

Complementary Slackness

Lemma 3

Assume a linear program $P = \max\{c^T x \mid Ax \leq b; x \geq 0\}$ has solution x^* and its dual $D = \min\{b^T y \mid A^T y \geq c; y \geq 0\}$ has solution y^* .

1. If $x_j^* > 0$ then the j -th constraint in D is tight.
2. If the j -th constraint in D is not tight then $x_j^* = 0$.
3. If $y_i^* > 0$ then the i -th constraint in P is tight.
4. If the i -th constraint in P is not tight then $y_i^* = 0$.

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If we say that a variable x_j^* (y_i^*) has slack if $x_j^* > 0$ ($y_i^* > 0$), (i.e., the corresponding variable restriction is not tight) and a constraint has slack if it is not tight, then the above says that for a primal-dual solution pair it is not possible that a constraint **and** its corresponding (dual) variable has slack.

Proof: Complementary Slackness

Analogous to the proof of weak duality we obtain

$$c^T x^* \leq y^{*T} A x^* \leq b^T y^*$$

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From the constraint of the dual it follows that $y^T A \geq c^T$. Hence the left hand side is a sum over the product of non-negative numbers. Hence, if e.g. $(y^T A - c^T)_j > 0$ (the j -th constraint in the dual is not tight) then $x_j^* = 0$ (2.). The result for (1./3./4.) follows similarly.

Interpretation of Dual Variables

- Brewer: find mix of ale and beer that maximizes profits

$$\begin{array}{lll} \max & 13a & + \quad 23b \\ \text{s.t.} & 5a & + \quad 15b \leq 480 \\ & 4a & + \quad 4b \leq 160 \\ & 35a & + \quad 20b \leq 1190 \\ & & a, b \geq 0 \end{array}$$

- Entrepreneur: buy resources from brewer at minimum cost
 C, H, M : unit price for corn, hops and malt.

$$\begin{array}{lll} \min & 480C & + \quad 160H \quad + \quad 1190M \\ \text{s.t.} & 5C & + \quad 4H \quad + \quad 35M \geq 13 \\ & 15C & + \quad 4H \quad + \quad 20M \geq 23 \\ & & C, H, M \geq 0 \end{array}$$

Note that brewer won't sell (at least not all) if e.g.

$5C + 4H + 35M < 13$ as then brewing ale would be advantageous.

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Interpretation of Dual Variables

Marginal Price:

- ▶ How much money is the brewer willing to pay for additional amount of Corn, Hops, or Malt?
- ▶ We are interested in the marginal price, i.e., what happens if we increase the amount of Corn, Hops, and Malt by ϵ_C , ϵ_H , and ϵ_M , respectively.

The profit increases to $\max\{c^T x \mid Ax \leq b + \epsilon; x \geq 0\}$. Because of strong duality this is equal to

$$\begin{array}{ll} \min & (b^T + \epsilon^T)y \\ \text{s.t.} & A^T y \geq c \\ & y \geq 0 \end{array}$$

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Interpretation of Dual Variables

If ϵ is “small” enough then the optimum dual solution y^* might not change. Therefore the profit increases by $\sum_i \epsilon_i y_i^*$.

Therefore we can interpret the dual variables as **marginal prices**.

Note that with this interpretation, complementary slackness becomes obvious.

• If the buyer has slack of some resource (e.g. food) then he is not willing to pay anything for it (corresponding dual variable is zero).

• If the dual variable for some resource is non-zero, then an increase of this resource increases the profit of the buyer. Hence, it makes no sense to have leftovers of this resource. Therefore its slack must be zero.

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If the primal has slack on some resource (e.g. only 1000 lbs of flour) then the primal is not willing to pay anything for it (corresponding dual variable is zero). If the primal is tight (e.g. 1000 lbs of flour) then the primal is willing to pay the corresponding dual value for it.

If we increase the right-hand side of a constraint (e.g. increase the amount of flour) then the primal is willing to pay the corresponding dual value for it.

If we decrease the right-hand side of a constraint (e.g. decrease the amount of flour) then the primal is willing to pay the corresponding dual value for it.

