

Brewery Problem

Brewery brews ale and beer.

- ▶ Production limited by supply of corn, hops and barley malt
- ▶ Recipes for ale and beer require different amounts of resources

	<i>Corn (kg)</i>	<i>Hops (kg)</i>	<i>Malt (kg)</i>	<i>Profit (€)</i>
ale (barrel)	5	4	35	13
beer (barrel)	15	4	20	23
supply	480	160	1190	

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How can brewer maximize profits?

- ▶ only brew ale: 34 barrels of ale $\Rightarrow 442 \text{ €}$
- ▶ only brew beer: 32 barrels of beer $\Rightarrow 736 \text{ €}$
- ▶ 7.5 barrels ale, 29.5 barrels beer $\Rightarrow 775 \text{ €}$
- ▶ 12 barrels ale, 28 barrels beer $\Rightarrow 800 \text{ €}$

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Brewery Problem

Linear Program

- Introduce variables a and b that define how much ale and stout to produce.
- Choose the variables in such a way that the profit is maximized.
- Make sure that no constraints (due to limited supply) are violated.

$$\begin{array}{ll}\max & 13a + 23b \\ \text{s.t.} & 5a + 15b \leq 480 \\ & 4a + 4b \leq 160 \\ & 35a + 20b \leq 1190 \\ & a, b \geq 0\end{array}$$

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Standard Form LPs

LP in standard form:

maximize $c^T x$
subject to
linear constraints, i.e. constraints
maximize linear objective function subject to linear
inequalities

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & Ax = b, \quad 1 \leq i \leq m \\ & x \geq 0, \quad 1 \leq j \leq n \end{aligned}$$

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Standard Form LPs

LP in standard form:

- ▶ input: numbers a_{ij} , c_j , b_i
- ▶ output: numbers x_j
- ▶ n = #decision variables, m = #constraints
- ▶ maximize linear objective function subject to linear (in)equalities

$$\begin{aligned} \max \quad & \sum_{j=1}^n c_j x_j \\ \text{s.t.} \quad & \sum_{j=1}^n a_{ij} x_j = b_i \quad 1 \leq i \leq m \\ & x_j \geq 0 \quad 1 \leq j \leq n \end{aligned}$$

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$$\begin{aligned} \max \quad & c_1 x_1 + c_2 x_2 + \dots + c_n x_n \\ \text{s.t.} \quad & a_{11} x_1 + a_{12} x_2 + \dots + a_{1n} x_n = b_1 \\ & a_{21} x_1 + a_{22} x_2 + \dots + a_{2n} x_n = b_2 \\ & \vdots \\ & a_{m1} x_1 + a_{m2} x_2 + \dots + a_{mn} x_n = b_m \\ & x_1, x_2, \dots, x_n \geq 0 \end{aligned}$$

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Original LP

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Standard Form

Add a **slack variable** to every constraint.

$$\begin{array}{llllllll} \max & 13a & + & 23b & & & & \\ \text{s.t.} & 5a & + & 15b & + & s_c & & = 480 \\ & 4a & + & 4b & & & + & s_h = 160 \\ & 35a & + & 20b & & & + & s_m = 1190 \\ & a & , & b & , & s_c & , & s_h , s_m \geq 0 \end{array}$$

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Standard Form LPs

There are different standard forms:

standard form

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standard
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Standard Form LPs

It is easy to transform variants of LPs into (any) standard form:

1. Maximize $c^T x$

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3. Maximize $c^T x$

4. Minimize $c^T x$ subject to $Ax = b$

5. Maximize $c^T x$

6. Minimize $c^T x$

7. Maximize $c^T x$

8. Minimize $c^T x$

9. Maximize $c^T x$

10. Minimize $c^T x$

Standard Form LPs

It is easy to transform variants of LPs into (any) standard form:

- ▶ **less or equal to equality:**

$$a - 3b + 5c \leq 12 \Rightarrow \begin{aligned} a - 3b + 5c + s &= 12 \\ s &\geq 0 \end{aligned}$$

- ▶ greater or equal to equality:

$$\begin{aligned} a - 3b + 5c - s &= 12 \\ s &\geq 0 \end{aligned}$$

- ▶ min to max:

$$\min z \Rightarrow \max -z$$

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$$x \text{ unrestricted} \Rightarrow x = x^+ - x^-, x^+ \geq 0, x^- \geq 0$$

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Observations:

- ▶ a linear program does not contain x^2 , $\cos(x)$, etc.
- ▶ transformations between standard forms can be done efficiently and only change the size of the LP by a small constant factor
- ▶ for the standard minimization or maximization LPs we could include the nonnegativity constraints into the set of ordinary constraints; this is of course not possible for the standard form

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Fundamental Questions

Definition 1 (Linear Programming Problem (LP))

Let $A \in \mathbb{Q}^{m \times n}$, $b \in \mathbb{Q}^m$, $c \in \mathbb{Q}^n$, $\alpha \in \mathbb{Q}$. Does there exist $x \in \mathbb{Q}^n$
s.t. $Ax = b$, $x \geq 0$, $c^T x \geq \alpha$?

