

## 8 Seidels LP-algorithm

- ▶ Suppose we want to solve  $\min\{c^T x \mid Ax \geq b; x \geq 0\}$ , where  $x \in \mathbb{R}^d$  and we have  $m$  constraints.
- ▶ In the worst-case Simplex runs in time roughly  $\mathcal{O}(m(m+d)\binom{m+d}{m}) \approx (m+d)^m$ . (slightly better bounds on the running time exist, but will not be discussed here).
- ▶ If  $d$  is much smaller than  $m$  one can do a lot better.
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## Setting:

- ▶ We assume an LP of the form

$$\begin{array}{ll}\min & c^T x \\ \text{s.t.} & Ax \geq b \\ & x \geq 0\end{array}$$

- ▶ We assume that the LP is **bounded**.

# Ensuring Conditions

Given a **standard minimization LP**

$$\begin{array}{ll}\min & c^T x \\ \text{s.t.} & Ax \geq b \\ & x \geq 0\end{array}$$

how can we obtain an LP of the required form?

- ▶ **Compute a lower bound on  $c^T x$  for any basic feasible solution.**

# Computing a Lower Bound

Let  $s$  denote the smallest common multiple of all denominators of entries in  $A, b$ .

Multiply entries in  $A, b$  by  $s$  to obtain integral entries. This does not change the feasible region.

Add slack variables to  $A$ ; denote the resulting matrix with  $\tilde{A}$ .

If  $B$  is an optimal basis then  $x_B$  with  $\tilde{A}_B x_B = \tilde{b}$ , gives an optimal assignment to the basis variables (non-basic variables are 0).

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### Theorem 3 (Cramers Rule)

Let  $M$  be a matrix with  $\det(M) \neq 0$ . Then the solution to the system  $Mx = b$  is given by

$$x_i = \frac{\det(M_j)}{\det(M)} ,$$

where  $M_i$  is the matrix obtained from  $M$  by replacing the  $i$ -th column by the vector  $b$ .

**Proof:**

## Proof:

- ▶ Define

$$X_i = \begin{pmatrix} | & & | & | & | & & | \\ e_1 & \cdots & e_{i-1} & x & e_{i+1} & \cdots & e_n \\ | & & | & | & | & & | \end{pmatrix}$$

Note that expanding along the  $i$ -th column gives that  $\det(X_i) = x_i$ .

- ▶ Further, we have

$$MX_i = \begin{pmatrix} | & & | & | & | & & | \\ Me_1 & \cdots & Me_{i-1} & Mx & Me_{i+1} & \cdots & Me_n \\ | & & | & | & | & & | \end{pmatrix} = M_i$$

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# Bounding the Determinant

Let  $Z$  be the maximum absolute entry occurring in  $\bar{A}$ ,  $\bar{b}$  or  $c$ . Let  $C$  denote the matrix obtained from  $\bar{A}_B$  by replacing the  $j$ -th column with vector  $\bar{b}$  (for some  $j$ ).

Observe that

$$|\det(C)|$$

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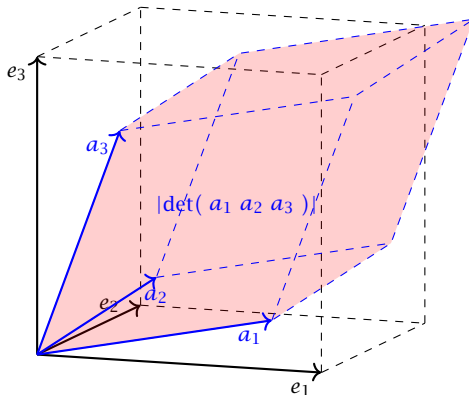
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$$\begin{aligned} |\det(C)| &\leq \prod_{i=1}^m \|C_{*i}\| \leq \prod_{i=1}^m (\sqrt{m}Z) \\ &\leq m^{m/2} Z^m . \end{aligned}$$

# Hadamards Inequality



Hadamard's inequality says that the volume of the red parallelepiped (**Spat**) is smaller than the volume in the black cube (if  $\|e_1\| = \|a_1\|$ ,  $\|e_2\| = \|a_2\|$ ,  $\|e_3\| = \|a_3\|$ ).

# Ensuring Conditions

Given a **standard minimization LP**

$$\begin{array}{ll} \min & c^T x \\ \text{s.t.} & Ax \geq b \\ & x \geq 0 \end{array}$$

how can we obtain an LP of the required form?

- ▶ **Compute a lower bound on  $c^T x$  for any basic feasible solution.** Add the constraint  $c^T x \geq -dZ(m! \cdot Z^m) - 1$ . **Note that this constraint is superfluous unless the LP is unbounded.**

# Ensuring Conditions

Compute an optimum basis for the new LP.

- ▶ If the cost is  $c^T x = -(dZ)(m! \cdot Z^m) - 1$  we know that the original LP is unbounded.
- ▶ Otw. we have an optimum basis.

In the following we use  $\mathcal{H}$  to denote the set of all constraints apart from the constraint  $c^T x \geq -dZ(m! \cdot Z^m) - 1$ .

We give a routine  $\text{SeidelLP}(\mathcal{H}, d)$  that is given a set  $\mathcal{H}$  of **explicit, non-degenerate** constraints over  $d$  variables, and minimizes  $c^T x$  over all feasible points.

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- 12: **if**  $\hat{x}^* = \text{infeasible}$  **then**
- 13:     **return** infeasible
- 14: **else**
- 15:     add the value of  $x_\ell$  to  $\hat{x}^*$  and return the solution

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- ▶ If  $d = 1$  we can solve the 1-dimensional problem in time  $\mathcal{O}(\max\{m, 1\})$ .
- ▶ If  $d > 1$  and  $m = 0$  we take time  $\mathcal{O}(d)$  to return  $d$ -dimensional vector  $x$ .
- ▶ The first recursive call takes time  $T(m-1, d)$  for the call plus  $\mathcal{O}(d)$  for checking whether the solution fulfills  $h$ .
- ▶ If we are unlucky and  $\hat{x}^*$  does not fulfill  $h$  we need time  $\mathcal{O}(d(m+1)) = \mathcal{O}(dm)$  to eliminate  $x_\ell$ . Then we make a recursive call that takes time  $T(m-1, d-1)$ .
- ▶ The probability of being unlucky is at most  $d/m$  as there are at most  $d$  constraints whose removal will decrease the objective function

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This gives the recurrence

$$T(m, d) = \begin{cases} \mathcal{O}(\max\{1, m\}) & \text{if } d = 1 \\ \mathcal{O}(d) & \text{if } d > 1 \text{ and } m = 0 \\ \mathcal{O}(d) + T(m-1, d) + \frac{d}{m}(\mathcal{O}(dm) + T(m-1, d-1)) & \text{otw.} \end{cases}$$

Note that  $T(m, d)$  denotes the **expected running time**.

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Let  $C$  be the largest constant in the  $\mathcal{O}$ -notations.

$$T(m, d) = \begin{cases} C \max\{1, m\} & \text{if } d = 1 \\ Cd & \text{if } d > 1 \text{ and } m = 0 \\ Cd + T(m - 1, d) + \frac{d}{m}(Cdm + T(m - 1, d - 1)) & \text{otw.} \end{cases}$$

Note that  $T(m, d)$  denotes the **expected running time**.

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if  $f(d) \geq df(d-1) + 2d^2$ .

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since  $\sum_{i \geq 1} \frac{i^2}{i!}$  is a constant.