

Brewery Problem

Brewery brews ale and beer.

- ▶ Production limited by supply of corn, hops and barley malt
- ▶ Recipes for ale and beer require different amounts of resources

	<i>Corn (kg)</i>	<i>Hops (kg)</i>	<i>Malt (kg)</i>	<i>Profit (€)</i>
ale (barrel)	5	4	35	13
beer (barrel)	15	4	20	23
supply	480	160	1190	

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How can brewer maximize profits?

- ▶ only brew ale: 34 barrels of ale $\Rightarrow 442 \text{ €}$
- ▶ only brew beer: 32 barrels of beer $\Rightarrow 736 \text{ €}$
- ▶ 7.5 barrels ale, 29.5 barrels beer $\Rightarrow 775 \text{ €}$
- ▶ 12 barrels ale, 28 barrels beer $\Rightarrow 800 \text{ €}$

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Brewery Problem

Linear Program

- Introduce variables a and b that define how much ale and stout to produce.
- Choose the variables in such a way that the profit is maximized.
- Make sure that no constraints (due to limited supply) are violated.

$$\begin{array}{ll}\max & 13a + 23b \\ \text{s.t.} & 5a + 15b \leq 480 \\ & 4a + 4b \leq 160 \\ & 35a + 20b \leq 1190 \\ & a, b \geq 0\end{array}$$

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Standard Form LPs

LP in standard form:

Maximize $c^T x$
subject to $Ax = b$
and $x \geq 0$
where x is a vector of n variables, A is a matrix of m constraints, b is a vector of m constants, and c is a vector of n constants.

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LP in standard form:

- ▶ input: numbers a_{ij} , c_j , b_i
- ▶ output: numbers x_j
- ▶ n = #decision variables, m = #constraints
- ▶ maximize linear objective function subject to linear (in)equalities

$$\begin{aligned} \max \quad & \sum_{j=1}^n c_j x_j \\ \text{s.t.} \quad & \sum_{j=1}^n a_{ij} x_j = b_i \quad 1 \leq i \leq m \\ & x_j \geq 0 \quad 1 \leq j \leq n \end{aligned}$$

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$$\begin{aligned} \max \quad & c_1 x_1 + c_2 x_2 + \dots + c_n x_n \\ \text{s.t.} \quad & a_{11} x_1 + a_{12} x_2 + \dots + a_{1n} x_n = b_1 \\ & a_{21} x_1 + a_{22} x_2 + \dots + a_{2n} x_n = b_2 \\ & \vdots \\ & a_{m1} x_1 + a_{m2} x_2 + \dots + a_{mn} x_n = b_m \\ & x_1, x_2, \dots, x_n \geq 0 \end{aligned}$$

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Original LP

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Standard Form

Add a **slack variable** to every constraint.

$$\begin{array}{llllllll}\max & 13a & + & 23b & & & & \\ \text{s.t.} & 5a & + & 15b & + & s_c & & = 480 \\ & 4a & + & 4b & & & + & s_h = 160 \\ & 35a & + & 20b & & & + & s_m = 1190 \\ & a & , & b & , & s_c & , & s_h , s_m \geq 0\end{array}$$

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There are different standard forms:

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It is easy to transform variants of LPs into (any) standard form:

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7. Maximize $c^T x$

8. Minimize $c^T x$

Standard Form LPs

It is easy to transform variants of LPs into (any) standard form:

- ▶ **less or equal to equality:**

$$a - 3b + 5c \leq 12 \Rightarrow \begin{aligned} a - 3b + 5c + s &= 12 \\ s &\geq 0 \end{aligned}$$

- ▶ greater or equal to equality:

$$\begin{aligned} a - 3b + 5c - s &= 12 \\ s &\geq 0 \end{aligned}$$

- ▶ min to max:

$$\min z \Rightarrow \max -z$$

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$$x \text{ unrestricted} \Rightarrow x = x^+ - x^-, x^+ \geq 0, x^- \geq 0$$

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Observations:

- ▶ a linear program does not contain x^2 , $\cos(x)$, etc.
- ▶ transformations between standard forms can be done efficiently and only change the size of the LP by a small constant factor
- ▶ for the standard minimization or maximization LPs we could include the nonnegativity constraints into the set of ordinary constraints; this is of course not possible for the standard form

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Fundamental Questions

Definition 1 (Linear Programming Problem (LP))

Let $A \in \mathbb{Q}^{m \times n}$, $b \in \mathbb{Q}^m$, $c \in \mathbb{Q}^n$, $\alpha \in \mathbb{Q}$. Does there exist $x \in \mathbb{Q}^n$
s.t. $Ax = b$, $x \geq 0$, $c^T x \geq \alpha$?