Questions:

1. Is LP in NP?

2. Is LP in co-NP?

3. Is LP in P?

Input size:

- ▶ n number of variables, m constraints, L number of bits to encode the input

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- ▶ n number of variables, m constraints, L number of bits to encode the input

Fundamental Questions

Definition 1 (Linear Programming Problem (LP))

Let $A \in \mathbb{Q}^{m \times n}$, $b \in \mathbb{Q}^m$, $c \in \mathbb{Q}^n$, $\alpha \in \mathbb{Q}$. Does there exist $x \in \mathbb{Q}^n$
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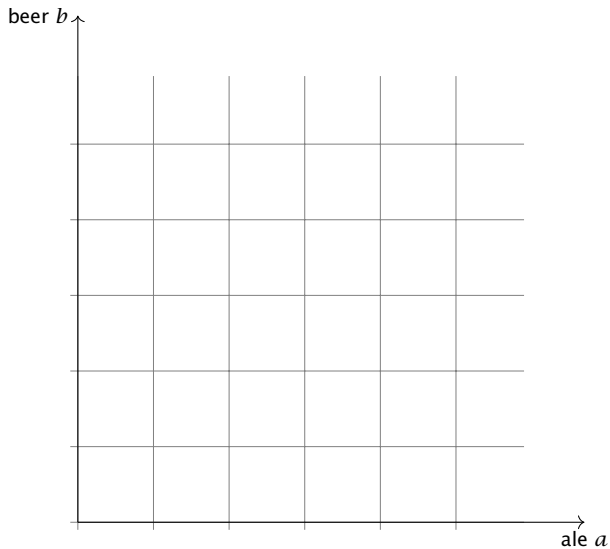
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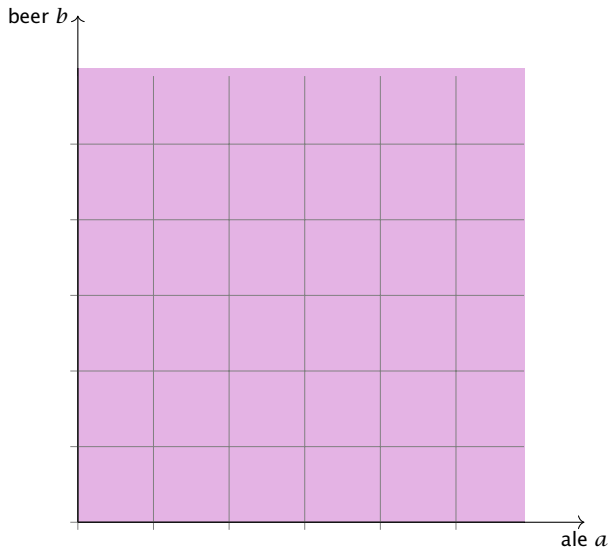
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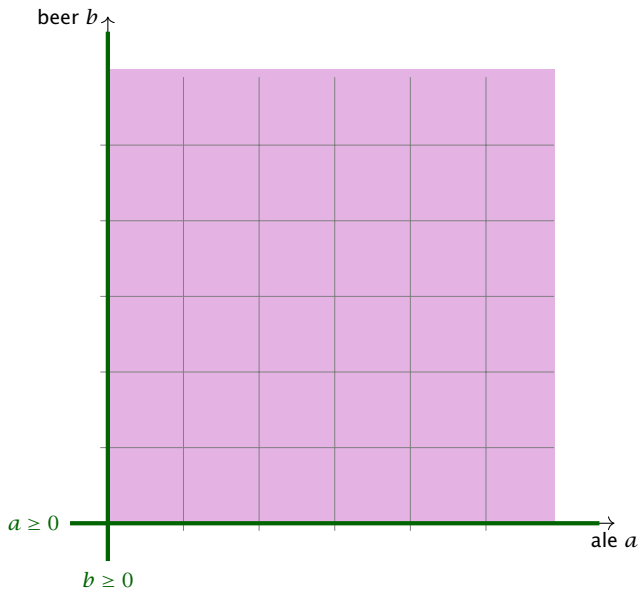
Geometry of Linear Programming



