

18 Cuts & Metrics

Shortest Path

$$\begin{array}{ll}\min & \sum_e c(e)x_e \\ \text{s.t.} & \forall S \in \mathcal{S} \quad \sum_{e \in \delta(S)} x_e \geq 1 \\ & \forall e \in E \quad x_e \in \{0,1\}\end{array}$$

$\mathcal{S}$ is the set of subsets that separate s from t .

The Dual:

$$\begin{array}{ll}\max & \sum_S y_S \\ \text{s.t.} & \forall e \in E \quad \sum_{S:e \in \delta(S)} y_S \leq c(e) \\ & \forall S \in \mathcal{S} \quad y_S \geq 0\end{array}$$

The **Separation Problem** for the Shortest Path LP is the Minimum Cut Problem.

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Minimum Cut

$$\begin{array}{ll}\min & \sum_e c(e)x_e \\ \text{s.t.} & \forall P \in \mathcal{P} \quad \sum_{e \in P} x_e \geq 1 \\ & \forall e \in E \quad x_e \in \{0,1\}\end{array}$$

$\mathcal{P}$ is the set of path that connect s and t .

The Dual:

$$\begin{array}{ll}\max & \sum_P y_P \\ \text{s.t.} & \forall e \in E \quad \sum_{P:e \in P} y_P \leq c(e) \\ & \forall P \in \mathcal{P} \quad y_P \geq 0\end{array}$$

The **Separation Problem** for the Minimum Cut LP is the Shortest Path Problem.

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Observations:

Suppose that ℓ_e -values are solution to Minimum Cut LP.

- ▶ We can view ℓ_e as defining the **length** of an edge.
- ▶ Define $d(u, v) = \min_{\text{path } P \text{ btw. } u \text{ and } v} \sum_{e \in P} \ell_e$ as the **Shortest Path Metric** induced by ℓ_e .
- ▶ We have $d(u, v) = \ell_e$ for every edge $e = (u, v)$, as otw. we could reduce ℓ_e without affecting the distance between s and t .

Remark for bean-counters:

d is not a metric on V but a semimetric as two nodes u and v could have distance zero.

How do we round the LP?

- ▶ Let $B(s, r)$ be the ball of radius r around s (w.r.t. metric d).
Formally:

$$B = \{v \in V \mid d(s, v) \leq r\}$$

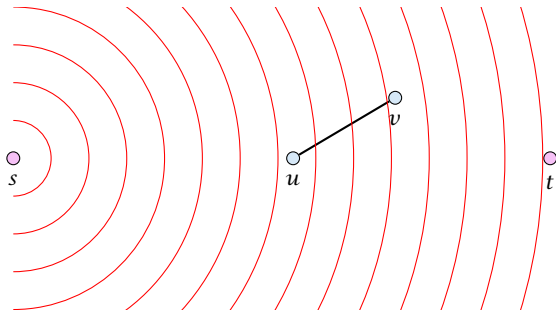
- ▶ For $0 \leq r < 1$, $B(s, r)$ is an s - t -cut.

Which value of r should we choose? **choose randomly!!!**

Formally:

choose r u.a.r. (uniformly at random) from interval $[0, 1)$

What is the probability that an edge (u, v) is in the cut?



► assume wlog. $d(s, u) \leq d(s, v)$

$$\begin{aligned}\Pr[e \text{ is cut}] &= \Pr[r \in [d(s, u), d(s, v)]] \leq \frac{d(s, v) - d(s, u)}{1 - 0} \\ &\leq \ell_e\end{aligned}$$

What is the expected size of a cut?

$$\begin{aligned} \mathbb{E}[\text{size of cut}] &= \mathbb{E}\left[\sum_e c(e) \Pr[e \text{ is cut}]\right] \\ &\leq \sum_e c(e) \ell_e \end{aligned}$$

On the other hand:

$$\sum_e c(e) \ell_e \leq \text{size of mincut}$$

as the ℓ_e are the solution to the Mincut LP *relaxation*.

Hence, our rounding gives an optimal solution.