Questions:

1. Is LP in NP?

2. Is LP in co-NP?

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Input size:

- ▶ n number of variables, m constraints, L number of bits to encode the input

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Let $A \in \mathbb{Q}^{m \times n}$, $b \in \mathbb{Q}^m$, $c \in \mathbb{Q}^n$, $\alpha \in \mathbb{Q}$. Does there exist $x \in \mathbb{Q}^n$
s.t. $Ax = b$, $x \geq 0$, $c^T x \geq \alpha$?

Questions:

- ▶ Is LP in NP?
- ▶ Is LP in co-NP?
- ▶ Is LP in P?

Input size:

- ▶ n number of variables, m constraints, L number of bits to encode the input

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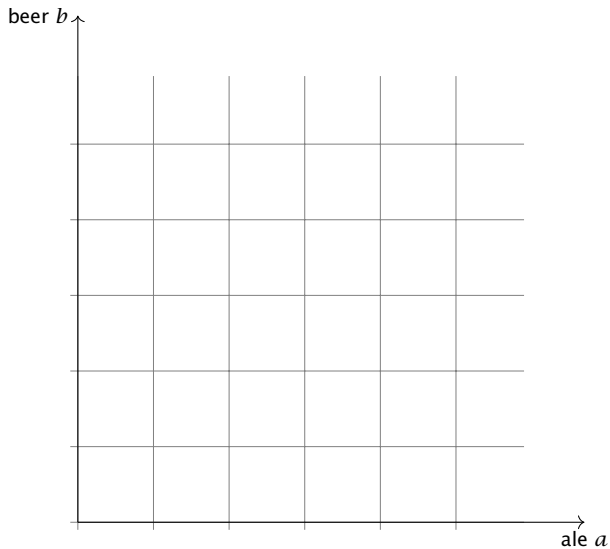
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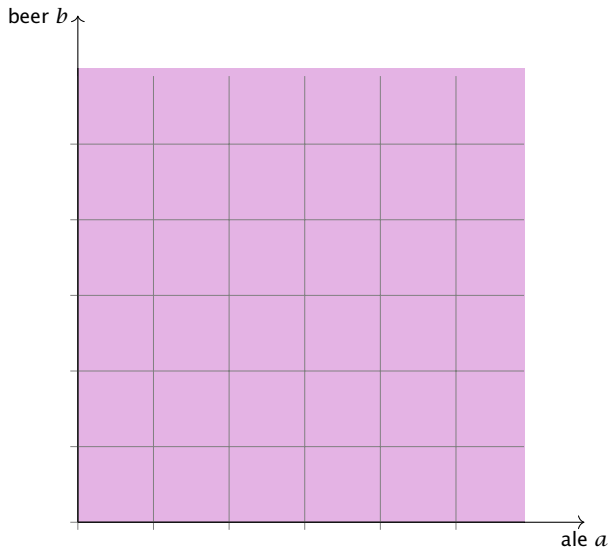
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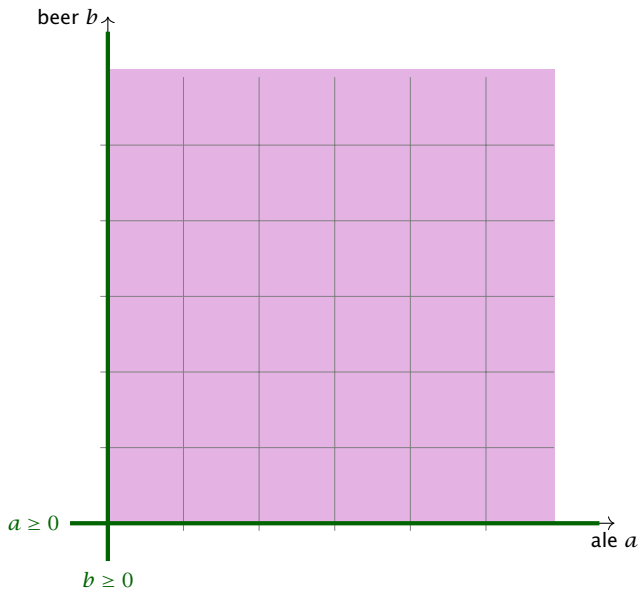
Geometry of Linear Programming