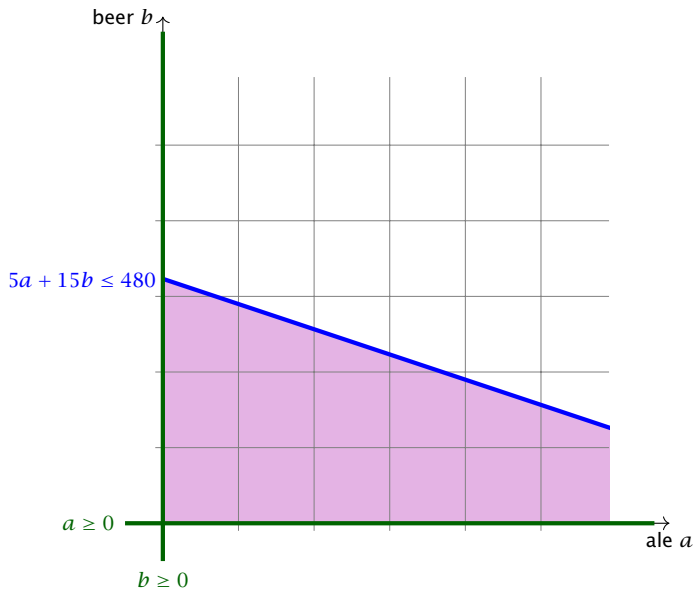
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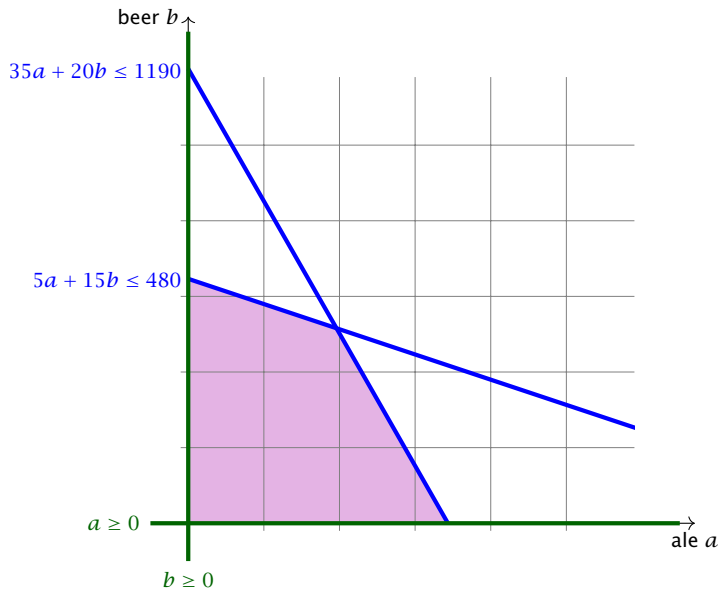
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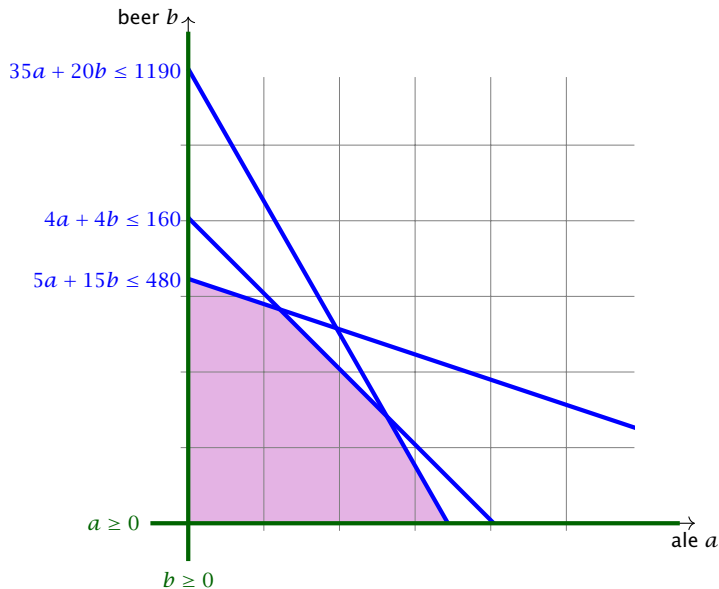
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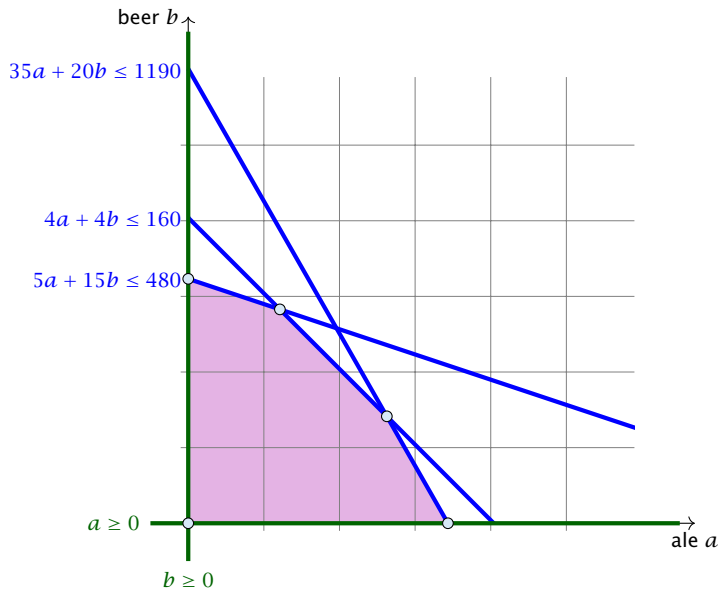
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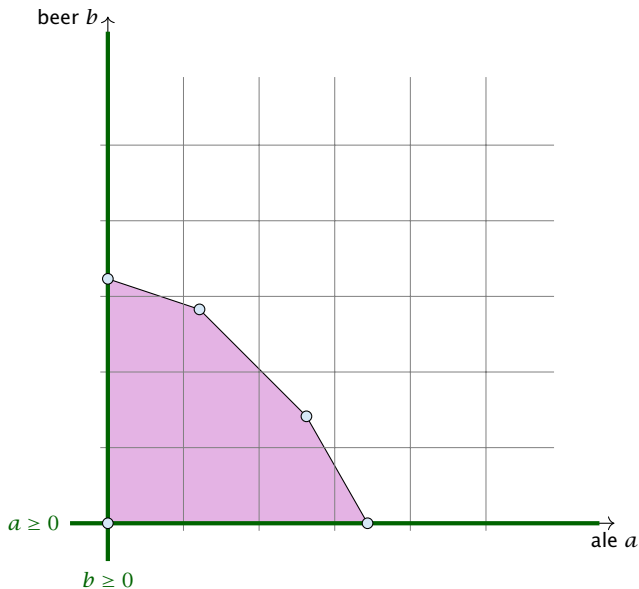
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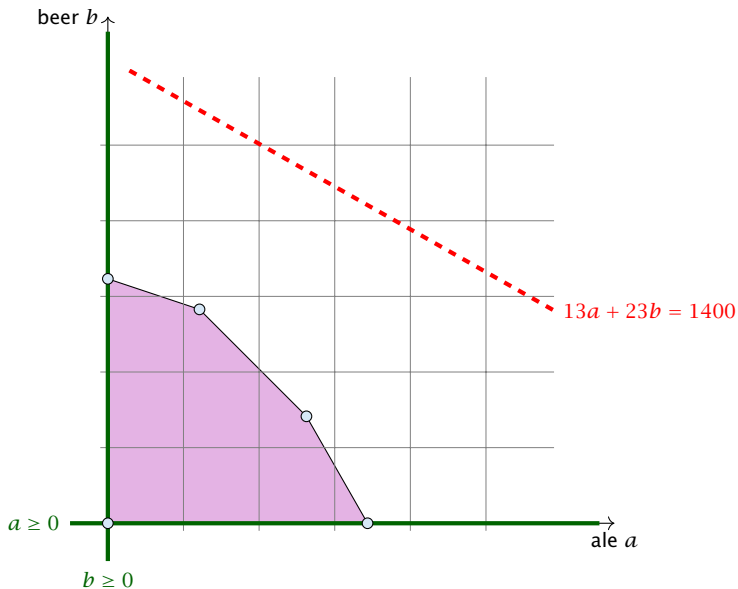
