\}$ with $A \in \mathbb{Z}$ and $b \in \mathbb{Z}$ be a bounded polytop.

Consider the following polyhedron

$$P_\lambda := \left\{ x \mid Ax \leq b + \frac{1}{\lambda} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \right\},$$

where $\lambda = \delta^2 + 1$.

Note that the volume of P_λ cannot be 0

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Note that the volume of P_λ cannot be 0

When can we terminate?

Let $P := \{x \mid Ax \leq b\}$ with $A \in \mathbb{Z}$ and $b \in \mathbb{Z}$ be a bounded polytop.

Consider the following polyhedron

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Making P full-dimensional

Lemma 53

P_λ is feasible if and only if P is feasible.

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$\Rightarrow$:

Consider the polyhedrons

$$\bar{P} = \{x \mid [A \ -A \ I_m]x = b; x \geq 0\}$$

and

$$\bar{P}_\lambda = \left\{x \mid [A \ -A \ I_m]x = b + \frac{1}{\lambda} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}; x \geq 0\right\}.$$

P is feasible if and only if $\bar{P}$ is feasible, and P_λ feasible if and only if $\bar{P}_\lambda$ feasible.

$\bar{P}_\lambda$ is bounded since P_λ and P are bounded.

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$\bar{P}_\lambda$ is bounded since P_λ and P are bounded.

Making P full-dimensional

Let $\bar{A} = \begin{bmatrix} A & -A & I_m \end{bmatrix}$.

$\bar{P}_\lambda$ feasible implies that there is a basic feasible solution represented by

$$x_B = \bar{A}_B^{-1}b + \frac{1}{\lambda}\bar{A}_B^{-1}\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

(The other x -values are zero)

The only reason that this basic feasible solution is not feasible for $\bar{P}$ is that one of the basic variables becomes negative.

Hence, there exists i with

$$(\bar{A}_B^{-1}b)_i < 0 \leq (\bar{A}_B^{-1}b)_i + \frac{1}{\lambda}(\bar{A}_B^{-1}\bar{1})_i$$

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By Cramers rule we get

$$(\bar{A}_B^{-1}b)_i < 0 \quad \Rightarrow \quad (\bar{A}_B^{-1}b)_i \leq -\frac{1}{\det(\bar{A}_B)} \leq -1/\delta$$

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But then

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Lemma 54

If P_λ is feasible then it contains a ball of radius $r := 1/\delta^3$. This has a volume of at least $r^n \text{vol}(B(0, 1)) = \frac{1}{\delta^{3n}} \text{vol}(B(0, 1))$.

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Hence, $x + \vec{\ell}$ is feasible for P_λ which proves the lemma.

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Algorithm 1 ellipsoid-algorithm

```
1: input: point  $c \in \mathbb{R}^n$ , convex set  $K \subseteq \mathbb{R}^n$ , radii  $R$  and  $r$   
2:         with  $K \subseteq B(c, R)$ , and  $B(x, r) \subseteq K$  for some  $x$   
3: output: point  $x \in K$  or “ $K$  is empty”  
4:  $Q \leftarrow \text{diag}(R^2, \dots, R^2)$  // i.e.,  $L = \text{diag}(R, \dots, R)$   
5: repeat  
6:     if  $c \in K$  then return  $c$   
7:     else  
8:         choose a violated hyperplane  $a$   
9:         
$$c \leftarrow c - \frac{1}{n+1} \frac{Qa}{\sqrt{a^T Q a}}$$
  
10:        
$$Q \leftarrow \frac{n^2}{n^2 - 1} \left( Q - \frac{2}{n+1} \frac{Qaa^T Q}{a^T Q a} \right)$$
  
11:     endif  
12: until  $\det(Q) \leq r^{2n}$  // i.e.,  $\det(L) \leq r^n$   
13: return “ $K$  is empty”
```

Separation Oracle

Let $K \subseteq \mathbb{R}^n$ be a convex set. A separation oracle for K is an algorithm A that gets as input a point $x \in \mathbb{R}^n$ and either

- ▶ certifies that $x \in K$,
- ▶ or finds a hyperplane separating x from K .

We will usually assume that A is a polynomial-time algorithm.

In order to find a point in K we need

- ▶ a guarantee that a ball of radius r is contained in K , and
- ▶ an initial ball $B(x, R)$ with radius R that contains K , and a separation oracle for K .

The Ellipsoid algorithm requires $\mathcal{O}(\text{poly}(n) \cdot \log(R/r))$ iterations. Each iteration is polytime for a polynomial-time Separation oracle.

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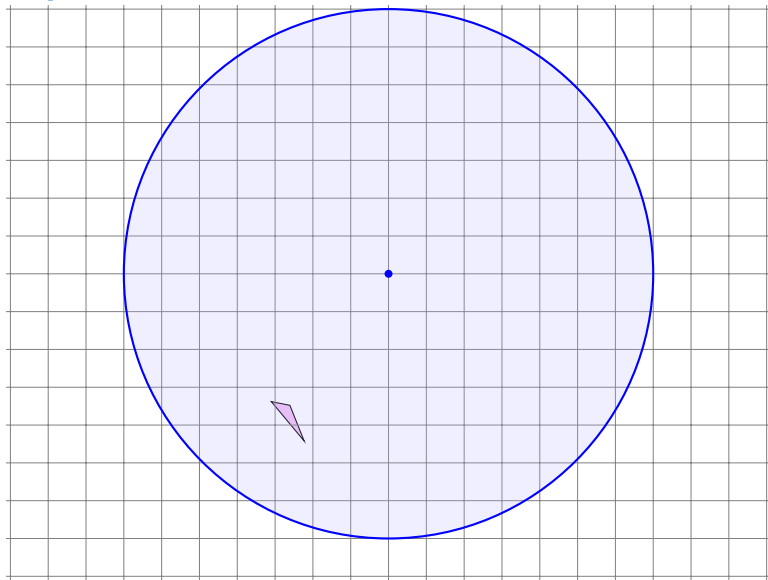
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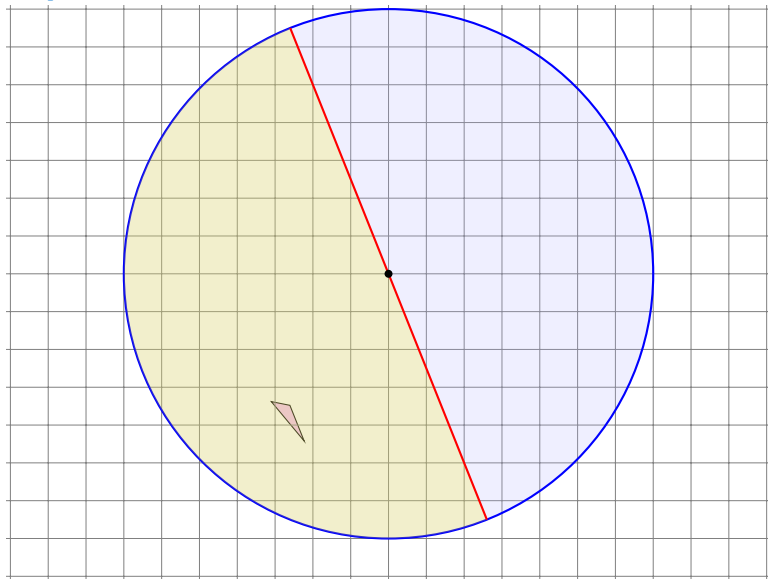
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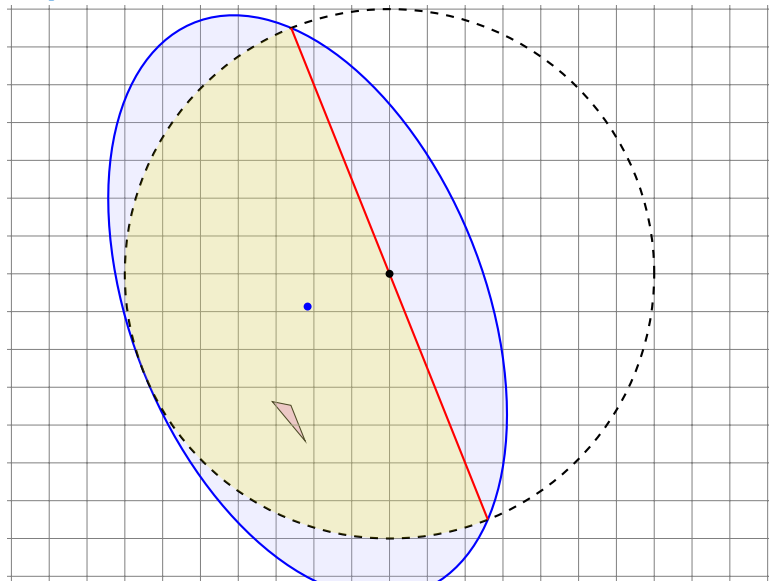
Example