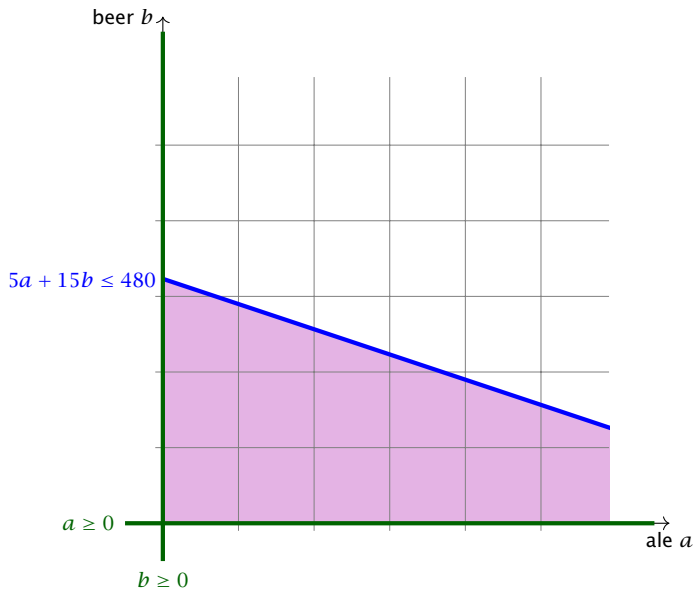
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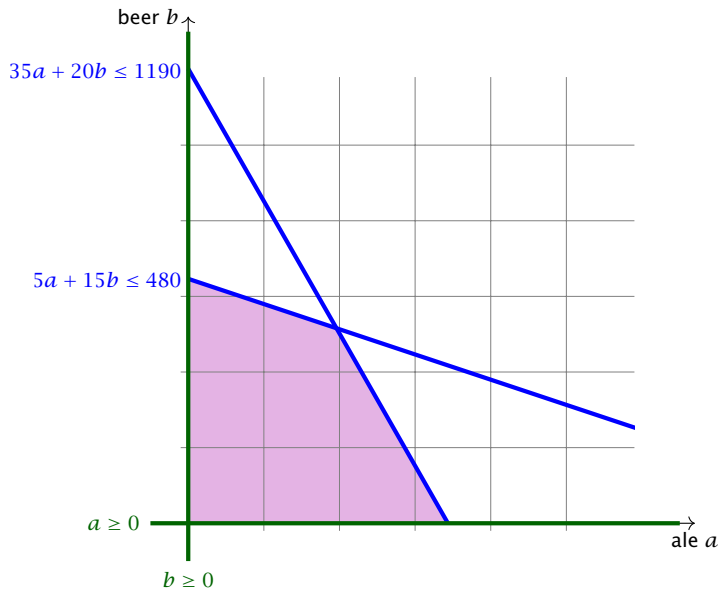
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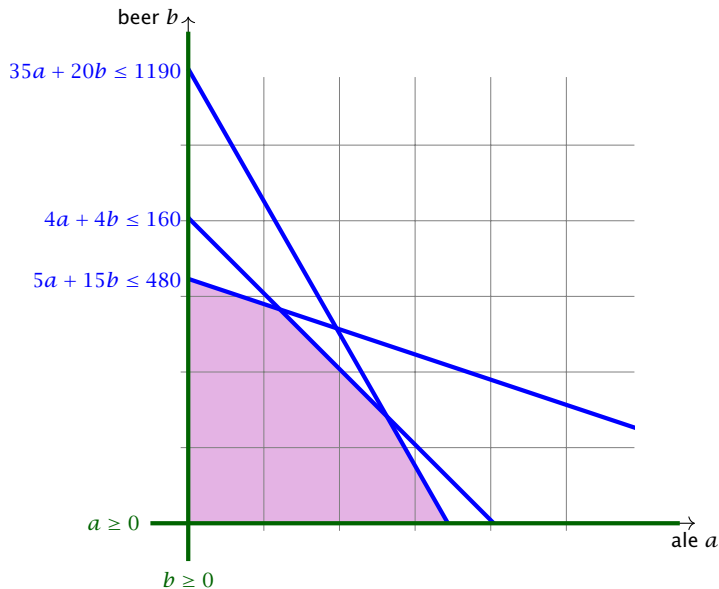
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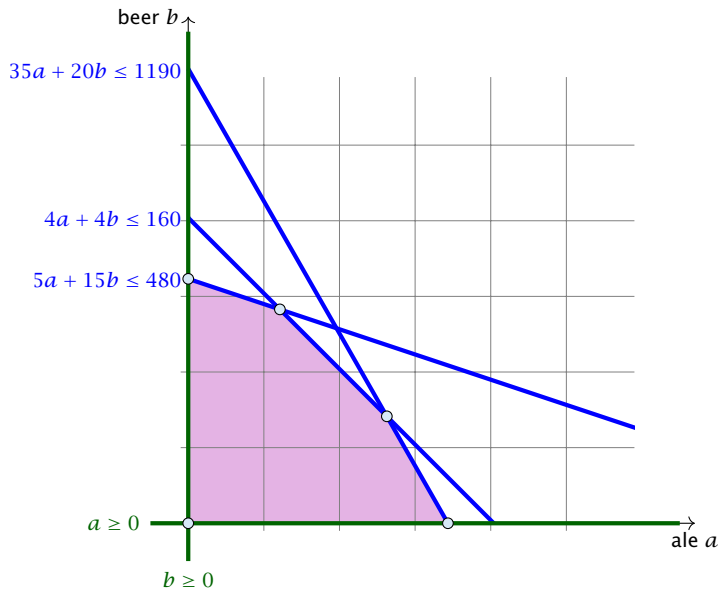
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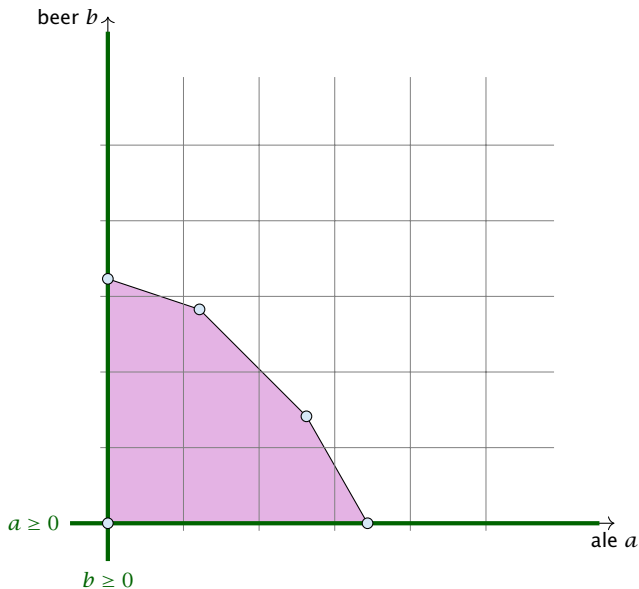