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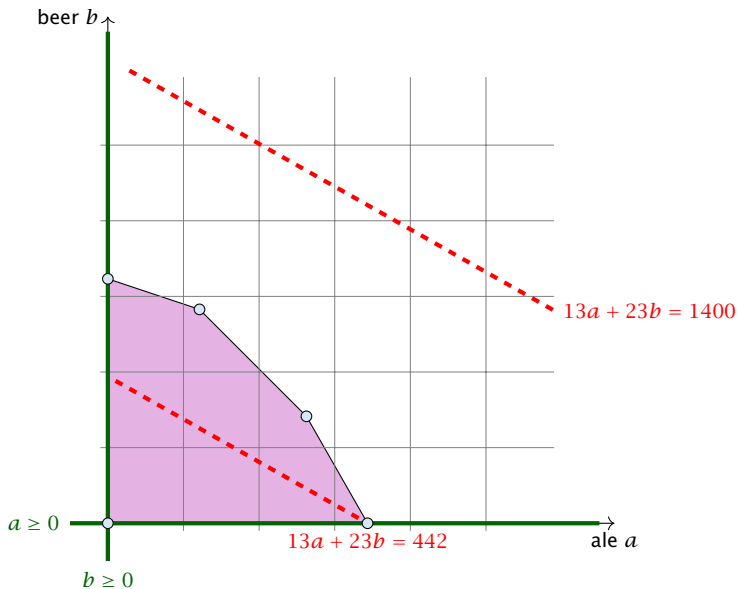
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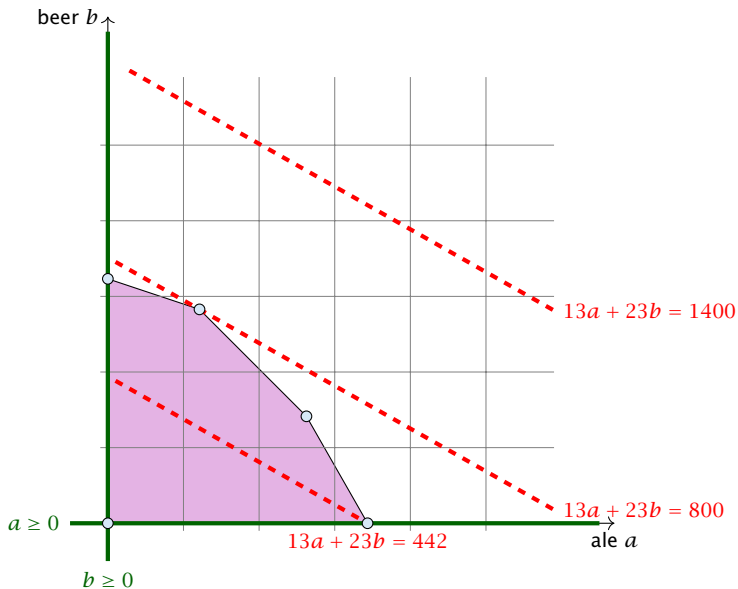
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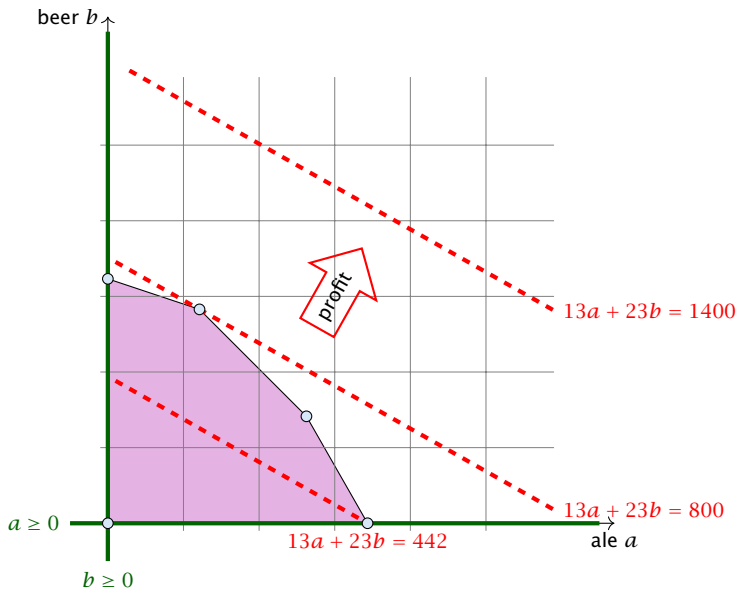
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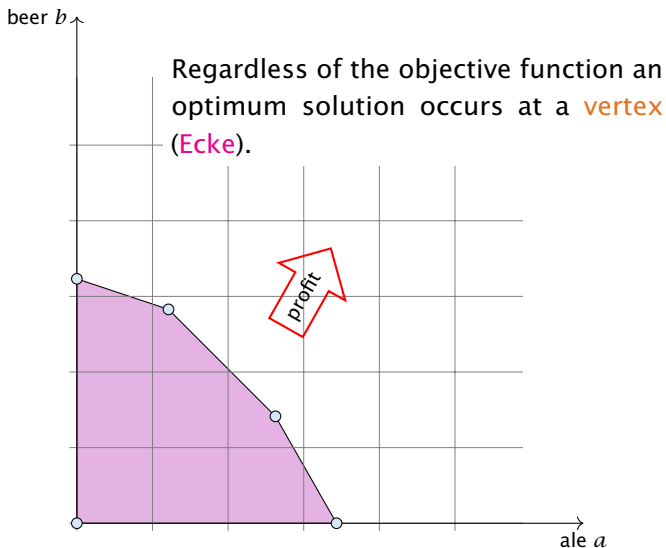
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$$P = \{x \mid Ax = b, x \geq 0\}.$$