Minimum Multicut:

Given a graph $G = (V, E)$, together with source-target pairs s_i, t_i , $i = 1, \dots, k$, and a capacity function $c : E \rightarrow \mathbb{R}^+$ on the edges. Find a subset $F \subseteq E$ of the edges such that all s_i - t_i pairs lie in different components in $G = (V, E \setminus F)$.

$$\begin{array}{ll} \min & \sum_e c(e) \ell_e \\ \text{s.t.} & \forall P \in \mathcal{P}_i \text{ for some } i \quad \sum_{e \in P} \ell_e \geq 1 \\ & \forall e \in E \quad \ell_e \in \{0, 1\} \end{array}$$

Here $\mathcal{P}_i$ contains all path P between s_i and t_i .

Re-using the analysis for the single-commodity case is difficult.

$$\Pr[e \text{ is cut}] \leq ?$$

- ▶ If for some R the balls $B(s_i, R)$ are disjoint between different sources, we get a $1/R$ approximation.
- ▶ However, this cannot be guaranteed.

- ▶ Assume for simplicity that all edge-length ℓ_e are multiples of $\delta \ll 1$.
- ▶ Replace the graph G by a graph G' , where an edge of length ℓ_e is replaced by ℓ_e/δ edges of length δ .
- ▶ Let $B(s_i, z)$ be the ball in G' that contains nodes v with distance $d(s_i, v) \leq z\delta$.

Algorithm 1 RegionGrowing(s_i, p)

```
1:  $z \leftarrow 0$ 
2: repeat
3:   flip a coin ( $\Pr[\text{heads}] = p$ )
4:    $z \leftarrow z + 1$ 
5: until heads
6: return  $B(s_i, z)$ 
```

Algorithm 1 Multicut(G')

```
1: while  $\exists s_i-t_i$  pair in  $G'$  do  
2:    $C \leftarrow \text{RegionGrowing}(s_i, p)$   
3:    $G' = G' \setminus C$  // cuts edges leaving  $C$   
4: return  $B(s_i, z)$ 
```

- ▶ probability of cutting an edge is only p
- ▶ a source either does not reach an edge during Region Growing; then it is not cut
- ▶ if it reaches the edge then it either **cuts** the edge or **protects** the edge from being cut by other sources
- ▶ if we choose $p = \delta$ the probability of cutting an edge is only its LP-value; our expected cost are at most **OPT**.

Problem:

We may not cut all source-target pairs.

A component that we remove may contain an s_i-t_i pair.

If we ensure that we cut before reaching radius $1/2$ we are in good shape.

- ▶ choose $p = 6 \ln k \cdot \delta$
- ▶ we make $\frac{1}{2\delta}$ trials before reaching radius $1/2$.
- ▶ we say a Region Growing is not successful if it does not terminate before reaching radius $1/2$.

$$\Pr[\text{not successful}] \leq (1-p)^{\frac{1}{2\delta}} = \left((1-p)^{1/p}\right)^{\frac{p}{2\delta}} \leq e^{-\frac{p}{2\delta}} \leq \frac{1}{k^3}$$

- ▶ Hence,

$$\Pr[\exists i \text{ that is not successful}] \leq \frac{1}{k^2}$$

What is expected cost?

$$\begin{aligned} E[\text{cutsize}] &= \Pr[\text{success}] \cdot E[\text{cutsize} \mid \text{success}] \\ &\quad + \Pr[\text{no success}] \cdot E[\text{cutsize} \mid \text{no success}] \end{aligned}$$

$$\begin{aligned} E[\text{cutsize} \mid \text{succ.}] &= \frac{E[\text{cutsize}] - \Pr[\text{no succ.}] \cdot E[\text{cutsize} \mid \text{no succ.}]}{\Pr[\text{success}]} \\ &\leq \frac{E[\text{cutsize}]}{\Pr[\text{success}]} \leq \frac{1}{1 - \frac{1}{k^2}} 6 \ln k \cdot \text{OPT} \leq 8 \ln k \cdot \text{OPT} \end{aligned}$$

Note: **success** means all source-target pairs separated

We assume $k \geq 2$.

If we are not successful we simply perform a trivial k -approximation.

This only increases the expected cost by at most $\frac{1}{k^2} \cdot kOPT \leq OPT/k$.

Hence, our final cost is $\mathcal{O}(\ln k) \cdot OPT$ in expectation.