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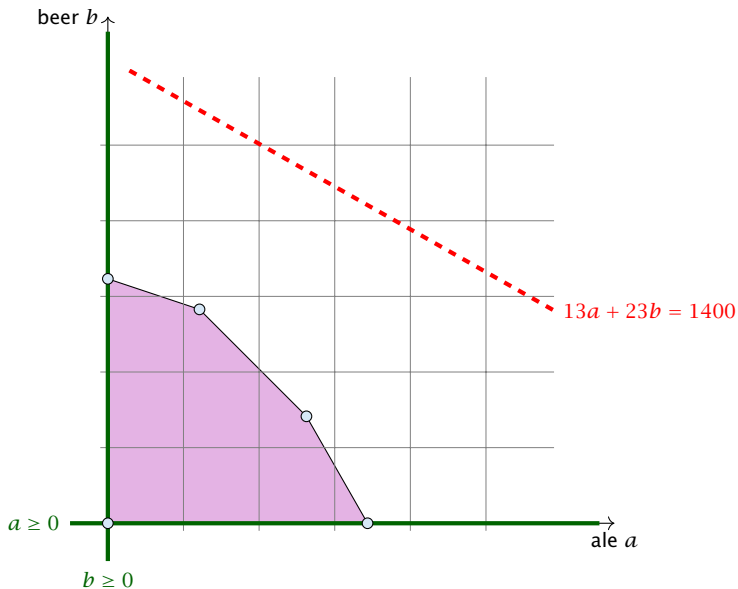
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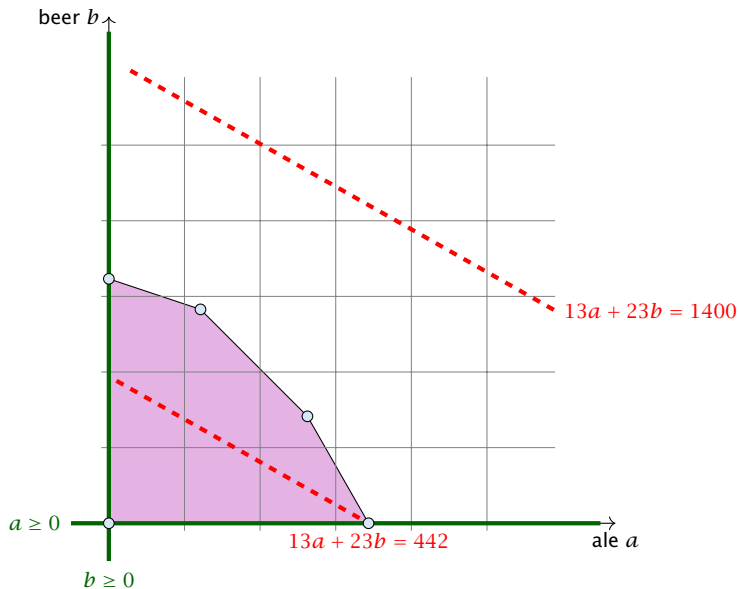
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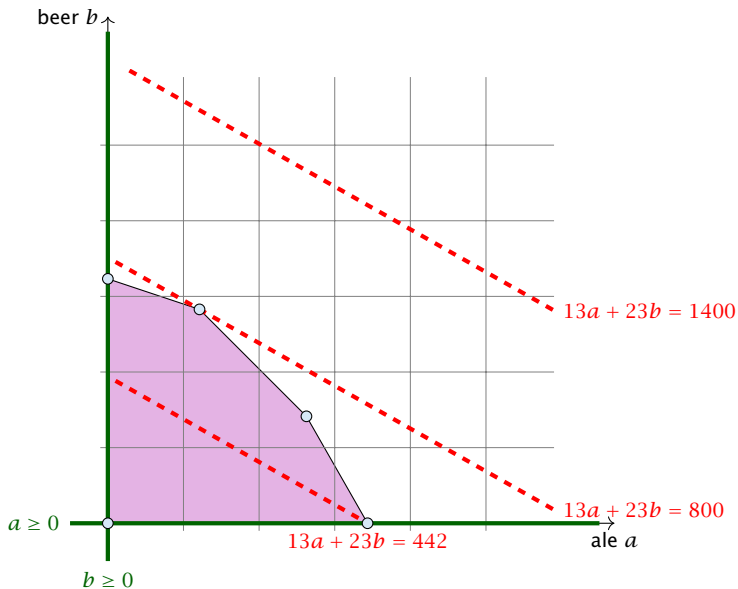