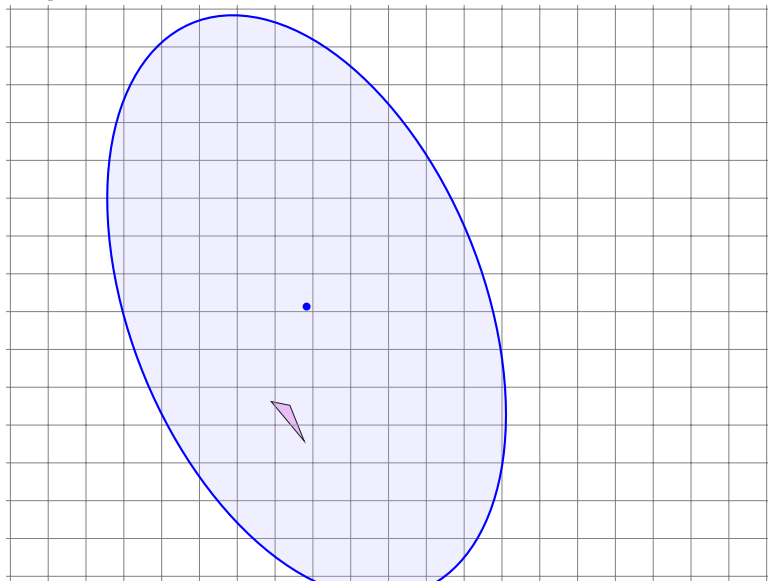
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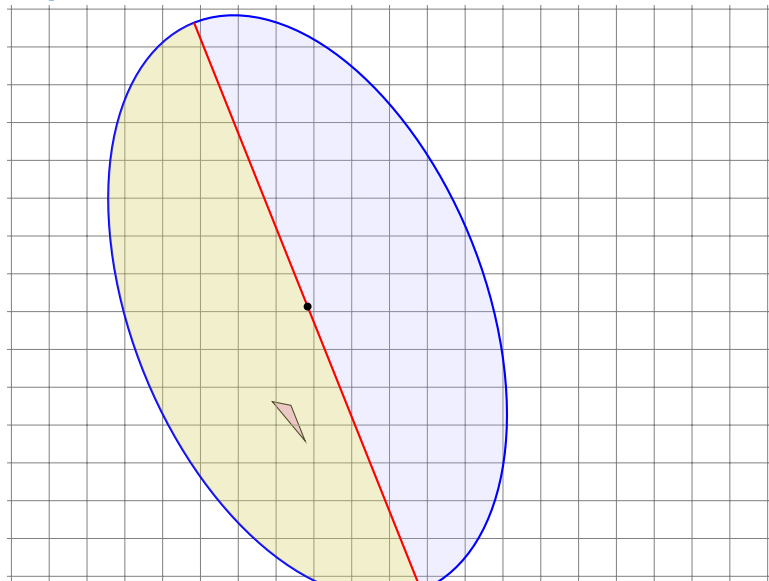
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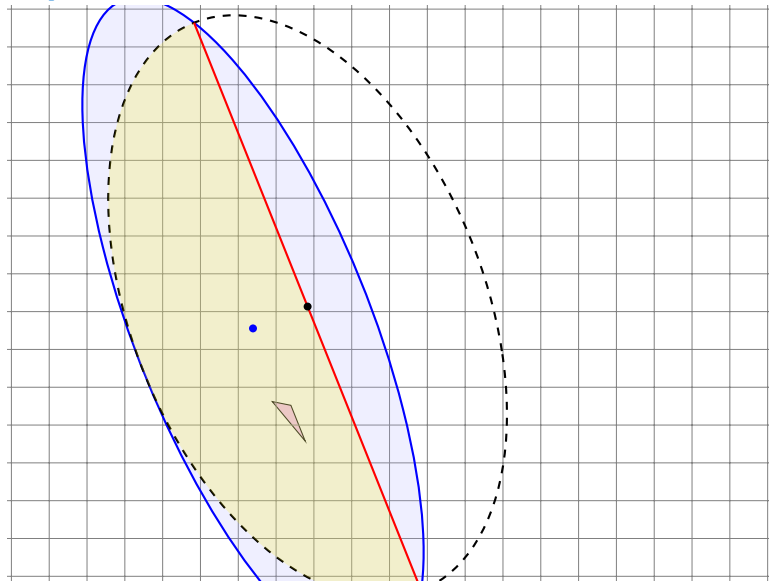
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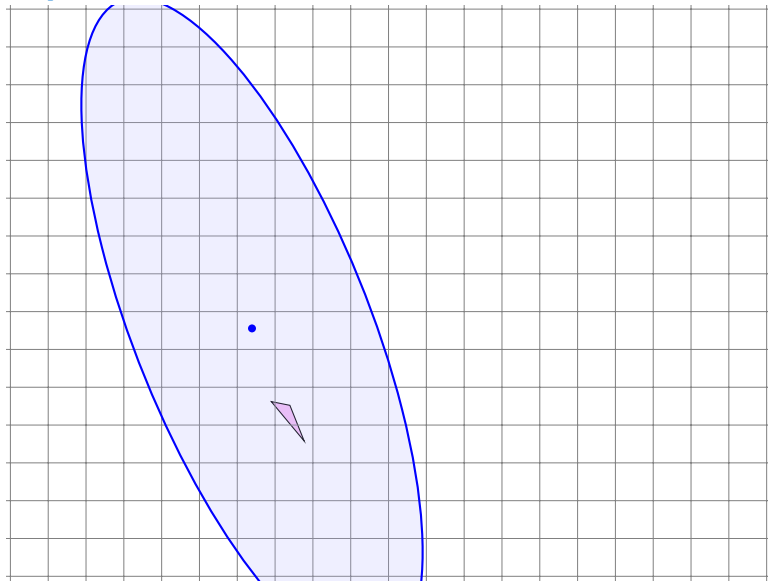
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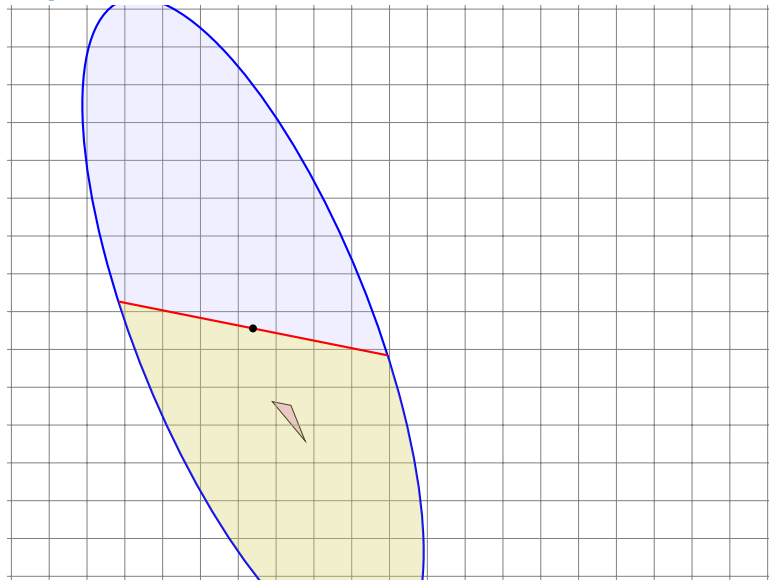
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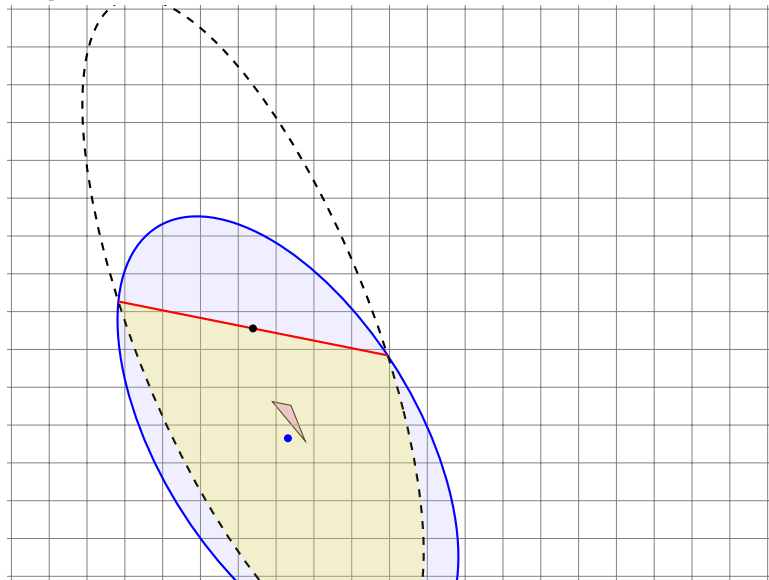
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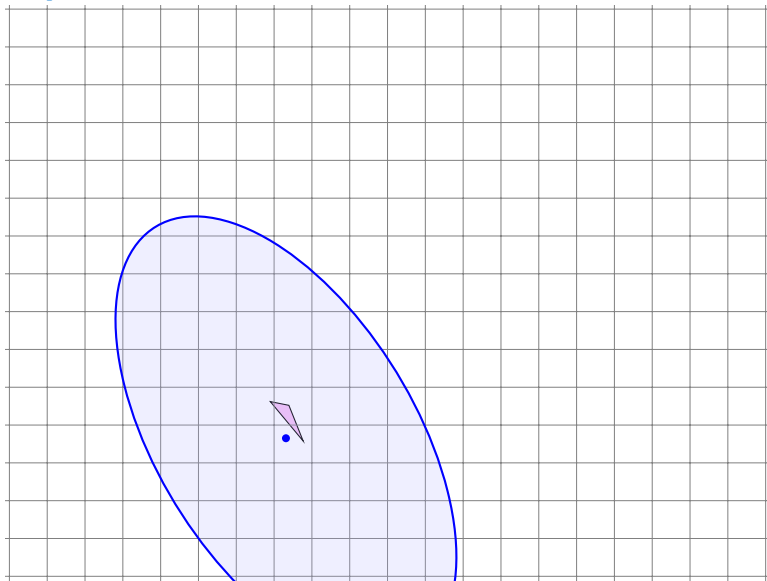
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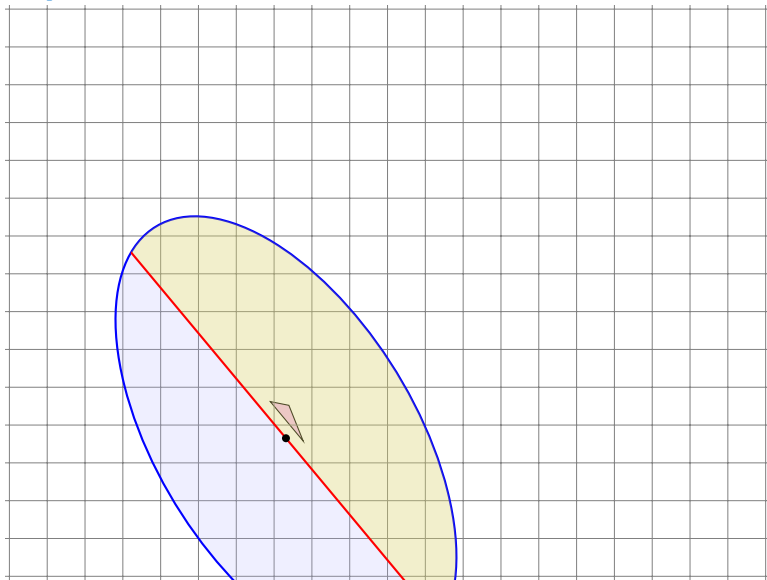