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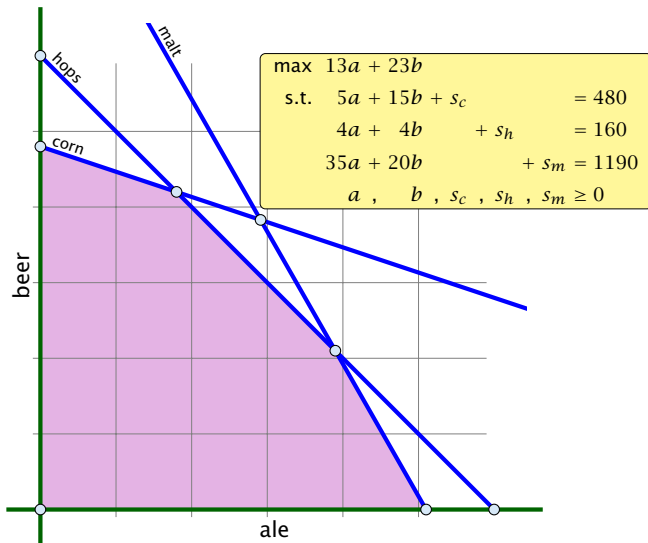
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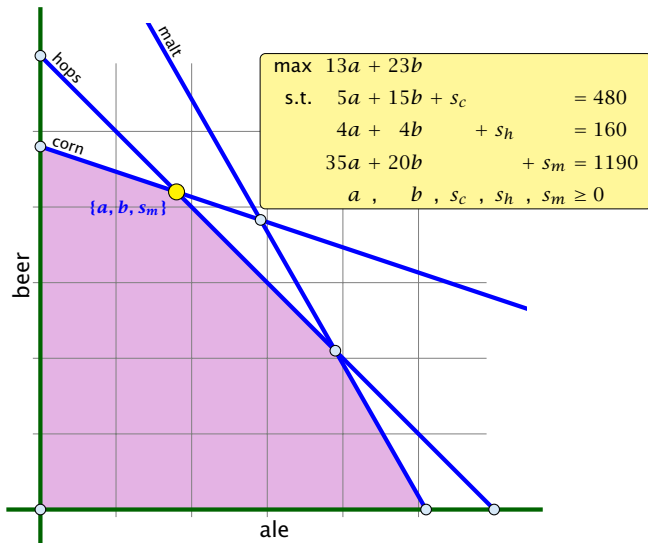
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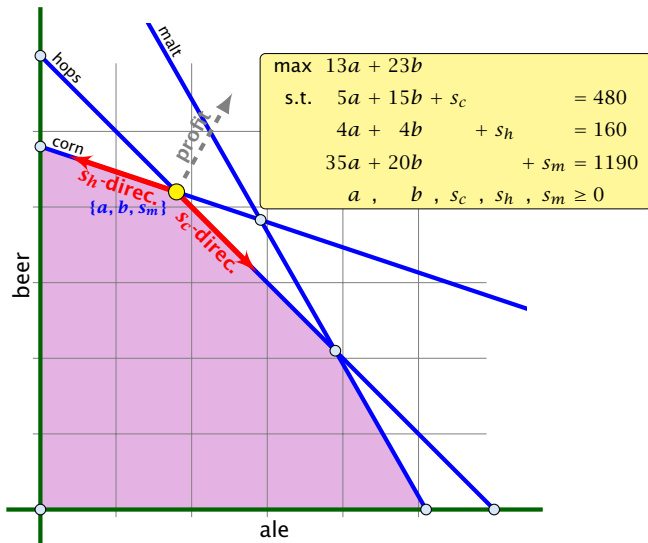
Example