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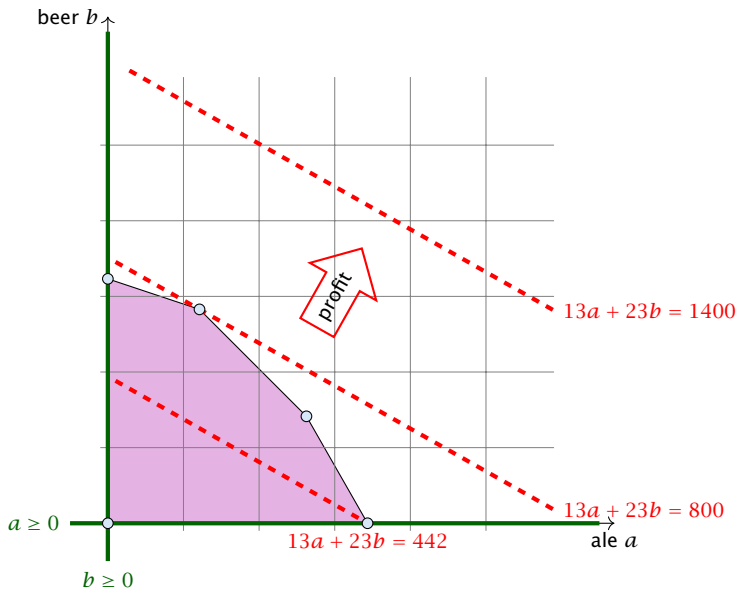
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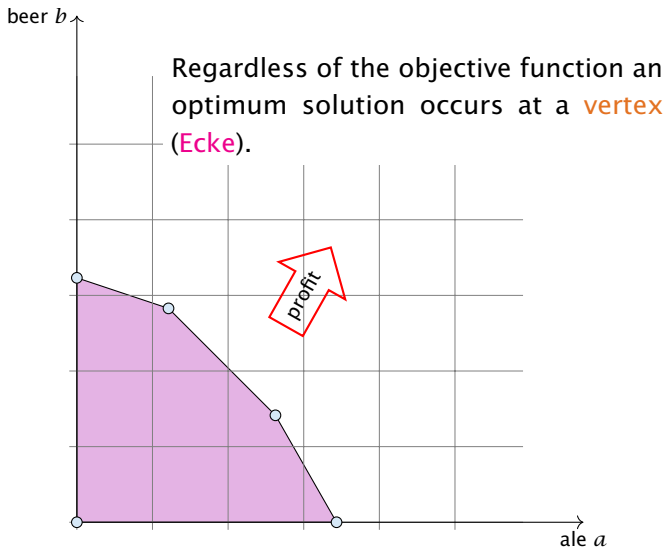
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 $\exists M \in \mathbb{R}$ s.t. $x_i \leq M$ for all $i \in \{1, \dots, n\}$ (for maximization problems)
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Definition 2