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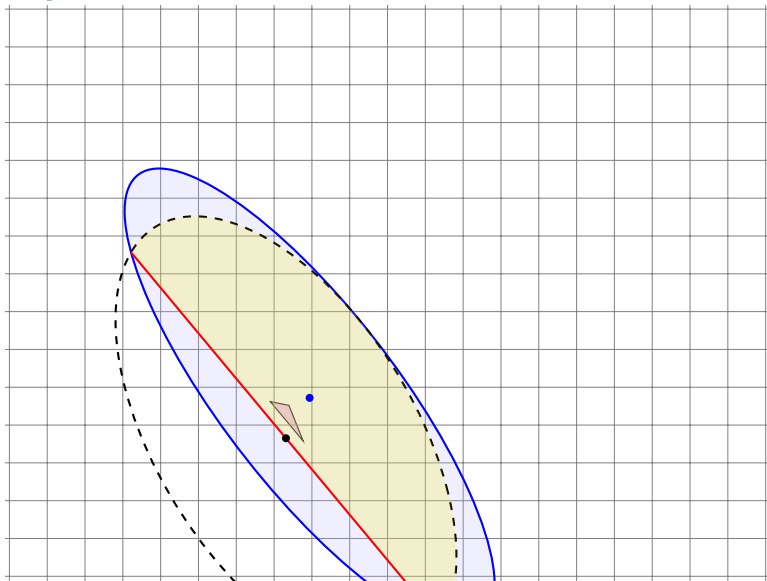
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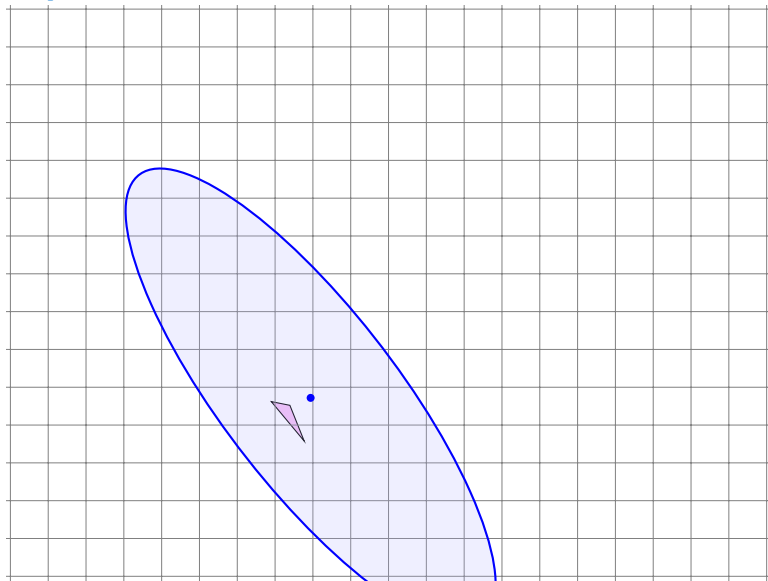
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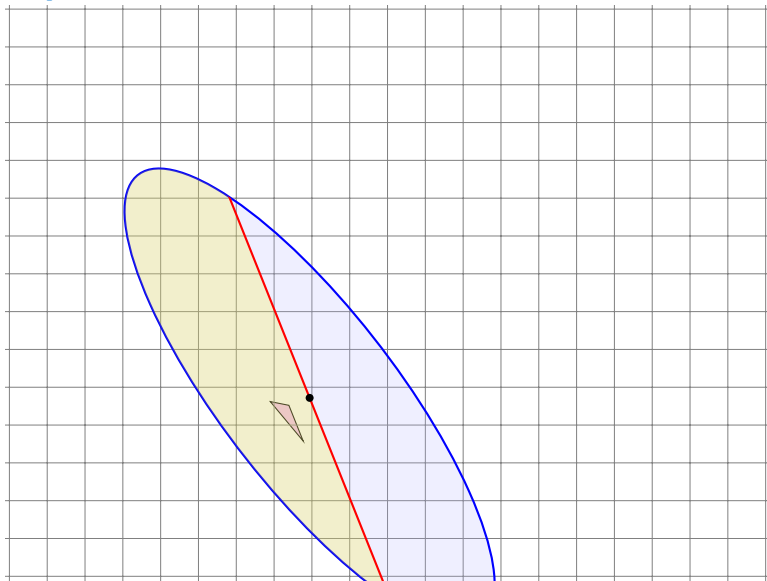
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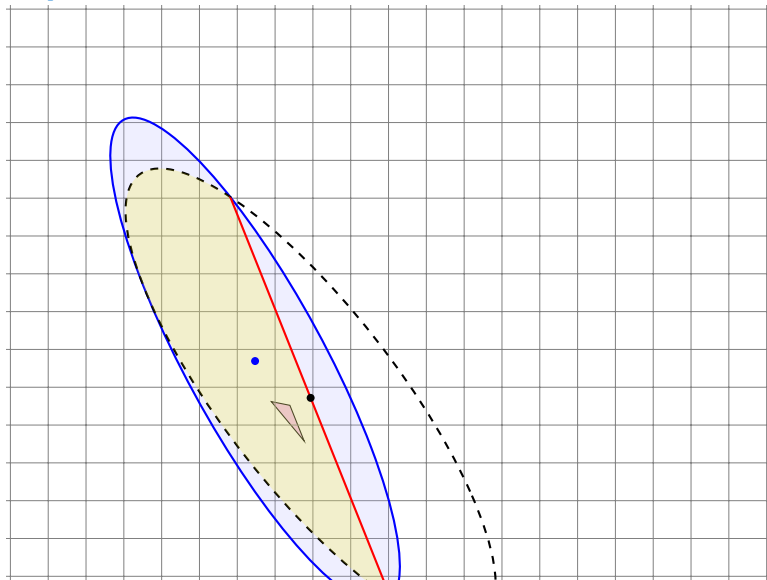
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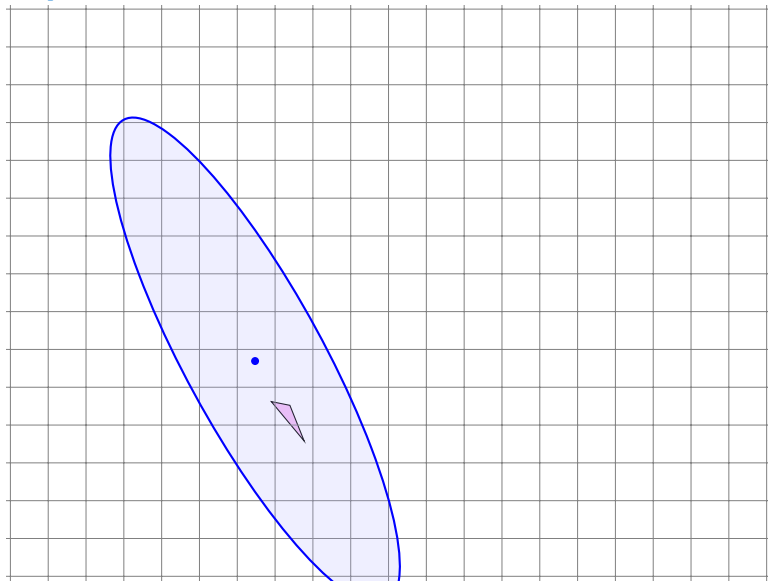
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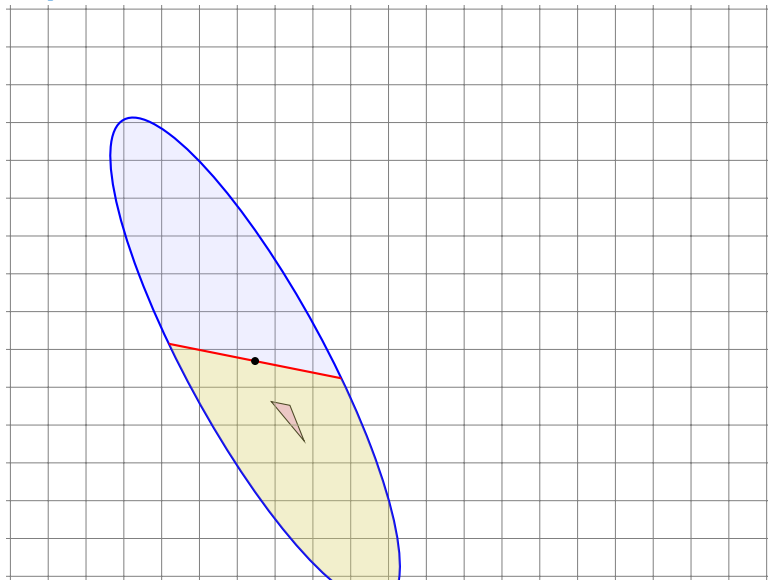
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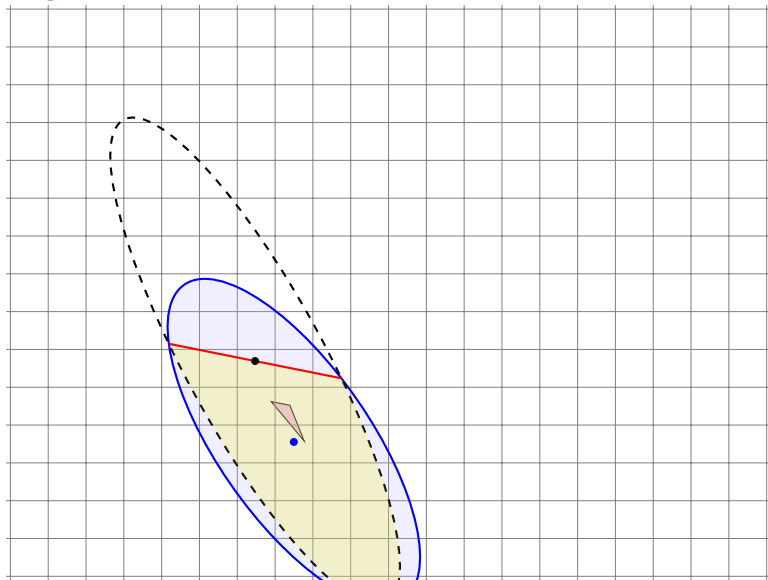
Example