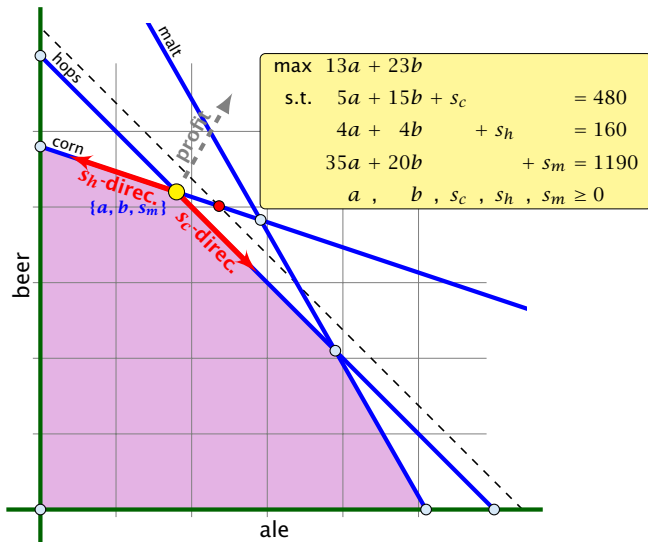
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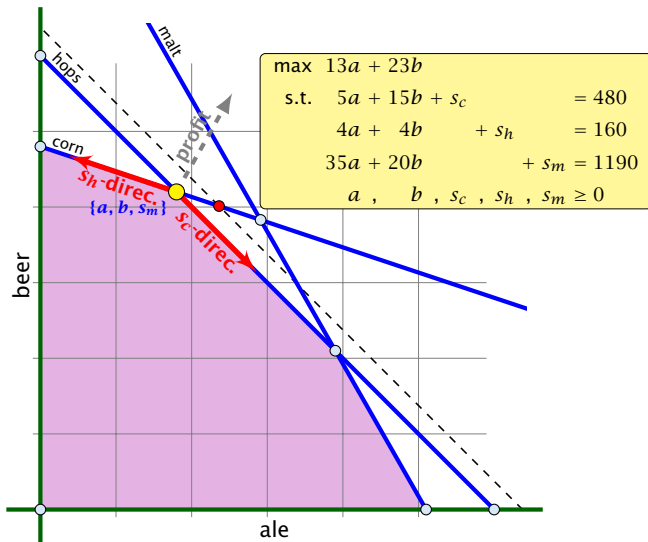
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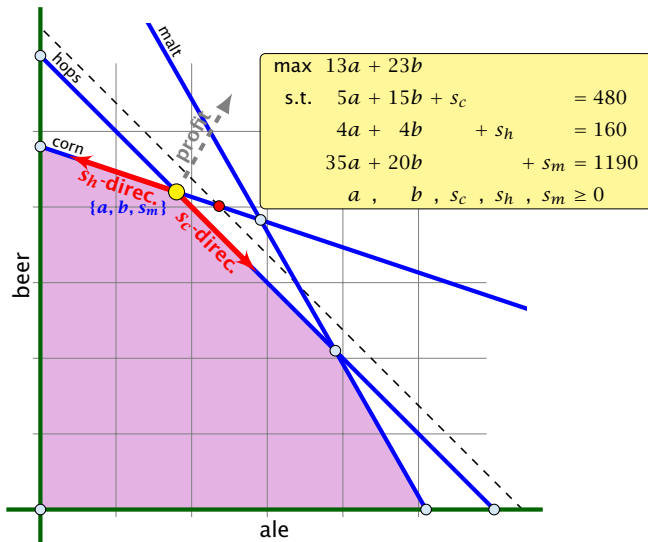
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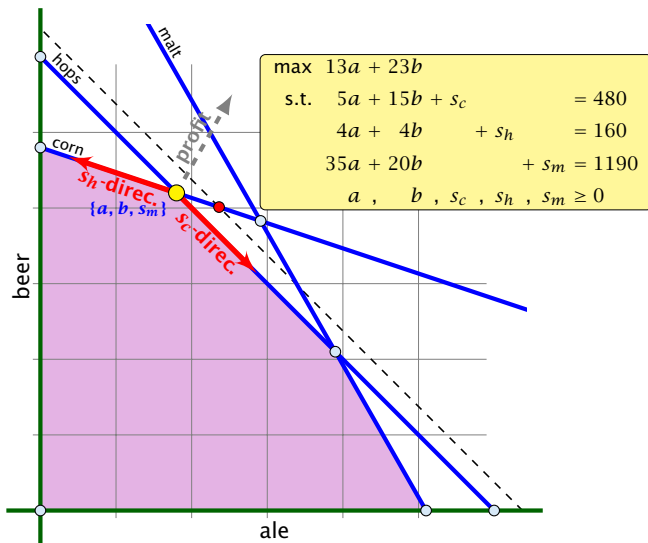
Example



The change in profit when increasing hops by one unit is

$$= c_B^T A_B^{-1} e_h.$$

Example



The change in profit when increasing hops by one unit is

$$= \underbrace{c_B^T A_B^{-1}}_{y^*} e_h.$$

Of course, the previous argument about the increase in the primal objective only holds for the non-degenerate case.

If the optimum basis is degenerate then increasing the supply of one resource may not allow the objective value to increase.