

Flows

Definition 3

An (s, t) -flow in a (complete) directed graph $G = (V, V \times V, c)$ is a function $f : V \times V \mapsto \mathbb{R}_0^+$ that satisfies

1. For each edge (x, y)

$$0 \leq f_{xy} \leq c_{xy} .$$

(capacity constraints)

2. For each $v \in V \setminus \{s, t\}$

$$\sum_x f_{vx} = \sum_x f_{xv} .$$

(flow conservation constraints)

Definition 4

The **value of an (s, t) -flow f** is defined as

$$\text{val}(f) = \sum_x f_{sx} - \sum_x f_{xs} .$$

Maximum Flow Problem:

Find an (s, t) -flow with maximum value.

LP-Formulation of Maxflow

$$\begin{array}{ll}
 \max & \sum_z f_{sz} - \sum_z f_{zs} \\
 \text{s.t.} & \forall (z, w) \in V \times V \quad f_{zw} \leq c_{zw} \quad \ell_{zw} \\
 & \forall w \neq s, t \quad \sum_z f_{zw} - \sum_z f_{wz} = 0 \quad p_w \\
 & f_{zw} \geq 0
 \end{array}$$

$$\begin{array}{ll}
 \min & \sum_{(xy)} c_{xy} \ell_{xy} \\
 \text{s.t.} & f_{xy} \ (x, y \neq s, t) : \quad 1\ell_{xy} - 1p_x + 1p_y \geq 0 \\
 & f_{sy} \ (y \neq s, t) : \quad 1\ell_{sy} \quad \quad + 1p_y \geq 1 \\
 & f_{xs} \ (x \neq s, t) : \quad 1\ell_{xs} - 1p_x \quad \quad \geq -1 \\
 & f_{ty} \ (y \neq s, t) : \quad 1\ell_{ty} \quad \quad + 1p_y \geq 0 \\
 & f_{xt} \ (x \neq s, t) : \quad 1\ell_{xt} - 1p_x \quad \quad \geq 0 \\
 & f_{st} : \quad \quad \quad 1\ell_{st} \quad \quad \geq 1 \\
 & f_{ts} : \quad \quad \quad 1\ell_{ts} \quad \quad \geq -1 \\
 & \quad \quad \quad \ell_{xy} \quad \quad \geq 0
 \end{array}$$

LP-Formulation of Maxflow

$$\begin{array}{ll} \min & \sum_{(x,y)} c_{xy} \ell_{xy} \\ \text{s.t.} & f_{xy} \ (x, y \neq s, t) : \quad 1\ell_{xy} - 1p_x + 1p_y \geq 0 \\ & f_{sy} \ (y \neq s, t) : \quad 1\ell_{sy} - \quad 1 + 1p_y \geq 0 \\ & f_{xs} \ (x \neq s, t) : \quad 1\ell_{xs} - 1p_x + \quad 1 \geq 0 \\ & f_{ty} \ (y \neq s, t) : \quad 1\ell_{ty} - \quad 0 + 1p_y \geq 0 \\ & f_{xt} \ (x \neq s, t) : \quad 1\ell_{xt} - 1p_x + \quad 0 \geq 0 \\ & f_{st} : \quad 1\ell_{st} - \quad 1 + \quad 0 \geq 0 \\ & f_{ts} : \quad 1\ell_{ts} - \quad 0 + \quad 1 \geq 0 \\ & \ell_{xy} \geq 0 \end{array}$$

LP-Formulation of Maxflow

$$\begin{array}{ll} \min & \sum_{(x,y)} c_{xy} \ell_{xy} \\ \text{s.t.} & f_{xy} \ (x, y \neq s, t) : \quad 1\ell_{xy} - 1p_x + 1p_y \geq 0 \\ & f_{sy} \ (y \neq s, t) : \quad 1\ell_{sy} - p_s + 1p_y \geq 0 \\ & f_{xs} \ (x \neq s, t) : \quad 1\ell_{xs} - 1p_x + p_s \geq 0 \\ & f_{ty} \ (y \neq s, t) : \quad 1\ell_{ty} - p_t + 1p_y \geq 0 \\ & f_{xt} \ (x \neq s, t) : \quad 1\ell_{xt} - 1p_x + p_t \geq 0 \\ & f_{st} : \quad 1\ell_{st} - p_s + p_t \geq 0 \\ & f_{ts} : \quad 1\ell_{ts} - p_t + p_s \geq 0 \\ & \ell_{xy} \geq 0 \end{array}$$

with $p_t = 0$ and $p_s = 1$.

LP-Formulation of Maxflow

$$\begin{array}{ll}\min & \sum_{(xy)} c_{xy} \ell_{xy} \\ \text{s.t.} & f_{xy} : \quad 1\ell_{xy} - 1p_x + 1p_y \geq 0 \\ & \ell_{xy} \geq 0 \\ & p_s = 1 \\ & p_t = 0\end{array}$$

We can interpret the ℓ_{xy} value as assigning a length to every edge.

The value p_x for a variable, then can be seen as the distance of x to t (where the distance from s to t is required to be 1 since $p_s = 1$).

The constraint $p_x \leq \ell_{xy} + p_y$ then simply follows from triangle inequality ($d(x, t) \leq d(x, y) + d(y, t) \Rightarrow d(x, t) \leq \ell_{xy} + d(y, t)$).

One can show that there is an optimum LP-solution for the dual problem that gives an integral assignment of variables.

This means $p_x = 1$ or $p_x = 0$ for our case. This gives rise to a cut in the graph with vertices having value 1 on one side and the other vertices on the other side. The objective function then evaluates the capacity of this cut.

This shows that the Maxflow/Mincut theorem follows from linear programming duality.