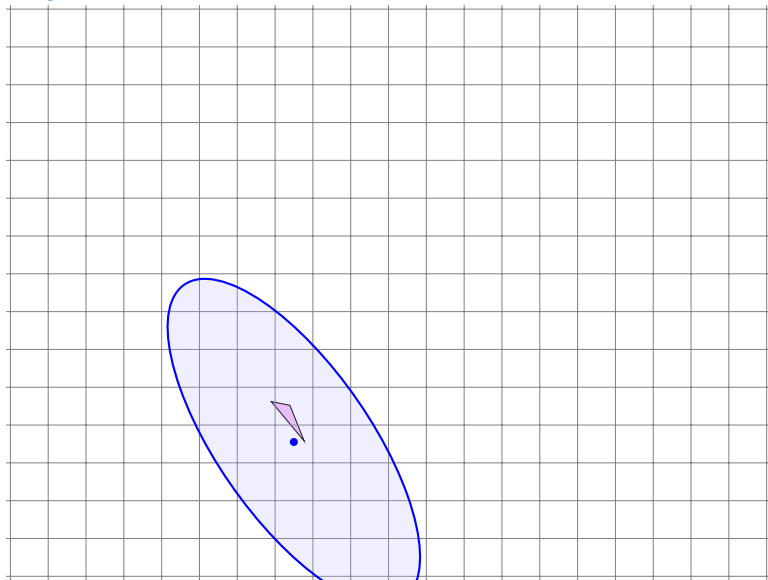
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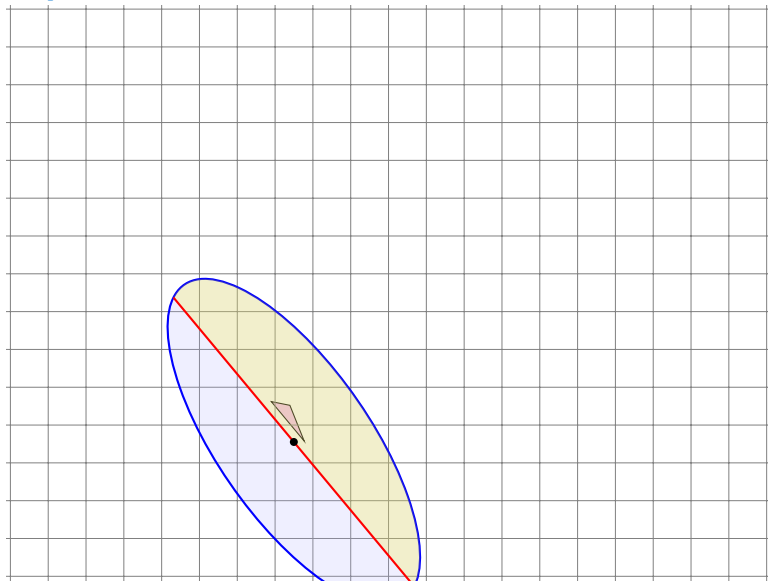
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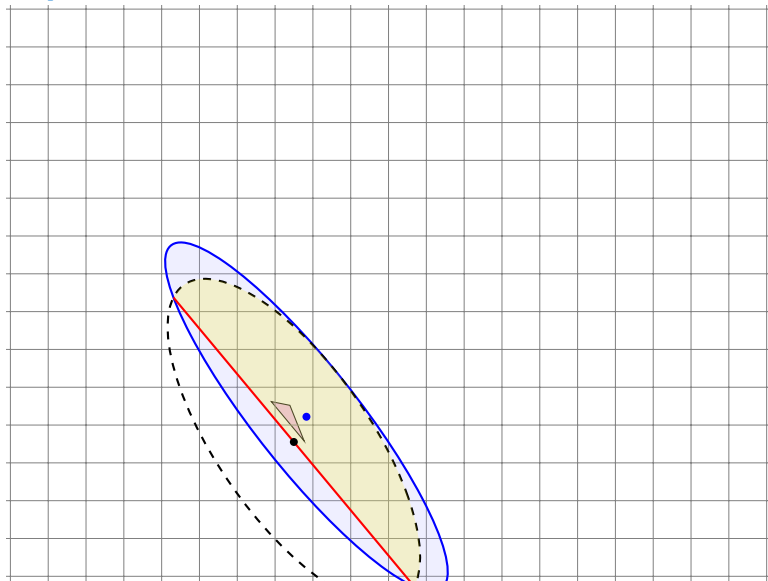
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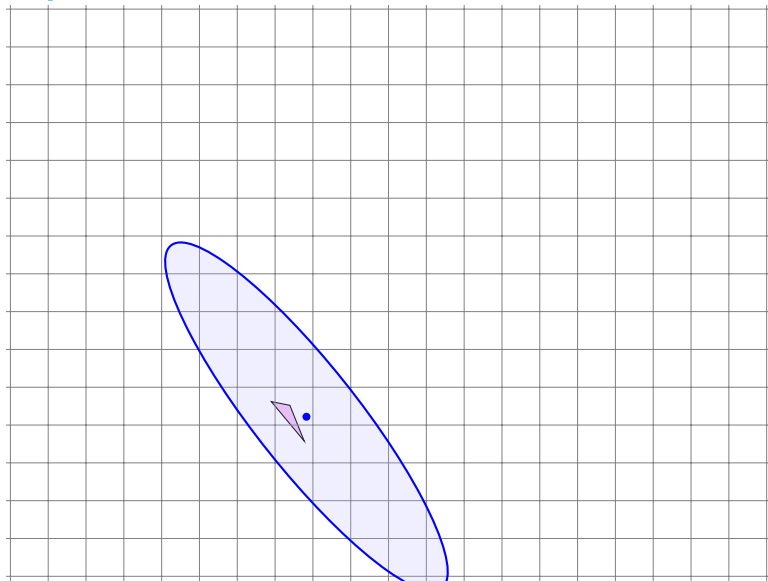
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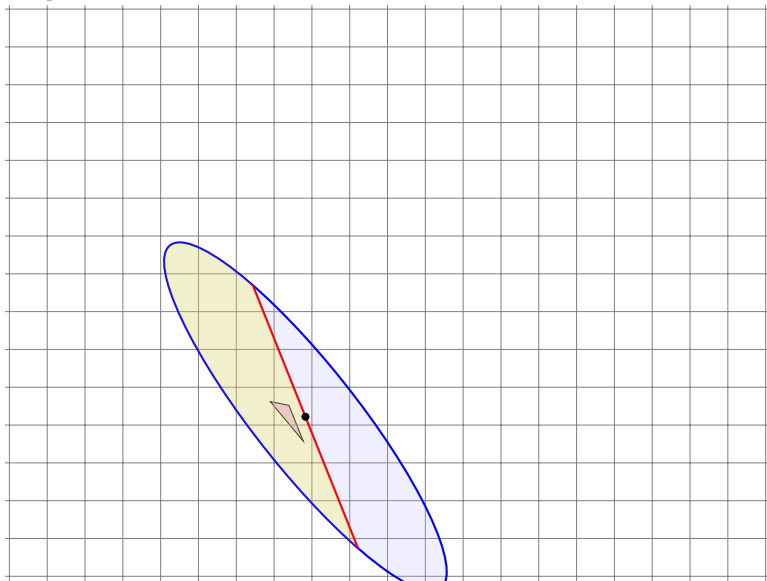
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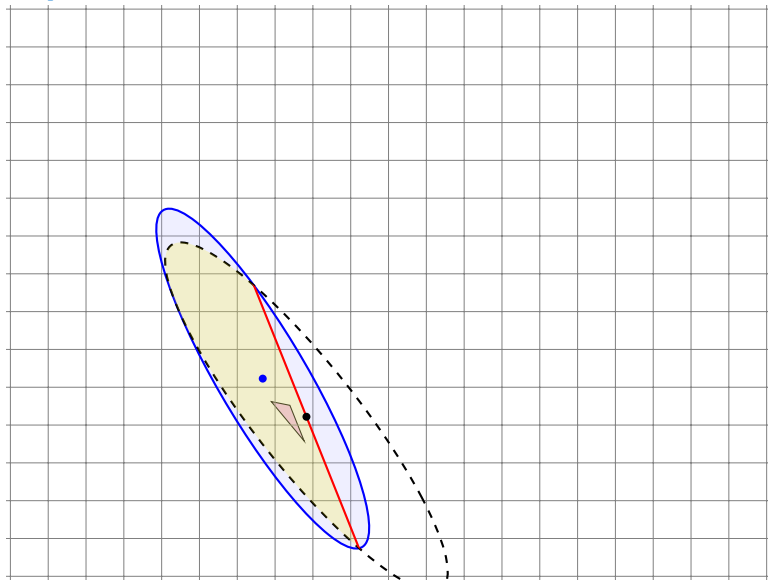
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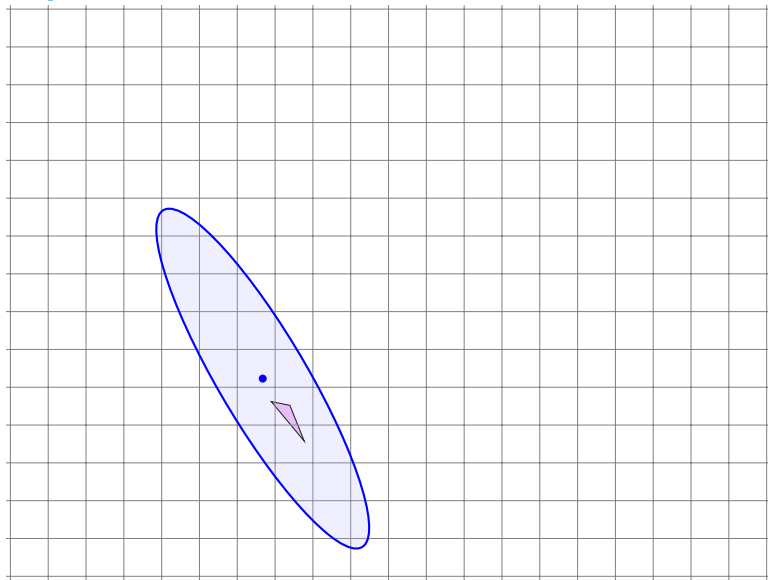