Given vectors/points $x_1, \dots, x_k \in \mathbb{R}^n$, $\sum \lambda_i x_i$ is called

- ▶ **linear combination** if $\lambda_i \in \mathbb{R}$.
- ▶ **affine combination** if $\lambda_i \in \mathbb{R}$ and $\sum_i \lambda_i = 1$.
- ▶ **convex combination** if $\lambda_i \in \mathbb{R}$ and $\sum_i \lambda_i = 1$ and $\lambda_i \geq 0$.
- ▶ **conic combination** if $\lambda_i \in \mathbb{R}$ and $\lambda_i \geq 0$.

Note that a combination involves only finitely many vectors.

Definition 3

A set $X \subseteq \mathbb{R}^n$ is called

- ▶ a **linear subspace** if it is closed under linear combinations.
- ▶ an **affine subspace** if it is closed under affine combinations.
- ▶ **convex** if it is closed under convex combinations.
- ▶ a **convex cone** if it is closed under conic combinations.

Note that an affine subspace is **not** a vector space

Definition 4

Given a set $X \subseteq \mathbb{R}^n$.

- ▶ $\text{span}(X)$ is the set of all linear combinations of X
(linear hull, span)
- ▶ $\text{aff}(X)$ is the set of all affine combinations of X
(affine hull)
- ▶ $\text{conv}(X)$ is the set of all convex combinations of X
(convex hull)
- ▶ $\text{cone}(X)$ is the set of all conic combinations of X
(conic hull)

Definition 5

A function $f: \mathbb{R}^n \rightarrow \mathbb{R}$ is **convex** if for $x, y \in \mathbb{R}^n$ and $\lambda \in [0, 1]$ we have

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

Lemma 6

If $P \subseteq \mathbb{R}^n$, and $f: \mathbb{R}^n \rightarrow \mathbb{R}$ convex then also

$$Q = \{x \in P \mid f(x) \leq t\}$$

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Dimensions

Definition 7

The **dimension** $\dim(A)$ of an affine subspace $A \subseteq \mathbb{R}^n$ is the dimension of the vector space $\{x - a \mid x \in A\}$, where $a \in A$.

Definition 8

The **dimension** $\dim(X)$ of a convex set $X \subseteq \mathbb{R}^n$ is the dimension of its affine hull $\text{aff}(X)$.

Definition 9

A set $H \subseteq \mathbb{R}^n$ is a **hyperplane** if $H = \{x \mid a^T x = b\}$, for $a \neq 0$.

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A set $H' \subseteq \mathbb{R}^n$ is a (closed) **halfspace** if $H = \{x \mid a^T x \leq b\}$, for $a \neq 0$.

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Definition 11

A **polytop** is a set $P \subseteq \mathbb{R}^n$ that is the convex hull of a **finite** set of points, i.e., $P = \text{conv}(X)$ where $|X| = c$.

Definitions

Definition 12

A **polyhedron** is a set $P \subseteq \mathbb{R}^n$ that can be represented as the intersection of **finitely** many half-spaces $\{H(a_1, b_1), \dots, H(a_m, b_m)\}$, where

$$H(a_i, b_i) = \{x \in \mathbb{R}^n \mid a_i x \leq b_i\} \quad .$$

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A polyhedron P is **bounded** if there exists B s.t. $\|x\|_2 \leq B$ for all $x \in P$.

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Theorem 14

P is a bounded polyhedron iff P is a polytop.

Definition 15

Let $P \subseteq \mathbb{R}^n$, $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$. The hyperplane

$$H(a, b) = \{x \in \mathbb{R}^n \mid a^T x = b\}$$

is a **supporting hyperplane** of P if $\max\{a^T x \mid x \in P\} = b$.

Definition 16

Let $P \subseteq \mathbb{R}^n$. F is a **face** of P if $F = P$ or $F = P \cap H$ for some supporting hyperplane H .

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Let $P \subseteq \mathbb{R}^n$.

- ▶ a face v is a **vertex** of P if $\{v\}$ is a face of P .
- ▶ a face e is an **edge** of P if e is a face and $\dim(e) = 1$.
- ▶ a face F is a **facet** of P if F is a face and $\dim(F) = \dim(P) - 1$.

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Equivalent definition for vertex:

Definition 18

Given polyhedron P . A point $x \in P$ is a **vertex** if $\exists c \in \mathbb{R}^n$ such that $c^T y < c^T x$, for all $y \in P$, $y \neq x$.

Definition 19

Given polyhedron P . A point $x \in P$ is an **extreme point** if $\nexists a, b \neq x$, $a, b \in P$, with $\lambda a + (1 - \lambda)b = x$ for $\lambda \in [0, 1]$.

Lemma 20

A vertex is also an extreme point.

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Observation

The feasible region of an LP is a Polyhedron.

Theorem 21

If there exists an optimal solution to an LP (in standard form) then there exists an optimum solution that is an extreme point.

Proof

- ▶ suppose x is optimal solution that is not extreme point
- ▶ there exists direction $d \neq 0$ such that $x \pm d \in P$
- ▶ $Ad = 0$ because $A(x \pm d) = b$
- ▶ Wlog. assume $c^T d \geq 0$ (by taking either d or $-d$)
- ▶ Consider $x + \lambda d, \lambda > 0$

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Convex Sets

Case 1. $[\exists j \text{ s.t. } d_j < 0]$

• Increase λ to λ^* until first component of $d + \lambda c$ hits 0.
• $d + \lambda^* c$ is feasible. Since $-c^T d = -c^T(d + \lambda^* c) \geq 0$ and $-c^T d > 0$,
• $d + \lambda^* c$ has the more zero-component $\lambda_j = 0$ for some j ($d_j < 0$ and $c_j > 0$).
• (Another λ exists, λ^* is the value λ such that $d + \lambda c \geq 0$.)