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- ▶ An LP is **bounded** (**beschränkt**) if it is feasible and
 $\exists M \in \mathbb{R}$ s.t. $x_i \leq M$ for all $i \in \{1, \dots, n\}$ (for maximization problems)
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Definition 2

Given vectors/points $x_1, \dots, x_k \in \mathbb{R}^n$, $\sum \lambda_i x_i$ is called

- ▶ **linear combination** if $\lambda_i \in \mathbb{R}$.
- ▶ **affine combination** if $\lambda_i \in \mathbb{R}$ and $\sum_i \lambda_i = 1$.
- ▶ **convex combination** if $\lambda_i \in \mathbb{R}$ and $\sum_i \lambda_i = 1$ and $\lambda_i \geq 0$.
- ▶ **conic combination** if $\lambda_i \in \mathbb{R}$ and $\lambda_i \geq 0$.

Note that a combination involves only finitely many vectors.

Definition 3

A set $X \subseteq \mathbb{R}^n$ is called

- ▶ a **linear subspace** if it is closed under linear combinations.
- ▶ an **affine subspace** if it is closed under affine combinations.
- ▶ **convex** if it is closed under convex combinations.
- ▶ a **convex cone** if it is closed under conic combinations.

Note that an affine subspace is **not** a vector space

Definition 4

Given a set $X \subseteq \mathbb{R}^n$.

- ▶ $\text{span}(X)$ is the set of all linear combinations of X
(linear hull, span)
- ▶ $\text{aff}(X)$ is the set of all affine combinations of X
(affine hull)
- ▶ $\text{conv}(X)$ is the set of all convex combinations of X
(convex hull)
- ▶ $\text{cone}(X)$ is the set of all conic combinations of X
(conic hull)

Definition 5

A function $f: \mathbb{R}^n \rightarrow \mathbb{R}$ is **convex** if for $x, y \in \mathbb{R}^n$ and $\lambda \in [0, 1]$ we have

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

Lemma 6

If $P \subseteq \mathbb{R}^n$, and $f: \mathbb{R}^n \rightarrow \mathbb{R}$ convex then also

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Dimensions

Definition 7

The **dimension** $\dim(A)$ of an affine subspace $A \subseteq \mathbb{R}^n$ is the dimension of the vector space $\{x - a \mid x \in A\}$, where $a \in A$.