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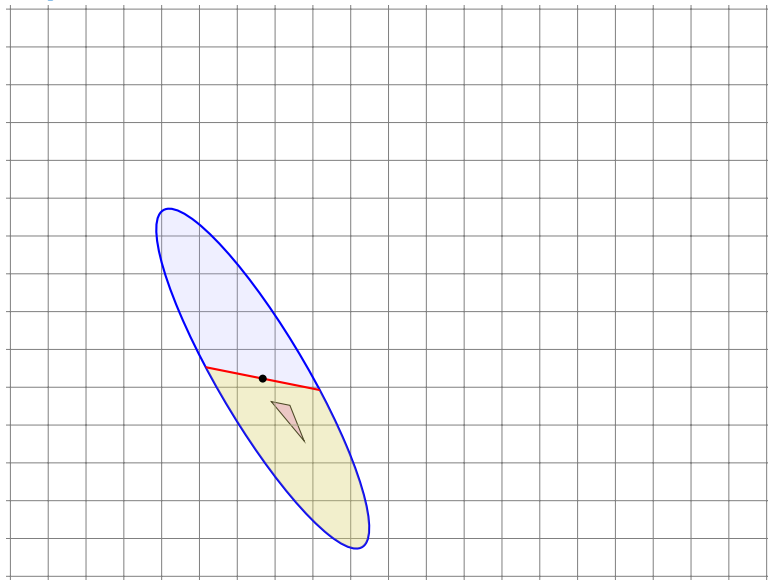
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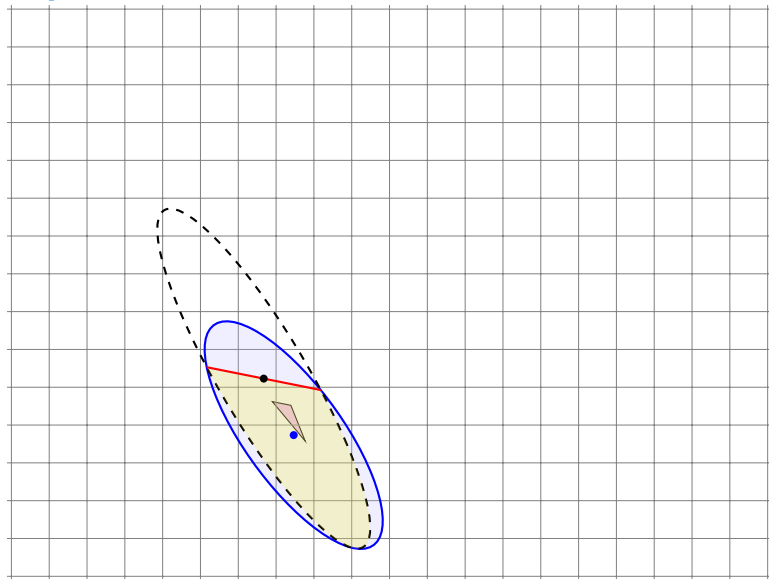
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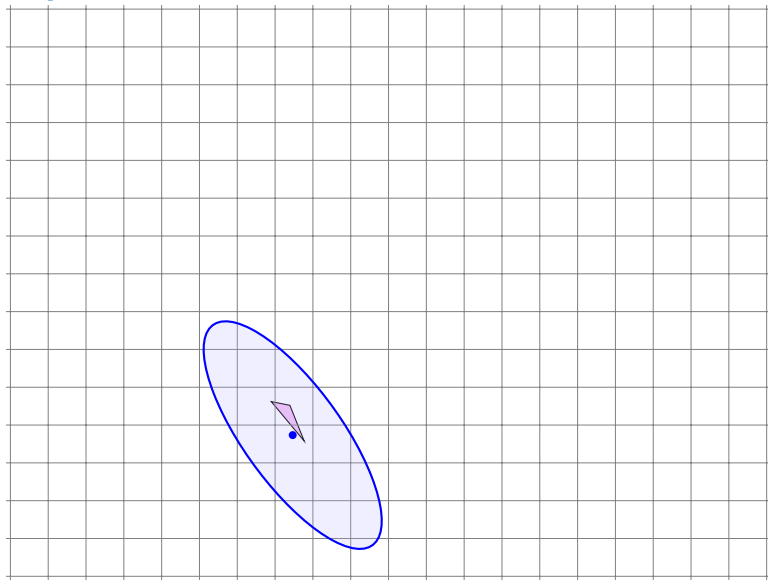
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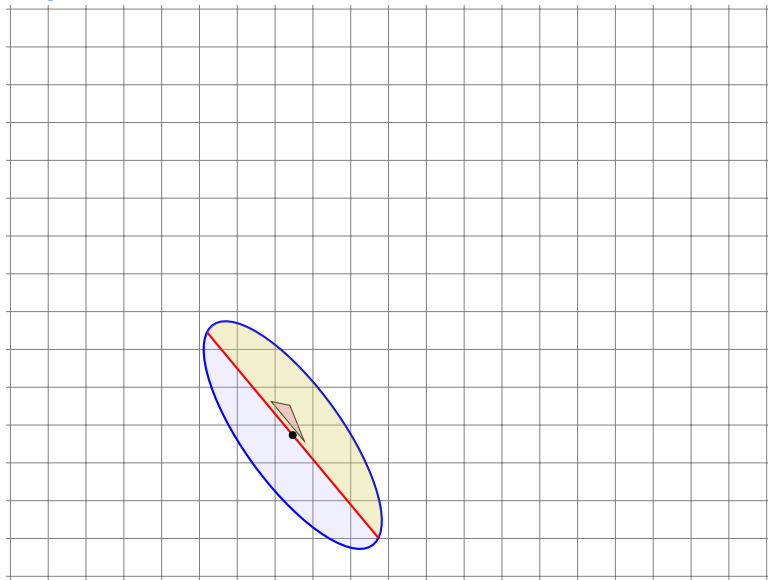
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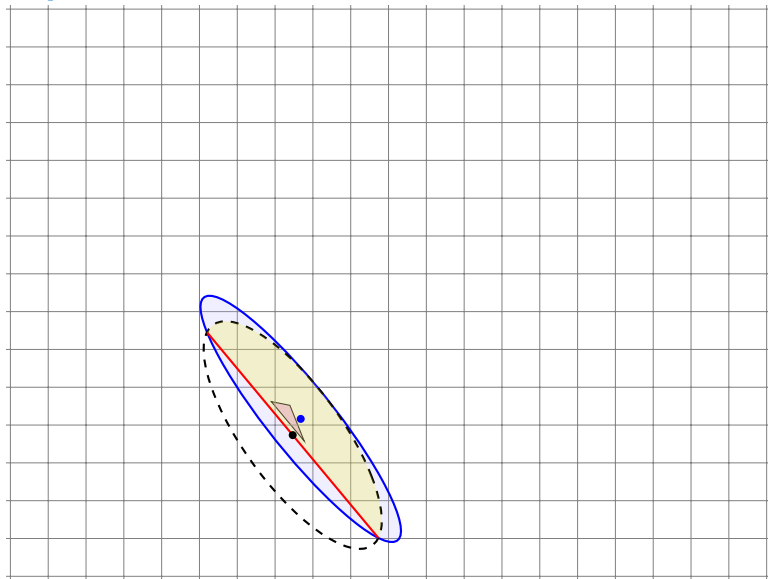
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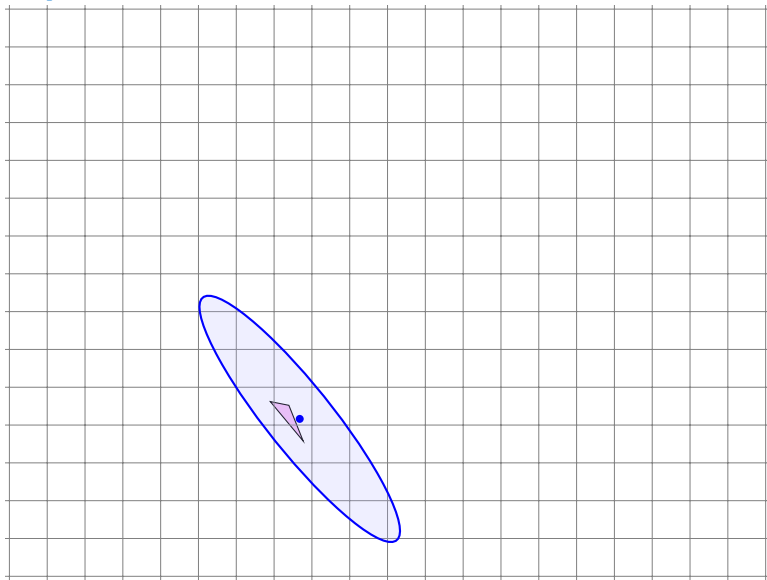