Case 2. $[d_j \geq 0 \text{ for all } j \text{ and } c^T d > 0]$

• $d + \lambda c$ is feasible for all $\lambda \geq 0$ since $d_j \geq 0$ and $c_j \geq 0$ and
• $-c^T d > 0$ means $-c^T(d + \lambda c) > 0$ for all $\lambda \geq 0$.
• as $\lambda \rightarrow \infty$, $d + \lambda c \rightarrow \infty$ as $c \neq 0$.

Case 1. $[\exists j \text{ s.t. } d_j < 0]$

Consider $y = 0$ until first component of d that is < 0 . This y is feasible. Since $c^T d > 0$ and $d_j < 0$, $c_j > 0$. y has the more zero-component $d_j = 0$ for some j (i.e. $c_j > 0$ and $d_j = 0$). The value $c^T y$ is smaller than $c^T d$. This is a contradiction. $\square$

Case 2. $[d_j \geq 0 \text{ for all } j \text{ and } c^T d > 0]$

$y = 0$ is feasible for all $\lambda \geq 0$ since $d_j \geq 0$ and $Ay = 0$. $c^T y = \lambda c^T d$ is as large as we want. $\square$

Case 1. [$\exists j$ s.t. $d_j < 0$]

- ▶ increase λ to λ' until first component of $x + \lambda d$ hits 0
- ▶ $x + \lambda' d$ is feasible. Since $A(x + \lambda' d) = b$ and $x + \lambda' d \geq 0$
- ▶ $x + \lambda' d$ has one more zero-component ($d_k = 0$ for $x_k = 0$ as $x \pm d \in P$)
- ▶ $c^T x' = c^T (x + \lambda' d) = c^T x + \lambda' c^T d \geq c^T x$

Case 2. [$d_j \geq 0$ for all j and $c^T d > 0$]

- ▶ $x + \lambda d$ is feasible for all $\lambda \geq 0$ since $A(x + \lambda d) = b$ and $x + \lambda d \geq 0$
- ▶ $c^T x + \lambda c^T d > c^T x$ for all $\lambda > 0$
- ▶ P is unbounded

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- ▶ $x + \lambda' d$ is feasible. Since $A(x + \lambda' d) = b$ and $x + \lambda' d \geq 0$
- ▶ $x + \lambda' d$ has one more zero-component ($d_k = 0$ for $x_k = 0$ as $x \pm d \in P$)
- ▶ $c^T x' = c^T (x + \lambda' d) = c^T x + \lambda' c^T d \geq c^T x$

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Convex Sets

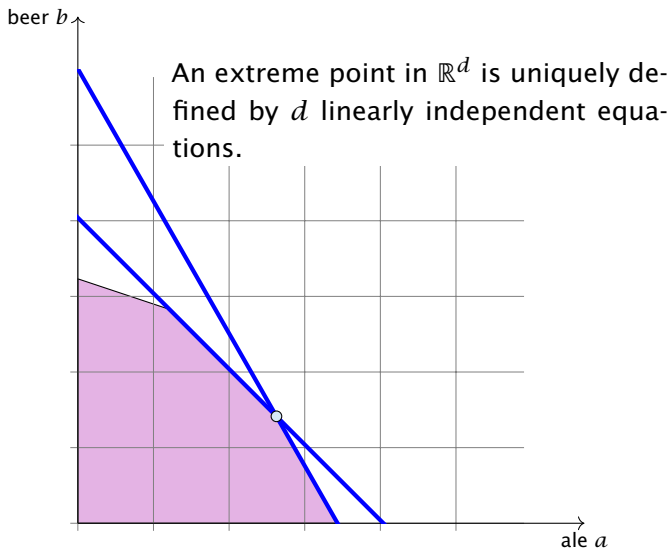
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Algebraic View



Notation

Suppose $B \subseteq \{1 \dots n\}$ is a set of column-indices. Define A_B as the subset of columns of A indexed by B .

Theorem 22

*Let $P = \{x \mid Ax = b, x \geq 0\}$. For $x \in P$, define $B = \{j \mid x_j > 0\}$.
Then x is extreme point iff A_B has linearly independent columns.*

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- ▶ $A_{B'}$ has linearly dependent columns as $Ad = 0$
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Let $P = \{x \mid Ax = b, x \geq 0\}$. For $x \in P$, define $B = \{j \mid x_j > 0\}$. If A_B has linearly independent columns then x is a vertex of P .

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- ▶ then $c^T x = 0$ and $c^T y \leq 0$ for $y \in P$
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- ▶ $b = Ay = A_B y_B = Ax = A_B x_B$ gives that $A_B(x_B - y_B) = 0$;
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$$A_1 x = \sum_{i=2}^m \lambda_i \cdot A_i x = \sum_{i=2}^m \lambda_i \cdot b_i \neq b_1$$

Observation

For an LP we can assume wlog. that the matrix A has full row-rank. This means $\text{rank}(A) = m$.