Definition 8

The **dimension** $\dim(X)$ of a convex set $X \subseteq \mathbb{R}^n$ is the dimension of its affine hull $\text{aff}(X)$.

Definition 9

A set $H \subseteq \mathbb{R}^n$ is a **hyperplane** if $H = \{x \mid a^T x = b\}$, for $a \neq 0$.

Definition 10

A set $H' \subseteq \mathbb{R}^n$ is a (closed) **halfspace** if $H = \{x \mid a^T x \leq b\}$, for $a \neq 0$.

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Definitions

Definition 11

A **polytop** is a set $P \subseteq \mathbb{R}^n$ that is the convex hull of a **finite** set of points, i.e., $P = \text{conv}(X)$ where $|X| = c$.

Definitions

Definition 12

A **polyhedron** is a set $P \subseteq \mathbb{R}^n$ that can be represented as the intersection of **finitely** many half-spaces $\{H(a_1, b_1), \dots, H(a_m, b_m)\}$, where

$$H(a_i, b_i) = \{x \in \mathbb{R}^n \mid a_i x \leq b_i\} \quad .$$

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A polyhedron P is **bounded** if there exists B s.t. $\|x\|_2 \leq B$ for all $x \in P$.

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Theorem 14

P is a bounded polyhedron iff P is a polytop.

Definition 15

Let $P \subseteq \mathbb{R}^n$, $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$. The hyperplane

$$H(a, b) = \{x \in \mathbb{R}^n \mid a^T x = b\}$$

is a **supporting hyperplane** of P if $\max\{a^T x \mid x \in P\} = b$.

Definition 16

Let $P \subseteq \mathbb{R}^n$. F is a **face** of P if $F = P$ or $F = P \cap H$ for some supporting hyperplane H .

Definition 17

Let $P \subseteq \mathbb{R}^n$.

- ▶ a face v is a **vertex** of P if $\{v\}$ is a face of P .
- ▶ a face e is an **edge** of P if e is a face and $\dim(e) = 1$.
- ▶ a face F is a **facet** of P if F is a face and $\dim(F) = \dim(P) - 1$.

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Equivalent definition for vertex:

Definition 18

Given polyhedron P . A point $x \in P$ is a **vertex** if $\exists c \in \mathbb{R}^n$ such that $c^T y < c^T x$, for all $y \in P$, $y \neq x$.

Definition 19

Given polyhedron P . A point $x \in P$ is an **extreme point** if $\nexists a, b \neq x$, $a, b \in P$, with $\lambda a + (1 - \lambda)b = x$ for $\lambda \in [0, 1]$.

Lemma 20

A vertex is also an extreme point.

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Observation

The feasible region of an LP is a Polyhedron.

Theorem 21

If there exists an optimal solution to an LP (in standard form) then there exists an optimum solution that is an extreme point.

Proof

Let x^* be an optimal solution to an LP (in standard form). If x^* is an extreme point, we are done. If not, then x^* can be written as a convex combination of two distinct feasible points x^1 and x^2 . Since x^* is optimal, the objective value at x^* is at least as large as the objective value at x^1 and x^2 . But since x^* is a convex combination of x^1 and x^2 , the objective value at x^* is a weighted average of the objective values at x^1 and x^2 . This implies that the objective value at x^1 and x^2 must both be equal to the objective value at x^* . Therefore, x^1 and x^2 are also optimal solutions. We can repeat this process until we find an extreme point that is an optimal solution.

Theorem 21

If there exists an optimal solution to an LP (in standard form) then there exists an optimum solution that is an extreme point.

Proof

- ▶ suppose x is optimal solution that is not extreme point
- ▶ there exists direction $d \neq 0$ such that $x \pm d \in P$
- ▶ $Ad = 0$ because $A(x \pm d) = b$
- ▶ Wlog. assume $c^T d \geq 0$ (by taking either d or $-d$)
- ▶ Consider $x + \lambda d, \lambda > 0$

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Convex Sets

Case 1. $[\exists j \text{ s.t. } d_j < 0]$

• Increase λ to λ^* until first component of $d + \lambda c$ hits 0.
• $d + \lambda^* c$ is feasible. Since $-c^T d = \lambda^* c^T d$ and $c^T d > 0$, $\lambda^* < 1$.
• $d + \lambda^* c$ has the more zero-component $\lambda_j = 0$ for some j (i.e. $d_j < 0$).
• (Another way to see $\lambda^* < 1$ is that with $\lambda = 1$ we have $d + \lambda c = 0$.)