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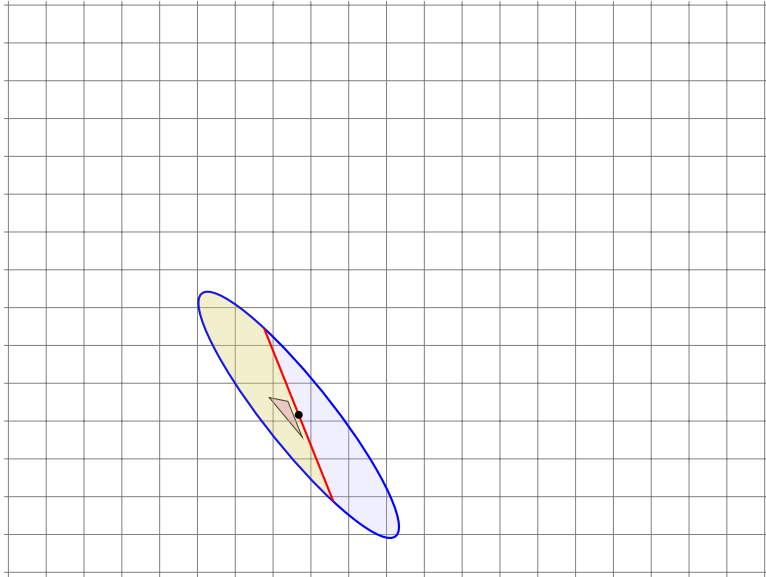
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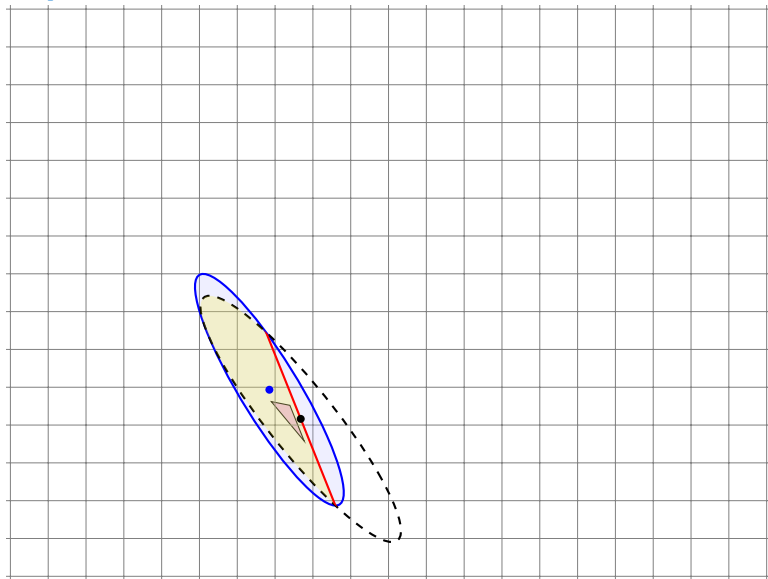
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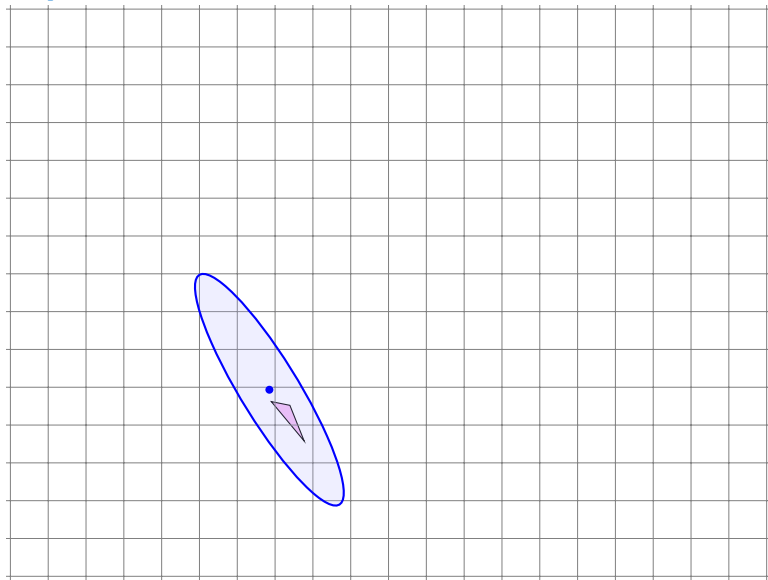
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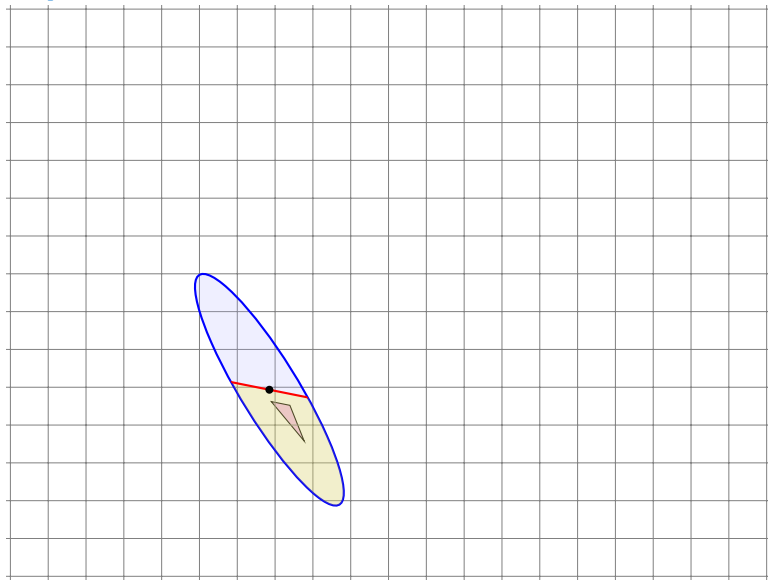
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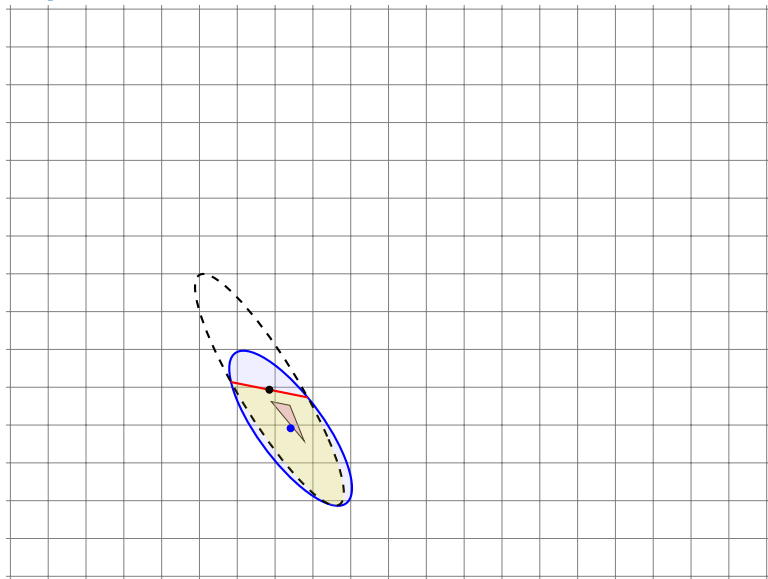