- ▶ assume that $\text{rank}(A) < m$
- ▶ assume wlog. that the first row A_1 lies in the span of the other rows $A_2, \dots, A_m$; this means

$$A_1 = \sum_{i=2}^m \lambda_i \cdot A_i, \text{ for suitable } \lambda_i$$

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From now on we will always assume that the constraint matrix of a standard form LP has full row rank.

Theorem 24

Given $P = \{x \mid Ax = b, x \geq 0\}$. x is extreme point iff there exists $B \subseteq \{1, \dots, n\}$ with $|B| = m$ and

- ▶ A_B is non-singular
- ▶ $x_B = A_B^{-1}b \geq 0$
- ▶ $x_N = 0$

where $N = \{1, \dots, n\} \setminus B$.

Proof

Take $B = \{j \mid x_j > 0\}$ and augment with linearly independent columns until $|B| = m$; always possible since $\text{rank}(A) = m$.

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Basic Feasible Solutions

$x \in \mathbb{R}^n$ is called **basic solution** (Basislösung) if $Ax = b$ and $\text{rank}(A_J) = |J|$ where $J = \{j \mid x_j \neq 0\}$;

x is a **basic feasible solution** (gültige Basislösung) if in addition $x \geq 0$.

A **basis** (Basis) is an index set $B \subseteq \{1, \dots, n\}$ with $\text{rank}(A_B) = m$ and $|B| = m$.

$x \in \mathbb{R}^n$ with $A_B x_B = b$ and $x_j = 0$ for all $j \notin B$ is **the basic solution associated to basis B** (die zu B assoziierte Basislösung)

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Basic Feasible Solutions

A BFS fulfills the m equality constraints.

In addition, at least $n - m$ of the x_i 's are zero. The corresponding non-negativity constraint is fulfilled with equality.

Fact:

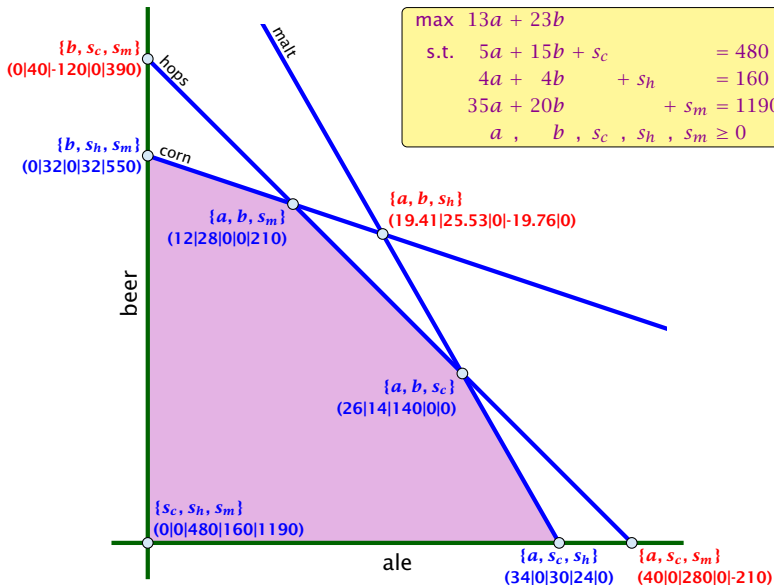
In a BFS at least n constraints are fulfilled with equality.

Basic Feasible Solutions

Definition 25

For a general LP ($\max\{c^T x \mid Ax \leq b\}$) with n variables a point x is a **basic feasible solution** if x is feasible and there exist n (linearly independent) constraints that are tight.

Algebraic View



Fundamental Questions

Linear Programming Problem (LP)

Let $A \in \mathbb{Q}^{m \times n}$, $b \in \mathbb{Q}^m$, $c \in \mathbb{Q}^n$, $\alpha \in \mathbb{Q}$. Does there exist $x \in \mathbb{Q}^n$
s.t. $Ax = b$, $x \geq 0$, $c^T x \geq \alpha$?

Questions:

- ▶ Is LP in NP? yes!
- ▶ Is LP in co-NP?
- ▶ Is LP in P?

Proof:

- ▶ Given a basis B we can compute the associated basis solution by calculating $A_B^{-1}b$ in polynomial time; then we can also compute the profit.

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- ▶ Given a basis B we can compute the associated basis solution by calculating $A_B^{-1}b$ in polynomial time; then we can also compute the profit.

Observation

We can compute an optimal solution to a linear program in time $\mathcal{O}\left(\binom{n}{m} \cdot \text{poly}(n, m)\right)$.

- ▶ there are only $\binom{n}{m}$ different bases.
- ▶ compute the profit of each of them and take the maximum

What happens if LP is unbounded?