Case 2. $[d_j \geq 0 \text{ for all } j \text{ and } c^T d > 0]$

• $d + \lambda c$ is feasible for all $\lambda \geq 0$ since $d_j + \lambda c_j \geq 0$ and $d_j \geq 0$.
• Since $c^T d > 0$, $c^T(d + \lambda c) = c^T d + \lambda c^T c$ increases as λ increases. Thus, $d + \lambda c$ is not optimal for any $\lambda \geq 0$.

Case 1. $[\exists j \text{ s.t. } d_j < 0]$

Choose $\lambda = 0$ until first component of d is < 0 . This λ is feasible. Since $\lambda = 0$ and $d_j < 0$, λ has the more zero-component $\lambda_j = 0$ for some j (i.e. $\lambda_j = 0$ for all j). The same argument can be repeated until all components of d are ≥ 0 .

Case 2. $[d_j \geq 0 \text{ for all } j \text{ and } c^T d > 0]$

$\lambda = 0$ is feasible for all $\lambda \geq 0$ since $d_j \geq 0$ and $c^T d > 0$. The problem is unbounded as λ can be chosen as large as we want.

Case 1. [$\exists j$ s.t. $d_j < 0$]

- ▶ increase λ to λ' until first component of $x + \lambda d$ hits 0
- ▶ $x + \lambda' d$ is feasible. Since $A(x + \lambda' d) = b$ and $x + \lambda' d \geq 0$
- ▶ $x + \lambda' d$ has one more zero-component ($d_k = 0$ for $x_k = 0$ as $x \pm d \in P$)
- ▶ $c^T x' = c^T (x + \lambda' d) = c^T x + \lambda' c^T d \geq c^T x$

Case 2. [$d_j \geq 0$ for all j and $c^T d > 0$]

- ▶ $x + \lambda d$ is feasible for all $\lambda \geq 0$ since $A(x + \lambda d) = b$ and $x + \lambda d \geq 0$
- ▶ $c^T x + \lambda c^T d > c^T x$ for all $\lambda > 0$ since $c^T d > 0$
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- ▶ $c^T x + \lambda c^T d > c^T x$ for all $\lambda > 0$

Case 1. $[\exists j \text{ s.t. } d_j < 0]$

- ▶ increase λ to λ' until first component of $x + \lambda d$ hits 0
- ▶ $x + \lambda' d$ is feasible. Since $A(x + \lambda' d) = b$ and $x + \lambda' d \geq 0$
- ▶ $x + \lambda' d$ has one more zero-component ($d_k = 0$ for $x_k = 0$ as $x \pm d \in P$)
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Convex Sets

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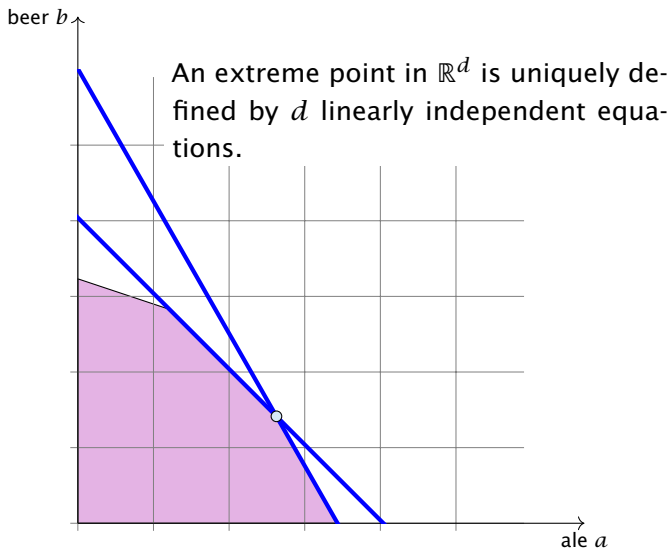
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Algebraic View



Notation

Suppose $B \subseteq \{1 \dots n\}$ is a set of column-indices. Define A_B as the subset of columns of A indexed by B .

Theorem 22

Let $P = \{x \mid Ax = b, x \geq 0\}$. For $x \in P$, define $B = \{j \mid x_j > 0\}$. Then x is extreme point iff A_B has linearly independent columns.

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- ▶ $A_{B'}$ has linearly dependent columns as $Ad = 0$
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Let $P = \{x \mid Ax = b, x \geq 0\}$. For $x \in P$, define $B = \{j \mid x_j > 0\}$. If A_B has linearly independent columns then x is a vertex of P .

- ▶ define $c_j = \begin{cases} 0 & j \in B \\ -1 & j \notin B \end{cases}$
- ▶ then $c^T x = 0$ and $c^T y \leq 0$ for $y \in P$
- ▶ assume $c^T y = 0$; then $y_j = 0$ for all $j \notin B$
- ▶ $b = Ay = A_B y_B = Ax = A_B x_B$ gives that $A_B(x_B - y_B) = 0$;
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For an LP we can assume wlog. that the matrix A has full row-rank. This means $\text{rank}(A) = m$.

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C1 if now $b_1 = \sum_{i=2}^m \lambda_i \cdot b_i$ then for all x with $A_2 x \leq b_2, \dots, A_m x \leq b_m$ we also have $A_1 x \leq b_1$, hence the first constraint is superfluous

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For an LP we can assume wlog. that the matrix A has full row-rank. This means $\text{rank}(A) = m$.