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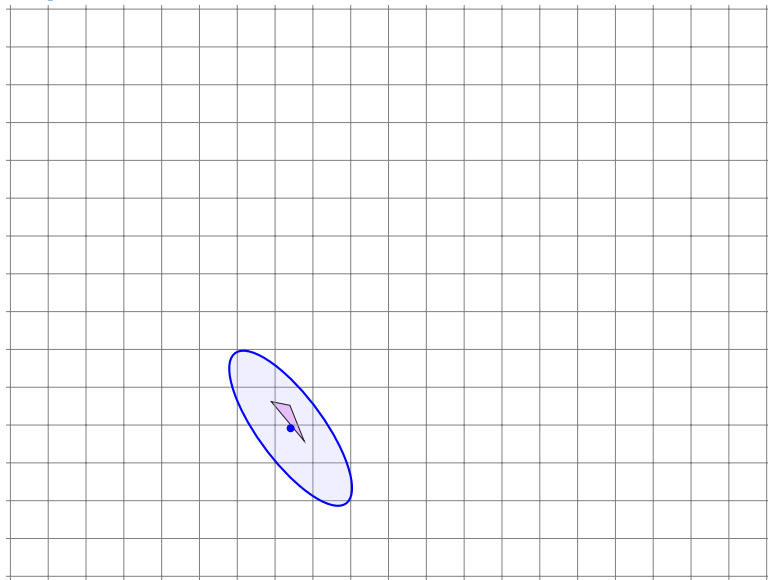
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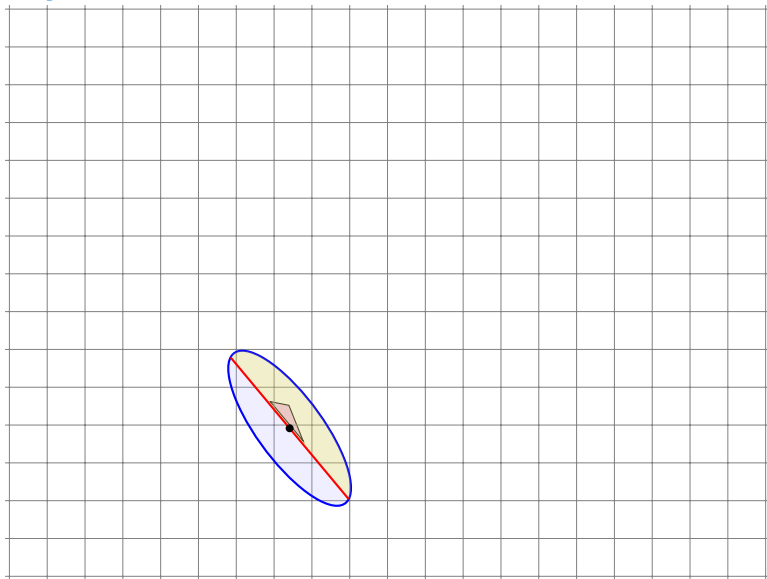
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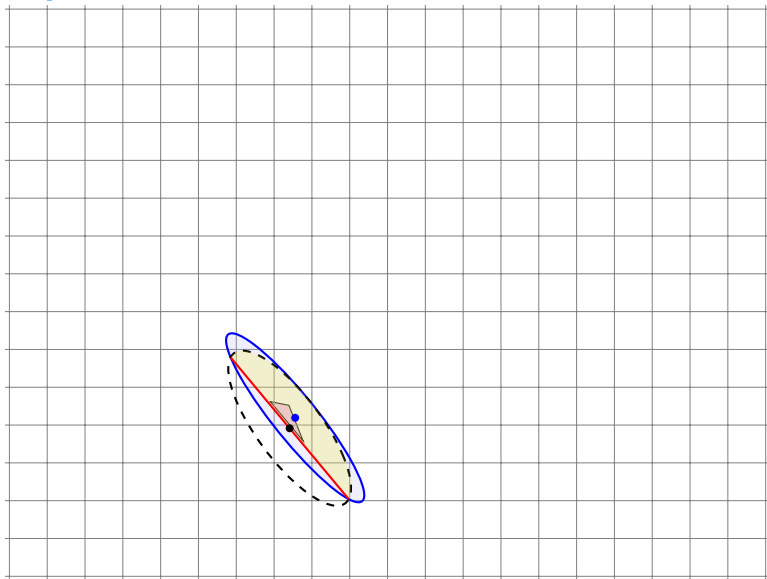
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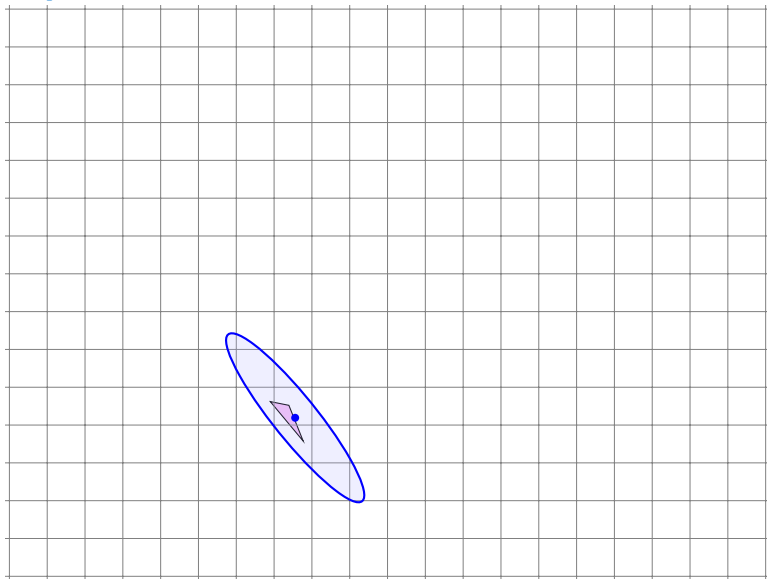
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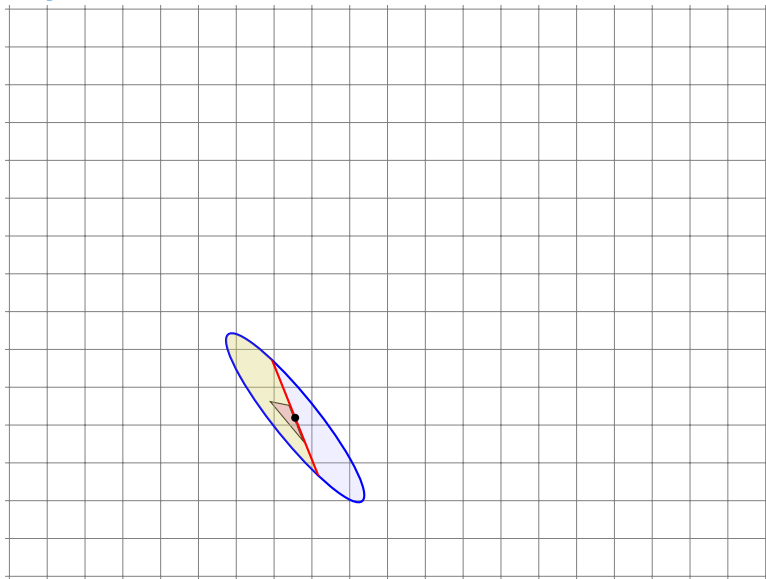
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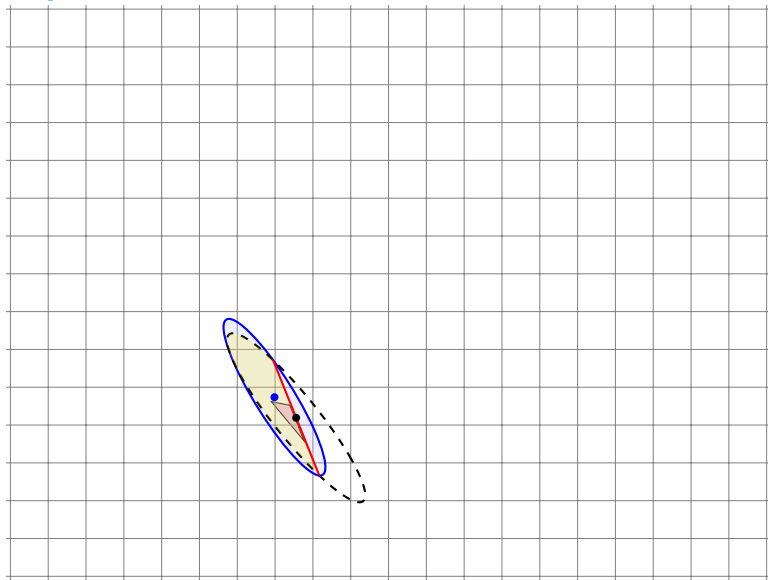
