

Flows

Definition 3

An (s, t) -flow in a (complete) directed graph $G = (V, V \times V, c)$ is a function $f : V \times V \mapsto \mathbb{R}_0^+$ that satisfies

1. For each edge (x, y)

$$0 \leq f_{xy} \leq c_{xy} .$$

(capacity constraints)

2. For each $v \in V \setminus \{s, t\}$

$$\sum_x f_{vx} = \sum_x f_{xv} .$$

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The **value of an (s, t) -flow f** is defined as

$$\text{val}(f) = \sum_x f_{sx} - \sum_x f_{xs} .$$

Maximum Flow Problem:

Find an (s, t) -flow with maximum value.

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LP-Formulation of Maxflow

$$\begin{array}{ll}\max & \sum_z f_{sz} - \sum_z f_{zs} \\ \text{s.t.} & \forall (z, w) \in V \times V \quad f_{zw} \leq c_{zw} \quad \ell_{zw} \\ & \forall w \neq s, t \quad \sum_z f_{zw} - \sum_z f_{wz} = 0 \quad p_w \\ & f_{zw} \geq 0\end{array}$$

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 \min & \sum_{(xy)} c_{xy} \ell_{xy} \\
 \text{s.t.} & f_{xy} \ (x, y \neq s, t) : \quad 1\ell_{xy} - 1p_x + 1p_y \geq 0 \\
 & f_{sy} \ (y \neq s, t) : \quad 1\ell_{sy} \quad + 1p_y \geq 1 \\
 & f_{xs} \ (x \neq s, t) : \quad 1\ell_{xs} - 1p_x \geq -1 \\
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with $p_t = 0$ and $p_s = 1$.

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We can interpret the ℓ_{xy} value as assigning a length to every edge.

The value p_x for a variable, then can be seen as the distance of x to t (where the distance from s to t is required to be 1 since $p_s = 1$).

The constraint $p_x \leq \ell_{xy} + p_y$ then simply follows from triangle inequality ($d(x, t) \leq d(x, y) + d(y, t) \Rightarrow d(x, t) \leq \ell_{xy} + d(y, t)$).

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One can show that there is an optimum LP-solution for the dual problem that gives an integral assignment of variables.

This means $p_x = 1$ or $p_x = 0$ for our case. This gives rise to a cut in the graph with vertices having value 1 on one side and the other vertices on the other side. The objective function then evaluates the capacity of this cut.

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