- ▶ assume that $\text{rank}(A) < m$
- ▶ assume wlog. that the first row A_1 lies in the span of the other rows $A_2, \dots, A_m$; this means

$$A_1 = \sum_{i=2}^m \lambda_i \cdot A_i, \text{ for suitable } \lambda_i$$

- C1** if now $b_1 = \sum_{i=2}^m \lambda_i \cdot b_i$ then for all x with $A_i x = b_i$ we also have $A_1 x = b_1$; hence the first constraint is superfluous
- C2** if $b_1 \neq \sum_{i=2}^m \lambda_i \cdot b_i$ then the LP is infeasible, since for all x that fulfill constraints $A_2, \dots, A_m$ we have

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From now on we will always assume that the constraint matrix of a standard form LP has full row rank.

Theorem 24

Given $P = \{x \mid Ax = b, x \geq 0\}$. x is extreme point iff there exists $B \subseteq \{1, \dots, n\}$ with $|B| = m$ and

- ▶ A_B is non-singular
- ▶ $x_B = A_B^{-1}b \geq 0$
- ▶ $x_N = 0$

where $N = \{1, \dots, n\} \setminus B$.

Proof

Take $B = \{j \mid x_j > 0\}$ and augment with linearly independent columns until $|B| = m$; always possible since $\text{rank}(A) = m$.

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Basic Feasible Solutions

$x \in \mathbb{R}^n$ is called **basic solution** (Basislösung) if $Ax = b$ and $\text{rank}(A_J) = |J|$ where $J = \{j \mid x_j \neq 0\}$;

x is a **basic feasible solution** (gültige Basislösung) if in addition $x \geq 0$.

A **basis** (Basis) is an index set $B \subseteq \{1, \dots, n\}$ with $\text{rank}(A_B) = m$ and $|B| = m$.

$x \in \mathbb{R}^n$ with $A_B x_B = b$ and $x_j = 0$ for all $j \notin B$ is **the basic solution associated to basis B** (die zu B assoziierte Basislösung)

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Basic Feasible Solutions

A BFS fulfills the m equality constraints.

In addition, at least $n - m$ of the x_i 's are zero. The corresponding non-negativity constraint is fulfilled with equality.

Fact:

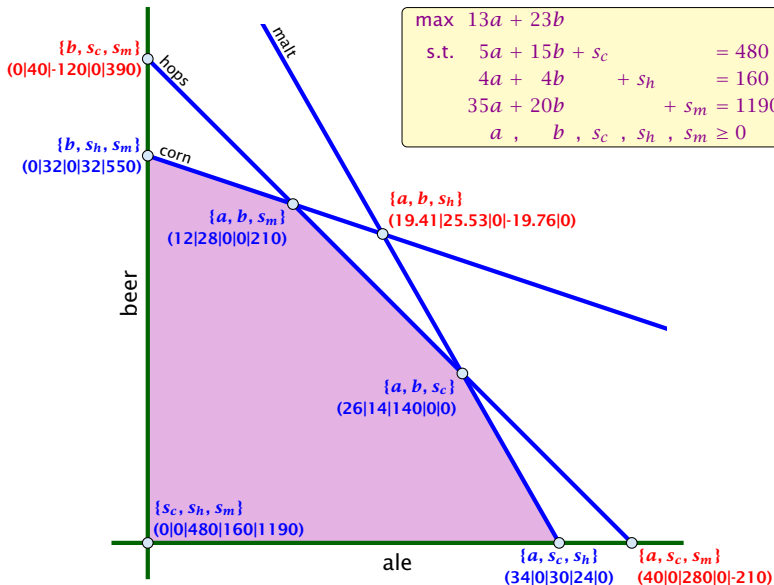
In a BFS at least n constraints are fulfilled with equality.

Basic Feasible Solutions

Definition 25

For a general LP ($\max\{c^T x \mid Ax \leq b\}$) with n variables a point x is a **basic feasible solution** if x is feasible and there exist n (linearly independent) constraints that are tight.

Algebraic View



Fundamental Questions

Linear Programming Problem (LP)

Let $A \in \mathbb{Q}^{m \times n}$, $b \in \mathbb{Q}^m$, $c \in \mathbb{Q}^n$, $\alpha \in \mathbb{Q}$. Does there exist $x \in \mathbb{Q}^n$
s.t. $Ax = b$, $x \geq 0$, $c^T x \geq \alpha$?

Questions:

- ▶ Is LP in NP? yes!
- ▶ Is LP in co-NP?
- ▶ Is LP in P?

Proof:

- ▶ Given a basis B we can compute the associated basis solution by calculating $A_B^{-1}b$ in polynomial time; then we can also compute the profit.

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Observation

We can compute an optimal solution to a linear program in time $\mathcal{O}\left(\binom{n}{m} \cdot \text{poly}(n, m)\right)$.

- ▶ there are only $\binom{n}{m}$ different bases.
- ▶ compute the profit of each of them and take the maximum

What happens if LP is unbounded?