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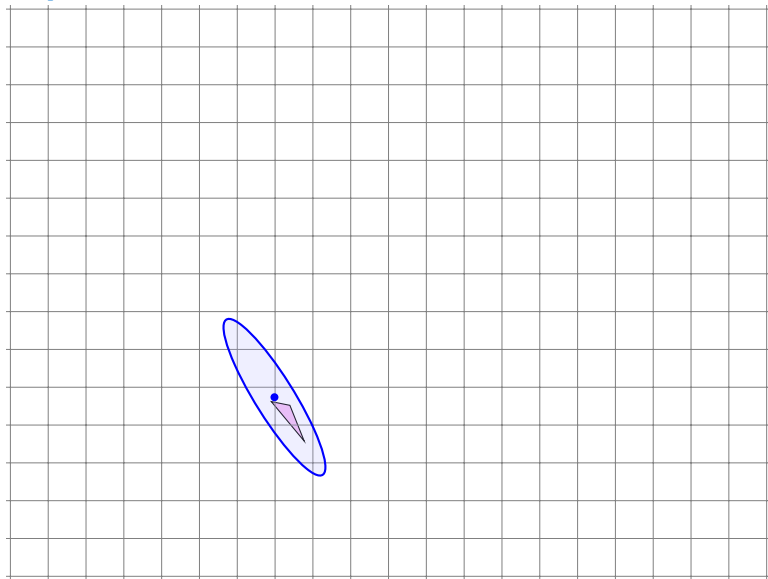
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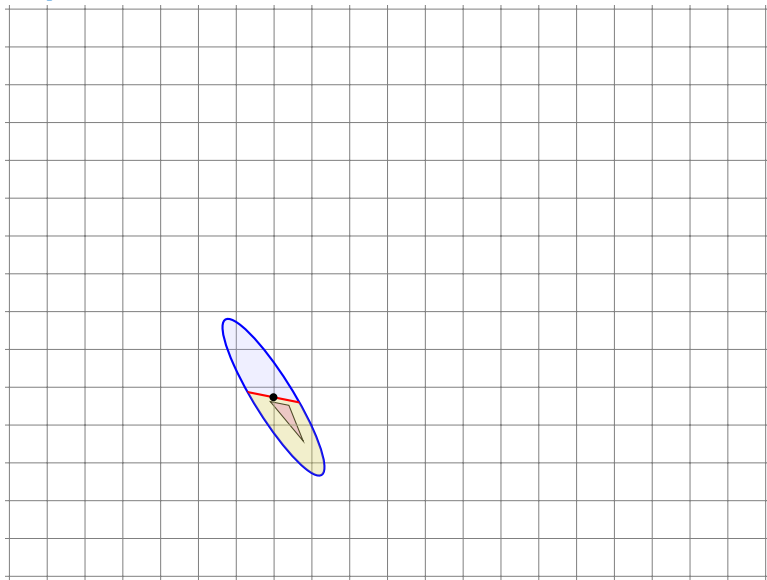
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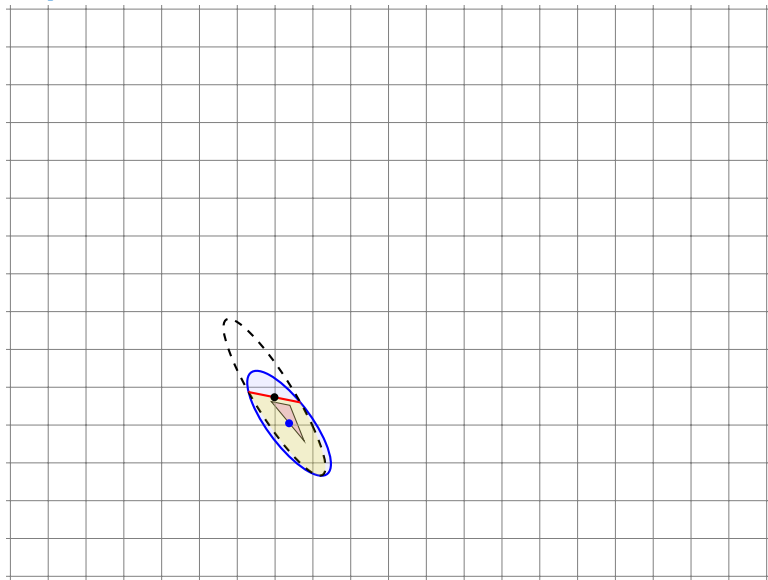
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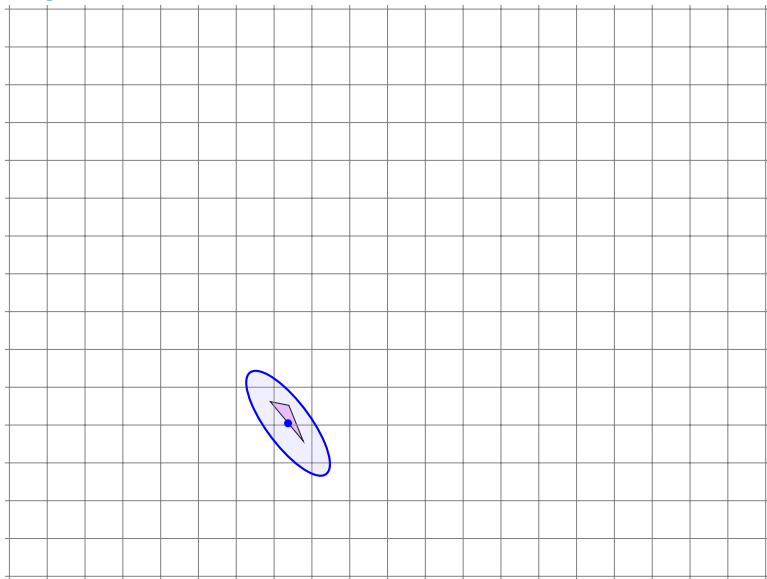
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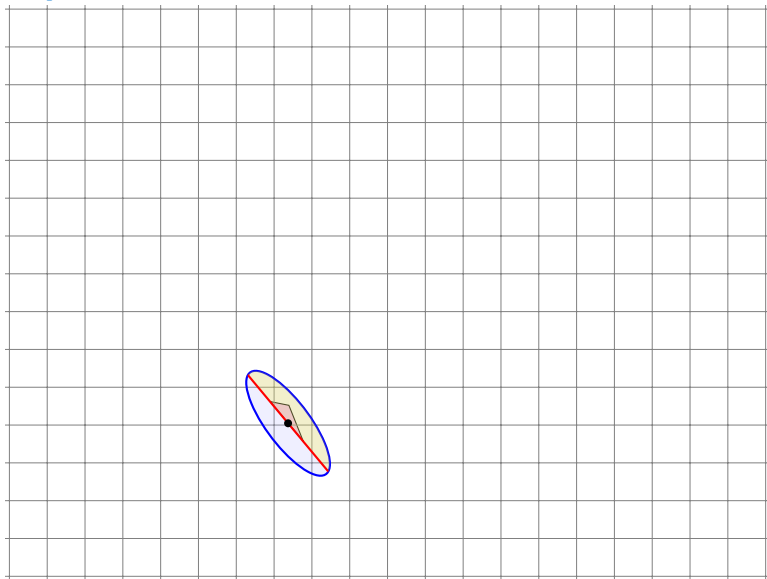
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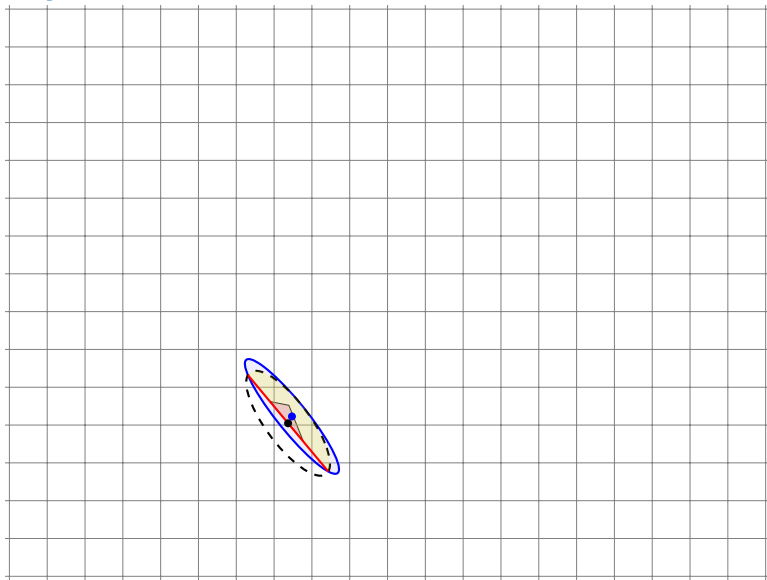
Example



Example



Example



Example

